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Advance in Algebraic curves: Application to Computer Graphing

Ahmed Mohsin Mahdi [a]

[a]*Faculty of Computer Science and Information Technology, University of Al-Qadisiyah*

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ABSTRACT

Differential geometry, a branch of mathematics that studies the geometry of curves, surfaces, and manifolds (the higher-dimensional analogs of surfaces). Curve is a smoothly flowing line (non sharp changes) or a curve must bend (change direction) but in Mathematics a straight line also a curve. We will present new definitions and theorems.

MSC. 16D10; 16D70; 16D99

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1. Introduction

We will present the racing of differential geometry and evolution since BC to reach such incredible development in our time and its relation to other scientific branches and its applications in the areas of life. It shows how the concept of geometry appeared in the works of both Riemann and Lobachevski [1] and [2]., which later proved the important of this geometry to many of life problems. At the end, we focus on the subject of the study of differential geometry because of its close link to other mathematic fields.

Definition, Postulates and Axioms: The geometric history back to the very early time, not only what we have of geometry facts, so that in this period directed to collect at the result together to be in logical order. Also the Greeks did a lot of work to develop geometry no thing appeared for us specially after Euclid's famous work appeared that named Euclid's famous elements.

2. Curve

We develop the mathematical tools needed to model and study a moving object.

The object might be moving in the plane[3] and [4].

Curve is a smoothly flowing line (non sharp changes) or a curve must bend (change direction) but in Mathematics a straight line also a curve.

2.1. A parameterized Curve in $\mathbb{R}^n$.

Definition 2.1. A parameterized *curve* in $\mathbb{R}^n$ is a smooth[1] function[2] $\psi: I \to \mathbb{R}^n$,where $I \subset \mathbb{R}$ is an interval.

Definition 2.2. The function $y = f(x)$ is *continuous at* the point x_o if $f(x_o)$ is defined and $\lim_{x\to x_o} f(x) = f(x_o)$.

The function $y = f(x)$ is *continuous* in the interval $(a, b) = x; a < x < b$ if $f(x)$is continuous at every point in that interval.

Definition 2.3 (1). If $\psi: I \to \mathbb{R}^n$ is a curve with components $\psi(t) = \big(x_1(t), x_2(t), \dots, x_n(t)\big)$, then its derivative $\psi': I \to \mathbb{R}^n$, is the curve defined as$\psi'(t) = \big(x_1'(t), x_2'(t), \dots, x_n'(t)\big)$.

Higher-order derivatives are defined analogously

Definition 2.4. Let $\psi: I \to \mathbb{R}^n$ be a curve. It is called *regular* if its speed is always nonzero ($|\psi(t)| = 0$ for all $t \in I$). It is called unit-speed or parameterized by arc length if its speed is always equal to 1 ($|\psi(t)| = 0$ for all $t \in I$).

Proposition 2.5. *The derivative of a curve $\psi: I \to \mathbb{R}^n$ at time t **I** is given by the formula*

$$\psi'(t) = \lim_{h\to 0} \frac{\psi(t+h) - \psi(t)}{h}$$

Proof. *slop* $= \frac{change\ in\ \psi}{change\ in\ t}$, where t changes from t to $t + h$ and ψ changes fron $\psi(t)$ to $\psi(t + h)$

$$\frac{\Delta\psi}{\Delta t} = \frac{\psi(t+h) - \psi(t)}{h} \Longrightarrow \psi'(t) = \lim_{h\to 0} \frac{\psi(t+h) - \psi(t)}{h}$$

Definition 2.6. A set I of $\mathbb{R}$ is an interval if I contains two real numbers, so it's contains all the numbers between them.

Example 2.7. *An interval means a nonempty connected subset of* $\mathbb{R}$. *Then that every interval has one of the following forms:*

(a, b), $[a, b]$, $(a, b]$, $[a, b)$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$, $(-\infty, \infty)$?

Solution. If an interval is a bounded and contains two element a and b we denoted by $a = \inf(I)$ (I) and $b = \sup(I)$ by the definition of *sup* and *inf* every element $x \in I$is between a and b and ($a < x < b$).

Now we prove that every ($a < x < b$) is in ***I***, if x is not upper bound or lower bound then, there exists two elements y and z in I such that $y < x < z$. So by the definition x in ***I***, according to a and b belong to I we obtain the fourth types.

Now, let I an interval has lower bounded but not upper bounded. Let a be the*inf* of ***I***, every element in I is $\geq a$. We are going to prove that I contains all real numbers $x > a$, hence x is not a lower bounded and I contain y such that $y < x$ by the same way, we have there exist z such that $z > x$ hence $y < x < z$, so x belongs to I. According to a belongs to I or not we obtain two types of non-upper boundedintervals.

Finally, if an interval I is not upper bounded or lower bounded and for every element x we can find two elements y and z in I such that $y < x < z$, that lead to$x$ in I. Hence I = R.

Example2.8. *A logarithmic spiral means a plane curve of the form*

$$\psi(t) = c(\exp(\lambda t)\cos(t), \exp(\lambda t)\sin(t)),$$

and $t \in \mathbb{R}$, where $c, \lambda \in \mathbb{R}$ with $c \neq 0$. It shows the restriction to $[0, \infty)$ of a logarithmic spiral with $\lambda < 0$. Use an improper integral to prove that such a restriction has finite arc length it makes infinitely many loops around the origin.

Solution.

$$\psi(t) = c(\exp(\lambda t)\cos(t), \exp(\lambda t)\sin(t))$$

The arc length between *0 and* ∞ equals $\int_0^\infty |\psi'|\, dt$.

$$the\ arc\ length = \int_0^\infty c^2 \exp(\lambda t) = c^2 \left[\frac{\exp(\lambda t)}{\lambda}\right]_0^\infty = -\frac{c^2}{\lambda} = \infty.$$

Example2.9. *Let* $\psi(t) = \left(\sin(t), \cos(t) + \ln\left(tan\left(\frac{t}{2}\right)\right)\right)$, $t \in \left(\frac{\pi}{2}, \pi\right)$, *be a curve. Demonstrate that for every point p of its image, the segment of the tangent line at* p *between p and the y-axis has length* 1.

Solution. The first we want to prove that $[\psi'(t) = 0 \leftrightarrow \cos(t) = 0]$

$\psi(t) = (\cos(t), -\sin(t) + 1/\sin(t))$, so it is differentiable at $(\pi/2, \pi)$. It is zero if and only if $\cos(t) = 0$, i.e. when $t = \pi/2$.

The tangent line at $\psi(t)$ is

$$y - \cos(t) - \ln\left(tan\left(\frac{t}{2}\right)\right) = \frac{\cos(t)}{\sin(t)}(x - \sin(t)).$$

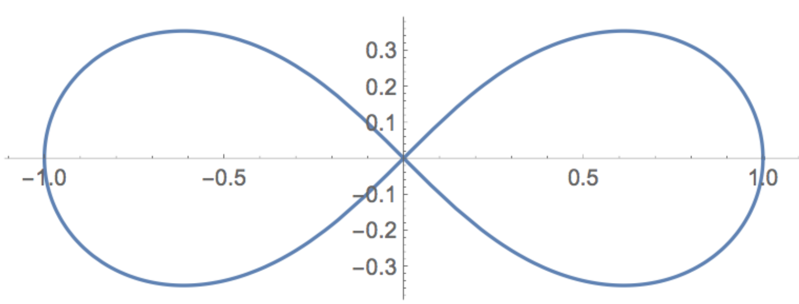
So the intersection with y-axis is $(0, \ln(\tan(t/2)))$.

Hence the distance between the point and tangency to intersection of tangent linewith y-axis is

$$sin^2(t) + \left(\cos(t) + \ln\left(tan\left(\frac{t}{2}\right)\right) - \ln\left(tan\left(\frac{t}{2}\right)\right)\right)^2 = 1.$$

Example 2.10. *Use a computer graphing application to plot the following planecurves (all with domain* $[0, 2\pi]$*)*

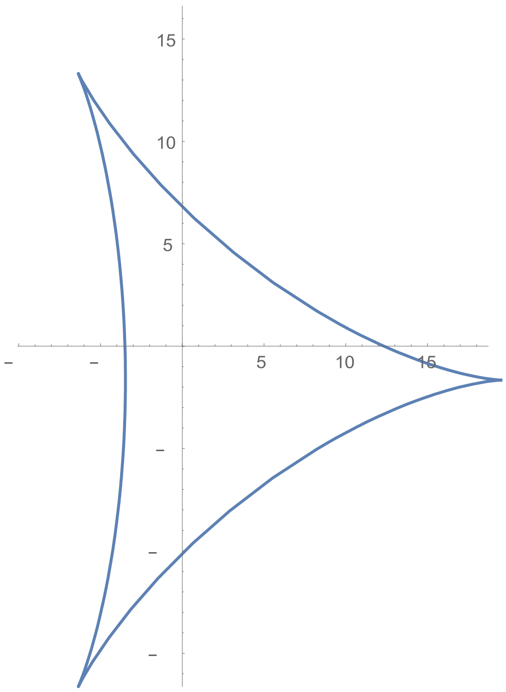
(1) The lemniscate of Bernoulli
(2) The deltoid curve
(3) The astroid curve
The epitrochoid

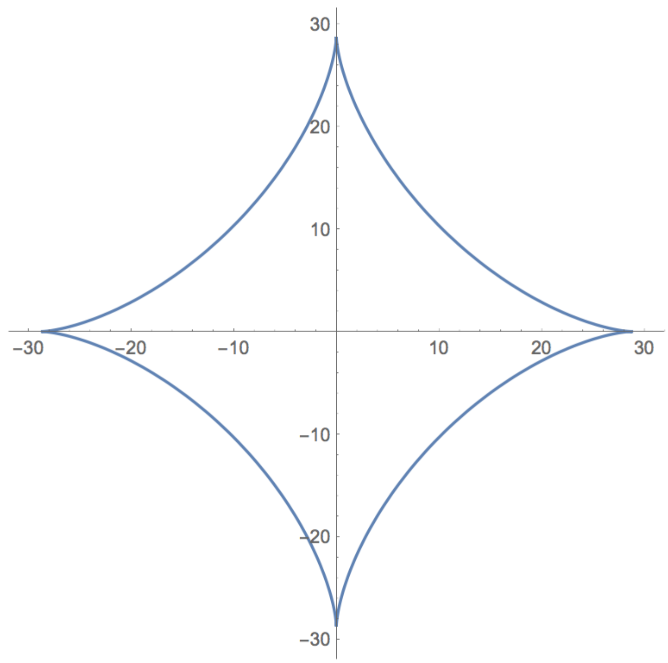


Solution.

2.2. The inner product.

Definition 2.11. The inner product of a pair of vectors$x, y \in \mathbb{R}^n$ (with componentsdenoted by $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is$\langle x, y\rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n \in \mathbb{R}^n$.





Lemma 2.12. *If $x, y, z \in \mathbb{R}^n$, and $\lambda, \mu \in \mathbb{R}$ then*

1- $\langle x, y\rangle = \langle y, x\rangle$.

2- $\langle x, x\rangle = |x|^2$, *which equals zero if and only if* $x = 0$.

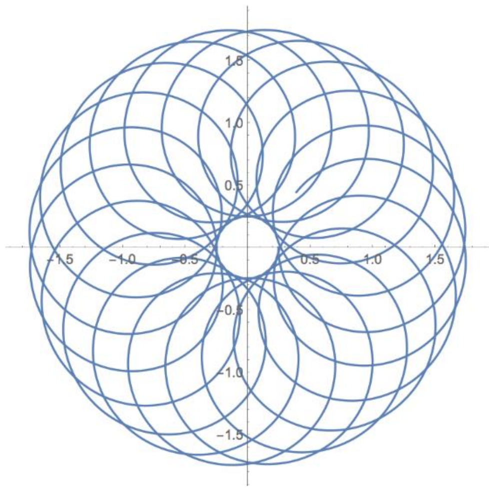
3- $\langle \lambda x + \mu y, x\rangle = \lambda \langle x, z\rangle + \mu \langle y, z\rangle$.

4- $\langle x, y\rangle \leq |\langle x, y\rangle| \leq |x||y|$.

Proof. 1- $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n = y_1x_1 + x_2y_2 + \cdots + y_nx_n = \langle y, x\rangle$

$2 - \langle x, x\rangle = x_1x_1 + x_2x_2 + \cdots + x_nx_n = x_1^2 + x_2^2 + \cdots + x_n^2 = |x|^2$
$3 - \langle \lambda(x_1 + x_2 + \cdots + x_n) + \mu(y_1 + y_2 + \cdots + y_n), (z_1 + z_2 + \cdots + z_n)\rangle = (\lambda x_1 + \mu y_1)z_1 + (\lambda x_2 + \mu y_2)z_2 + \cdots + (\lambda x_n + \mu y_n)z_n = \lambda(x_1z_1 + x_2z_2 + \cdots + x_nz_n) + \mu(y_1z_1 + y_2z_2 + \cdots + y_nz_n) = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$

4- $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \leq |x_1y_1 + x_2y_2 + \cdots + x_ny_n| \leq |x_1y_1| + |x_2y_2| + \cdots + |x_ny_n| = |x_1||y_1| + |x_2||y_2| + \cdots + |x_n||y_n| \leq |x||y|$



Definition 2.13. The angle between nonzero vectors x and y is defined as:

$\angle(x,y) = cos^{-1}\frac{\langle x,y\rangle}{|x||y|} \in \mathbb{R}$.

Recall that x and y are called orthogonal if $\langle x,y\rangle = 0$. They are called parallel ifone of them is a scalar multiple of the other.

Definition 2.14. If $X, Y \in \mathbb{R}$ with $|Y| \neq 0$, then there is a unique way to write x as a sum of two vectors:

$$X = X^{\parallel} + X^{\perp},$$

the first of which is parallel to y and the second of which is orthogonal to y. Thevector$X^{\parallel}$ is called the projection of X in the direction of Y. The signed length of $X^{\perp}$ (that is, the scalar $\lambda \in \mathbb{R}$ such that

$$X^{\perp} = \lambda\frac{Y}{|Y|}$$

is called the component of X in the direction of Y.

Definition 2.15. A set $Y = \{y_1, \dots, y_k\} \subset \mathbb{R}$ is called orthonormal if

$\langle Y_i, Y_j\rangle = 1$if $i = j$

Definition 2.16. Let $Y = (Y_1, \dots, Y_k)$ be a nonempty subset of a vector space $\mathbb{R}^n$. The span of Y, denoted by span(Y), is the set containing of all linear combinationsof vectors in Y.

$$\text{span}(Y) = \left\{\sum_{i=1}^{k} \lambda_i Y_i \mid k \in \mathbb{N}, Y_i \in Y, \lambda_i \in \mathbb{R}\right\}.$$

Notation 2.17. A basis B of a vector space V over a field F is a linearly independent subset of V that spans V.

Notation 2.18. The vector in the set $Y = \{Y_1, \dots, Y_n\}$ are said to be linearlyindependent if the equation

$$a_1Y_1 + a_2Y_2 + \dots + a_nY_n = 0,$$

can only be satisfied by $a_i = 0$ for $i = 1, 2, ..., n$.

Example 2.19. *Prove that every orthonormal set in $\mathbb{R}^n$ must be linearly independent.*

Solution. Suppose $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^n$, let

$$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = 0,$$

from som $\lambda_1, ..., \lambda_n \in \mathbb{R}$.
Then for any index $1 \le i \le k$ we have by orthonormal that

$$\langle \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n, x_i \rangle = \lambda_i \langle x_i, x_i \rangle = 0 \Rightarrow \lambda_i = 0.$$

Example 2.20. *Let* $V = \mathbb{R}^n$, $\langle X, Y \rangle = X.YV_1 = (3, 5, 4)$, $V_2 = (3, -5, 4)$, $V_3(4, 0, -3)$.
$V_1.V_2 = 0$, $V_1V_3 = 0$, $V_2V_3 = 0$
$V_1.V_1 = 50$, $V_2V_2 = 50$, $V_3V_3 = 25$.
Thus the set $\{V_1, V_2, V_3\}$ *is orthogonal but not orthonormal.*

Lemma 2.21. *If $\beta\gamma: I \to \mathbb{R}^n$ is a pair of curves, and $c: I \to \mathbb{R}$ is a smoothfunction then:*

(1) $-\frac{d}{dt}\langle \gamma(t), \beta(t) \rangle = \langle \gamma'(t), \beta(t) \rangle + \langle \gamma(t), \beta'(t) \rangle.$

(2) $-\frac{d}{dt}\big(c(t)\gamma(t)\big) = c'(t)\gamma(t) + c(t)\gamma'(t).$

Proof. (1) - Let define $\gamma(t)$ and $\beta(t)$ as in the definition [2.3]

$\frac{d}{dt}\langle \gamma(t), \beta(t) \rangle = \frac{d}{dt}\big(x_1(t)y_1(t) + x_2(t)y_2(t) + \dots + x_n(t)y_n(t)\big) = \frac{d}{dt}x_1(t)y_1(t) + \frac{d}{dt}x_2(t)y_2(t) + \dots + \frac{d}{dt}x_n(t)y_n(t) = x_1(t)y_1'(t) + x_1'(t)y_1(t) + \dots + x_n(t)y_n'(t) + x_n'(t)y_n(t) = [x_1'(t)y_1(t) + \dots + x_n'(t)y_n(t)] + [x_1(t)y_1'(t) + \dots + x_n(t)y_n'(t)] = \langle \gamma'(t), \beta(t) \rangle + \langle \gamma(t), \beta'(t) \rangle.$
(2)- Direct from the definition of the derivative.

Proposition 2.22. *Let If $\gamma, \beta: I \to \mathbb{R}^n$be a pair of curves.*

(1) If γ has constant norm (that is $|\gamma(t)|, = C$, for all $t \in I$), then $\gamma'(t)$ isorthogonal to $\gamma(t)$for all $t \in I$.

(2) If $\gamma(t)$ is orthogonal to $\beta(t)$ for all $t \in I$, then

$$\langle \gamma'(t), \beta(t) \rangle = -\langle \gamma(t)'(t), \beta(t) \rangle,$$

for all $t \in I$. Notice that the hypotheses of (1) and (2) are both true if $\{\gamma(t), \beta(t)\}$ is orthonormal for all $t \in I$. ,

(3) If $\gamma(t): I \to \mathbb{R}^n$ is a curve with constant speed, then $\gamma\ (t)$ is orthogonal to If $\gamma''(t)$ for all $t \in I$.

Proof. (1) We have $\langle\gamma(t),\gamma(t)\rangle = |\gamma(t)|^2 = C$, the derivative is zero.

$$0=\frac{d}{dt}\langle\gamma(t),\gamma(t)\rangle=\langle\gamma'(t),\gamma(t)\rangle+\langle\gamma(t),\gamma'(t)\rangle.$$

Thus $\langle\gamma(t),\gamma'(t)\rangle=0$, which mean it is orthogonal.
(2) Let $\gamma(t)$ is orthogonal to $\beta(t)$ for all $t\in I$, which means $\langle\gamma(t),\beta(t)\rangle=0$, and also the derivative must be zero.

$$0=\frac{d}{dt}\langle\gamma(t),\beta(t)\rangle=\langle\gamma'(t),\beta(t)\rangle+\langle\gamma'(t)(t),\beta(t)\rangle.$$

(3) We have $\langle\gamma'(t),\gamma'(t)\rangle = |\gamma'(t)|^2 = C$, the derivative is zero.

$$0=\frac{d}{dt}\langle\gamma'(t),\gamma'(t)\rangle=\langle\gamma'(t),\gamma''(t)\rangle+\langle\gamma''(t),\gamma'(t)\rangle.$$

Thus $\langle\gamma'(t),\gamma''(t)\rangle=0$, which mean it is orthogonal

Example 2.23. *Is the converse of part (1) of Proposition 2.22 true?*

Proof. Let $\gamma'(t)$ is orthogonal to $\gamma(t)$ for all $t\in I$.

$$0=\langle\gamma'(t),\gamma(t)\rangle=\langle\gamma'(t),\gamma(t)\rangle+\langle\gamma(t),\gamma'(t)\rangle==\frac{d}{dt}\langle\gamma(t),\gamma(t)\rangle.$$

Thus $\langle\gamma(t),\gamma(t)\rangle = |\gamma(t)|^2 = C$, *then* $\gamma(t)$ *has constant norm.*

Example 2.24. *If* $\gamma(t)\colon I\to\mathbb{R}^n$ *is a regular curve, prove that*

$$\frac{d}{dt}|\gamma(t)|=\left\langle\gamma'(t),\frac{\gamma(t)}{|\gamma(t)|}\right\rangle.$$

Solution.

$$\frac{d}{dt}|\gamma(t)|=\frac{d}{dt}\langle\gamma(t),\gamma(t)\rangle^{\frac{1}{2}}=1/2\langle\gamma(t),\gamma(t)\rangle^{-\frac{1}{2}}\{\langle\gamma'(t),\gamma(t)\rangle+\langle\gamma(t),\gamma'(t)\rangle\}=\frac{2\langle\gamma(t),\gamma'(t)\rangle}{2\langle\gamma(t),\gamma(t)\rangle^{\frac{1}{2}}}$$
$$=\left\langle\gamma'(t),\frac{\gamma(t)}{|\gamma(t)|}\right\rangle.$$

2.3. Acceleration.

Definition 2.25. Acceleration is a vector quantity that is defined as the rate at which an object changes its velocity. An object is accelerating if it is changing its velocity. Velocity is defined as a vector measurement of the rate and direction of motion or, in other terms, the rate and direction of the change in the position of an object. The following notational convention from physics: If $\gamma(t)\colon I\to\mathbb{R}^n$, is a regular curve.

$v(t)=\gamma'(t)$ (the velocity function)

$a(t)=v'(t)=\gamma''(t)$ (the acceleration function).

Also we have by the physics interpretation of a(t) comes from the vector version ofNewton's law:

$$F(t)=ma(t),$$

where m is the objects mass, and $F(t)$ is the vector-valued force acting on the object at time t.

Example 2.26. *Find velocity, acceleration and speed of particle described by* $\gamma(t)=\langle t,t^2,t^3\rangle$, *at* $t=1$.

Solution. $v(t)=\gamma'(t)=\langle 1,2t,3t^2\rangle\Rightarrow v(1)=\langle 1,2,3\rangle$

speed: $|v(t)| = \sqrt{1+4+9} = 2\sqrt{3}$.
$a(t) = v'(t) = \gamma''(\mathrm{t}) = \langle 0, 2, 6\mathrm{t}\rangle = \langle 0, 2, 6\rangle$.

Example 2.27. *Let* $\gamma(t) = \langle (1, -2t^2), t^2, (-2+2t^2)\rangle$.

1- *Compute velocity, speed, acceleration and find the unit-tangent vector.*
2- *Compute the arc length.*

Solution. 1-$\mathrm{v(t)} = \langle -4\mathrm{t}, 2\mathrm{t}, 4\mathrm{t}\rangle$.

$|v(t)| = \sqrt{16t^2 + 4t2 + 16t2} = 6t$
$a(t) = \langle -4,2,4\rangle$.

The unit-tangent: $\frac{v(t)}{|v(t)|} = \frac{1}{6t}\langle -4t, 2t, 4t\rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right\rangle$.

2- The arc length $= \int_0^2 |v(t)|dt = \int_0^2 (6t)dt = 12$ units.

Proposition 2.28. $\frac{\boldsymbol{d}}{\boldsymbol{dt}}|v(t)| = \frac{\langle a(t), v(t)\rangle}{|v(t)|} =$ *the component of a(t) in the direction of* $v(t)$.

Proof. Similarly as example 2.24.

Example 2.29. *If* γ *is a curve in* $\mathbb{R}^n$ *with* $|\gamma(t)| = c$ *(a constant), prove that* $\langle a(t), -\gamma(t)\rangle = |\gamma(t)|^2$. *Rewrite this as* $\left\langle a(t), -\frac{\gamma(t)}{|\gamma(t)|}\right\rangle = \frac{|\gamma(t)|^2}{c}$, *and notice that the left side is the component of* $a(t)$ *in the direction of the center-pointing vector. Interpret this physically in terms of centripetal force.*

Solution. The expression $\langle \gamma(t), \gamma(t)\rangle = |\gamma(t)|^2 = c^2$, is constant, the derivative of this expression must be zero:

$0 = \frac{d}{dt}\langle \gamma(t), \gamma(t)\rangle = \langle \gamma(t), v(t)\rangle + \langle v(t), \gamma(t)\rangle = 2\langle \gamma(t), v(t)\rangle$, which means $\langle \gamma(t), v(t)\rangle = 0$.

$0 = \frac{d}{d}\langle \gamma(t), v(t)\rangle = \langle v(t), v(t)\rangle + \langle \gamma(t), a(t)\rangle \Longrightarrow= \langle v(t), v(t)\rangle = -\langle \gamma(t), a(t)\rangle \Longrightarrow |v(t)|^2 = \langle a(t), -\gamma(t)\rangle$.

Example 2.30. *Find a space curve $\gamma(t) = \mathbb{R} \to \mathbb{R}^n$ with acceleration function $a(t) = (t^2 - 1, t^3, t^2 + 1)$. How unique is the solution?*
Solution.

$$v(t) = \int a(t)dt = \int (t^2 - 1, t^3, t^2 + 1)dt = \left(\left(\frac{t^3}{3} - t + c, \frac{t^4}{4} + d, \left(\frac{t^3}{3} + t + e\right)\right)\right),$$

the velocity at time t.
Space curve:

$$\gamma(t) = \int v(t)dt = \left(\left(\frac{t^4}{12} - \frac{t^2}{2} + ct + c_1\right), \left(\frac{t^5}{20} + dt + d_1\right)\left(\frac{t^4}{12} + \frac{t^2}{2} + et + e_1\right)\right),$$

the position at time t.

2.4. Reparametrization. During the previous week, i studied the section reparameterization and i read:

Definition 2.31. Suppose that $: I \to \mathbb{R}^n$ is a regular curve. A reparametrization of γ is a function of the form $\widetilde{\gamma} = \gamma\, o\, \varphi \to \mathbb{R}^n$, where $\tilde{I}$ is an interval and $\varphi : \tilde{I} \to I$ is a smooth bijection with nowhere-vanishing derivative ($\varphi'(t)$ 0 for all $t \in \tilde{I}$). And $\widetilde{\gamma}$ is called orientation-preserving if $\tilde{\varphi} > 0$, and orientation-reversing if $\tilde{\varphi} < 0$.

Proposition 2.32. *A regular curve $\gamma: I \to \mathbb{R}^n$ can be reparametrized by arc length. That is, there exists a unit-speed reparametrization of γ.*

Definition 2.33. A closed curve means a regular curve of the form $\gamma: [a, b] \to \mathbb{R}^n$ such that $\gamma(a) = \gamma(b)$ and all derivatives match:

$$\gamma(a)' = \gamma(b)', \quad \gamma(a)'' = \gamma(b)'' \quad elc.$$

If additionally is one-to-one on the domain $[a, b)$, then it is called a simple closedcurve (

Proposition 2.34. *A regular curve $\gamma: [a, b] \to \mathbb{R}^n$ is a closed curve if and only if there exists a periodic regular curve $\gamma: R \to \mathbb{R}^n$ with period ba such that $\widehat{\gamma}$ for all $t \in [a, b]$.*

Definition 2.35. Let $\gamma: [a, b] \to \mathbb{R}^n$ be a closed curve. A reparametrization ofγis a function of the form $\widehat{\gamma} = \gamma_\lambda\, o\, \varphi$: $[c, d] \to \mathbb{R}^n$, where $\lambda \in \mathbb{R}$ *and* φ: $[c, d] \to$ $[a +,b +]$ is a smooth bijection with now here vanishing derivative, whose derivativesall match at c and d; that is,

$$\phi(c)' = \phi(d)', \quad \phi(c)'' = \phi(d)'' \quad elc.$$

Proposition 2.36. *Two simple closed curves have the same trace if and only if each is a reparametrization of the other..*

1.5. Curvature. To date. I have studied in this section the curvature and its properties and I read some theorems that connect between the unit tangent, unit normal and the curvature through the rate of change of velocity (Acceleration).

Definition 2.37. Let $\gamma: I \to \mathbb{R}^n$ be a regular curve. Its curvature function $\kappa : I \to [0, \infty)$, is define as

$$\kappa(t) = \frac{|a^\perp(t)|}{|v(t)|^2}.$$

Proposition 2.38. *If γ is parameterized by arc length, then $\kappa = |a(t)|$.*

Definition 2.39. Let $\gamma: I \to \mathbb{R}^n$ be a regular curve. Define the unit tangent and the unit normal vectors at $t \in I$ as

$$T(t) = \frac{v(t)}{|v(t)|}, \quad N(t) = \frac{a^\perp(t)}{|a^\perp(t)|}.$$

Proposition 2.40. *If $\gamma: I \to \mathbb{R}^n$ is a regular curve (not necessarily of unit speed), then for all $t \in I$,*

$$\kappa = \frac{|\mathrm{T}'(t)|}{|\mathrm{v}(t)|}.$$

Proposition 2.41. *If $\gamma: I \to \mathbb{R}^n$ is a regular curve (not necessarily of unit speed), then at every time when $\kappa \neq 0$, we have $T' = \kappa|v|N$. Consequently,*

$$-\langle N', T\rangle = \langle T', N\rangle = \kappa|v|.$$

Definition 2.42. (A critical Point). We say that $x = t$ is a critical point of the function $\gamma(x)$. If $\gamma(t)$- exists and if the following are true $\gamma'(t) = 0$ or $\gamma'(t)$ does not exist.

Notation 2.43. The n-th degree of Taylor Polynomials of $\gamma(x)$ is define as

$$T_n(x) = \sum_{i=0}^{n} \frac{\gamma^i(x)}{i!}(x-a)^i.$$

Example 2.44. *For constant $a, b, c > 0$, consider the generalized helix define as*

$$\gamma(t) = (a\cos t, b\sin t, ct) \qquad t \in \mathrm{R}.$$

Where is the curvature maximal and minimal?

1.6. Plane Curve. [A plane curve is any curve, which can be drawn on the plane. Some curves are fairly simple, like a circle, and will have fairly simple algebraic equations. Some are very complex, like your signature, and may be very difficult to describe with an equation. Plane curves were studied intensively from the seven- tenth through the nineteenth centuries.] In This section, I studied some specializedproperties of regular plane curve (regular curves in the plan $\mathbb{R}^2$). Let as first con-sider the linear isomorphism [An isomorphism between two vector space V and Wis a map $f: V \to W$ such that:

1- f is one to one.
2- $f(v_1 + v_2) = f(v_1) + f(v_2)$ for all $v_1, v_2 \in V$.
3- $f(rv) = rf(v)$ for $r \in \mathbb{R}$.
$R_{90} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$R_{90}(x, y) = (-y, x)$$

whose effect is to rotate the vector (x, y) by 90 degrees counterclockwise

Definition 2.45. Let $\gamma : I \to \mathbb{R}^2$ is a unit-speed plane curve. At any time $t \in I$. We call $\kappa_s : I \to \mathbb{R}$ the signed curvature function which define as

$$\kappa(t) = \frac{a(t)}{R_{90}\big(v(t)\big)},$$

its negative if the curve is turning clockwise at t, and its positive if counterclockwise.

Lemma 2.46. *Let $\gamma : I \to \mathbb{R}^2$ is a unit-speed plane curve. At any time $t \in I$ With $\kappa_s : I \to \mathbb{R}$ then $|\kappa_s(t)| = |\kappa(t)|$*

Definition 2.47. If $: I \to \mathbb{R}^2$ is a regular plane curve (not necessarily parameterized by arc length), then for all $t \in I$,

$$\kappa_s(t) = \frac{\left\langle a(t), R_{90}\left(\frac{v(t)}{|v(t)|}\right)\right\rangle}{|v(t)|^2} = \frac{\langle a(t), R_{90}\big(v(t)\big)\rangle}{|v(t)|^3}.$$

Proposition 2.49. *If $\gamma : I \to \mathbb{R}^2$ is a unit-speed plane, then there exists a smooth angle function, $\theta : I \to \mathbb{R}$, such that for all $t \in I$, we have*

$$v(t) = (\cos\theta(t), \sin\theta(t)).$$

This function is unique up to adding an integer multiple of 2π.

Definition 2.50. The rotation index of a unit-speed closed plane curve $\gamma : [a, b] \to \mathbb{R}^2$ equals $\frac{1}{2\pi}\big(\theta(b) - \theta(a)\big)$, where is the angle function from Proposition 3.5. The rotation index of a regular closed plane curve (not necessarily of unit speed) means the rotation index of an orientation preserving unit-speed re parameterization of it.

2.7. Space Curves. In this short subsection we concern a smooth curve γ in the standard three dimensional Euclidean space E. Let this curve be defined (up to translations and rotations of E) by its curvature $\kappa(s)$ and its torsion τ (s), the arguments is the arc-length parameter.

Definition 2.51. If $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{R}^3$, then

$$a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \in \mathbb{R}^3.$$

Lemma 2.52. *Let* $a, b \in \mathbb{R}^3$

. (1) $a \times b$ is orthogonal to both a and b.

. (2) $|a \times b| = |a||b| \sin(\theta) = \sqrt{|a|^2|b|^2 - \langle a, b\rangle^2}$ = the area of the parallelogram spanned by a *and* b (where $\theta = \angle(a, b)$).

. (3) *The direction of* $a \times b$ *is given by the right-hand rule.*

Lemma 2.53. *If* $a, b, c \in \mathbb{R}^3$ *and* $\mu, \lambda\mathbb{R}$, *then :*

(1) $a \times b = -(b \times a)$.

(2) $(\lambda + \mu b) \times c = \lambda\ (a \times c) + \mu(b \times c)$,
$a \times (\lambda\ b + \mu c) = \lambda\ (a \times b) + \mu(a \times c)$.

Lemma 2.54. *If* $\gamma, \beta \to \mathbb{R}^3$ *is a pairs of space curves, then*

$$\frac{d}{dt}\big(\gamma(t) \times \beta(t)\big) = \gamma'(t) \times \beta(t) + \gamma(t) \times \beta'(t).$$

Proposition 2.55. *If* $\gamma : I \to \mathbb{R}^3$ *is a regular space curve, then for all* $t \in I$,

$$\kappa(t) = \frac{|v(t) \times a(t)|}{|v(t)|^3}.$$

Definition 2.56. Let $\gamma : I \to \mathbb{R}^3$ be a regular space curve. Let $t \in I$ with $\kappa(t) \neq 0$. Then the frenet frame at t is the basis $T(t), N(t), B(t)$ of $\mathbb{R}^3$ define as

$$T(t) = \frac{v(t)}{|v(t)|}, N(t) = \frac{a^\perp(t)}{|a^\perp(t)|} = \frac{T'(t)}{|T'(t)|}, B(t) = T(t) \times N(t)$$

Individually they are called the unit tangent, unit normal, and unit binormal vectorsat t.

Definition 2.57. Let $\gamma : I \to \mathbb{R}^3$ be a regular curve. Let $t \in I$ with $\kappa(t) \neq 0$. The torsion of γ at t, denoted by $\tau(t)$, is

$$\tau(t) = \frac{-\langle B'(t), N(t)\rangle}{|v(t)|}.$$

Proposition 2.58. *Let* $\gamma : I \to \mathbb{R}^3$ *be a regular curve. Let* $t \in I$ *with* $\kappa(t) \neq 0$*The trace of* γ *is constrained to a plane if and only if* $\tau(t) = 0$ *for all* $t \in I$.

Proposition 2.59. (The Frenet Equations.) *Let* $\gamma : I \to \mathbb{R}^3$ *be a regular curve. At every time* $t \in I$ *with* $\kappa(t) \neq 0$*, the derivatives of the vectors in the Frenet frame are*

(1) $T' = |v|\kappa N$.

(2) $N' = -|v|\kappa T$.

(3) $B' = -|v|\tau N$.

2.8. Rigid Motion. A Rigid Motion (motions are the orientation preserving isometrics.) is the action of taking an object and moving it to a different location withoutaltering its shape or size. For examples of a Rigid motion are translation and rotation . But reflection and glide reflection are isometrics, but are not motions. Where,

Translation: It is a shifting of a shape, where all the shapes are moved in the same direction and the same distance. Shapes are simply translated in a direction without loss of orientation.

Reflection occurs when an image is flipped over along an axis. A way to en- visage this is by placing a small mirror along an object to act as an axis of reflection.

Rotation To understand rotation, imagine sticking a pin through the duplicate copy of tracing paper and moving it around the pin, which serves as the center of rotation.

Glide reflection A glide reflection is a reflection around an axis, combined witha translation along the same axis.

3. Additional Topics in Curve

This section studied more deeply into the geometry of curves, including some of the famous theorems in the field. The theory of curves is an old and extremely well developed mathematical topic.

3.1. **Theorems of Hopf and Jordan.**

Definition 3.1. Let $\gamma : [a, b] \to \mathbb{R}^3$ be a simple plane curve . Let $C = \gamma([a, b])$, denote its trace, Then the rotation index of γ is define

$$\mathrm{ind}(\gamma) = \frac{1}{2\pi}\big(\phi(b) - \phi(a)\big).$$

Theorem 3.2. (Hopf's Umlaufsatz) *Let γ be a regular simple close plane curve.Then*

$$\text{ind}(\gamma) = \pm 1 .$$

Theorem 3.3. (The Jordan curve theorem)

$$\mathbb{R}^2 - C = \{p \in \mathbb{R}^2 \mid P \notin C\}$$

has exactly two path-connected components. Their common boundary is C. One component (which we call the interior) is bounded, while the other (which we callthe exterior) is unbounded.

Definition 3.4. A simple closed plane curve γ :$[a, b] \to \mathbb{R}^2$ is called positively oriented if it satisfies the following equivalent conditions:

(1) The rotation index of γ equals 1.

(2) The interior is on ones left as one traverses γ; more precisely, for each $t \in [a, b]$, $R_{90}(\gamma(t)')$ points toward the interior in the sense that there exists $\delta > 0$ such that $\gamma(t)$ + $sR_{90}(\gamma)'$lies in the interior for all $s \in (0, \delta)$. Otherwise, γ is negatively oriented, in which case its rotation index equals 1, and $R_{90}(\gamma)'$ points toward the exterior for all $t \in [a, b]$.

Proposition 3.5. *If f : $[a, b] \to S'$ is a continuous function with $f(a) = f(b)$, then there exists a continuous angle function φ : $[a, b] \to \mathbb{R}$ such that for all $t \in [a, b]$, we have*

$$f(t) = (\cos\varphi(t), \sin\varphi(t)).$$

This function is unique up to adding an integer multiple of 2π. The degree of f isdefined as the integer $\frac{1}{2\pi}(\phi(b) - \phi(a))$.

Definition 3.6. A piecewise-regular curve in R' is a continuous function γ : $[a, b] \to \mathbb{R}^n$ with a partition, $a = t_0 < t_1 < \ldots < t_n = b$, such that the restriction, γ_i, ofγ to each subinterval $[t_i, t_{i+1}]$ is a regular curve. It is called closed if additionally $\gamma(a) = \gamma(b)$, and simple if γ is one-to-one on the domain $[a, b)$. It is said to be ofunit speed if each γ_i is of unit speed.

Theorem 3.7. *(Generalized Hopf s Umlaufsatz) Let : [a, b]→ $\mathbb{R}^2$be a unit-speed positively oriented piecewise-regular simple closed plane curve. Let s denote its signed curvature function, and let i be the list of signed angles at its corners. Then*

$$\int_a^b k_s(t) + \sum_i \alpha_i = 2\pi.$$

Definition 3.8. Let γ : $[a, b] \to \mathbb{R}^2$ be a piecewise-smooth simple closed plane curve with signed angles denoted by ℵ The ith interior angle of γ, denoted by$\beta_i \in [0, 2\pi]$, is defined as

$$\beta_i = \begin{cases} \pi - \alpha & \text{if } \gamma \text{ is positively oriented} \\ \pi + \alpha & \text{if } \gamma \text{ is negatively oriented} \end{cases}$$

In theorem 3.7, γ is assumed to be positively oriented, so the theorem becomes

$$\int_a^b k_s(t) = \sum_i \beta_i - (n-2)\pi,$$

where n is the number of corners.

If the smooth segments of γ are straight-line segments, then this becomes

$$\sum_i \beta_i = (n-2).$$

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