Journal of Al-Qadisiyah for Computer Science and Mathematics Vol. 14(3) 2022 , pp Math. 65–74

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Almost Approximaitly Nearly Quasiprime Submodules

Ali Sh. Ajeel[a], Haibat K. Mohammadali[b]

[a]*Department of Mathematics, College of Computer Science and Mathematics, Univ. of Tikrit, Iraq . Email: Ali.shebl@st.tu.edu.iq*

[b]*Department of Mathematics, College of Computer Science and Mathematics, Univ. of Tikrit, Iraq . Email: H.mohammadali@tu.edu.iq*

ARTICLE INFO

Article history:
Received: 15 /07/2022
Rrevised form:19 /08/2022
Accepted : 31 /08/2022
Available online: 24 /09/2022

Keywords:

Prime submodules.
Quasi prime submodules.
Nearly quasi prime submodules.
Approximaitly quasi prime submodules.
Almost approximaitly nearly quasiprime submodules.

ABSTRACT

Let R be a commutative ring with identity and U be an R-module. This study deals the concept of almost approximaitly nearly quasiprime submodules as a new generalization of quasiprime submodules, also generalizations of (prime, nearly quasiprime, nearly prime, approximaitly quasiprime and approximaitly prime) submodules. We give same basic properties, and characterizations of this concept. Furthermore, we study the behaviour of almost approximaitly nearly quasiprime submodules under R-homomorphisms. Also, we explain the relationships between almost approximaitly nearly quasiprime submodules with the above concepts.

MSC..

https://doi.org/10.29304/jqcm.2022.14.3.1028

1. Introduction

Quasiprime submodules of left unitary R-module U over commutative ring have been introduced and studied in 1999 by [1], where a proper submodule F for U is called quasiprime if whenever $acu \in F$, for $a, c \in R, u \in U$, implies that either $au \in F$ or $cu \in F$ as generalization of a prime submodule [2]. Recently, many authors have focused on generalizing the concept of a quasiprime submodule such as "Nearly quasiprime, Approximaitly quasiprime, Approximaitly quasi-primary, Weakly approximaitly quasiprime, and Weakly nearly quasiprime" submodules see [3, 4, 5, 6, 7]. In this paper we introduce and study a new generalization of quasiprime submodule called almost approximaitly nearly quasiprime submodules as a proper submodule F for an R-module U is called almost approximaitly nearly quasiprime (simply Alappnq-prime) submodule if whenever $acu \in F$, for $a, c \in R, u \in U$, implies that either $au \in F + (soc(U) + J(U))$ or $cu \in F + (soc(U) + J(U))$. Where $soc(U)$ is the socle of U defined by the intersection of all essential submodule in U, and $J(U)$ is the Jacobson radical of U defined to be the intersection of all maximal submodules in U[8]. The concept of almost approximaitly nearly quasiprime submodule is also, generalizations of concepts "nearly prime and approximaitly prime" submodules which appear in [9, 10].
Recall that an R-module U is cyclic if there exist $u \in U$ such that $U = uR$ [11]. An R-module U is a semi simple if and only if $soc(U) = U$ [8]. A submodule F of an R-module U is called small if $F + K = U$ implies that $K = U$ for any proper submodule K of U [12]. A subset S of a ring R is called multiplicatively closed if $1 \in S$ and $ab \in S$ for every

∗Corresponding author

Email addresses:

Communicated by 'sub etitor'

$a,b \in S$, let T be the set of all order pairs (u,s) where $u \in U$ and $s \in S$, the relation on T is defined by $(u,s)\sim(u',s')$ if there exists $t \in S$ such that $t(su' - s'u) = 0$ is an equivalence relation, and we denoted the equivalence classes of (u,s) by $\frac{u}{s}$. Let $S^{-1}U$ denoted the set of all equivalence classes T with respect to this relation. $S^{-1}U$ is an R-module [13]. An R-module U is multiplication if every submodule F of U is of the form $F = IU$ for some ideal I of R [14].

2. Basic Properties and Characterizations of Alappnq-prime Submodules.

Definition 2.1 A proper submodule F of an R-module U is called almost approximaitly nearly quasiprime (simply Alappnq-prime) submodule, if for any $acu \in F$, for $a,c \in R, u \in U$, implies that either $au \in F + (soc(U) + J(U))$ or $cu \in F + (soc(U) + J(U))$. And an ideal J of a ring R is called Alappnq-prime ideal of R if J is an Alappnq-prime R-submodule of an R-module R.

Remarks and Examples 2.2

1. Let $U = Z_{72}, R = Z$, the submodule $F = \langle\bar{4}\rangle$ is an Alappnq-prime submodule of Z_{72}. Thus for each $a,c \in Z, u \in Z_{72}$, if $acu \in F$, implies that either $au \in F + \big(soc(Z_{72}) + J(Z_{72})\big) = \langle\bar{4}\rangle + (\langle\overline{12}\rangle + \langle\bar{6}\rangle) = \langle\bar{2}\rangle$ or $cu \in F + \big(soc(Z_{72}) + J(Z_{72})\big) = \langle\bar{2}\rangle$. That is if $2.2.\bar{1} = \bar{4} \in F$, for $2 \in Z$, $\bar{1} \in Z_{72}$, implies that $2.\bar{1} \in F + \big(soc(Z_{72}) + J(Z_{72})\big) = \langle\bar{2}\rangle$.

2. It's obvious that every quasiprime submodule of an R-module U is an Alappnq-prime submodule of U, but contrariwise isn't true as in example:
   Let $U = Z_{72}, R = Z$, the submodule $X = \langle\bar{4}\rangle$ is an Alappnq-prime submodule of Z_{72}[see (1)]. But X is not quasiprime submodule of Z_{72}, because $2.2.\bar{1} \in X$, but $2.\bar{1} \notin X$.

3. It's obvious that every prime submodule of an R-module U is an Alappnq-prime submodule of U, but contrariwise isn't true as in example:
   Let $U = Z_{72}, R = Z$, the submodule $X = \langle\bar{4}\rangle$ is an Alappnq-prime submodule of Z_{72}[see (1)]. But X is not prime submodule of Z_{72}, because $2.\bar{2} \in X$, but neither $\bar{2} \in X$ nor $2 \in [X{:}Z_{72}] = 4Z$.

4. It's obvious that every nearly quasiprime submodule of an R-module U is an Alappnq-prime submodule of U, but contrariwise isn't true as in example:
   Let $U = Z_{12}, R = Z$ and the submodule $X = \langle\bar{6}\rangle$ is an Alappnq-prime submodule of Z_{12}. Thus for each $a,c \in Z, u \in Z_{12}$, if $acu \in X$, implies that either $au \in X + \big(soc(Z_{12}) + J(Z_{12})\big) = \langle\bar{6}\rangle + (\langle\bar{2}\rangle + \langle\bar{6}\rangle) = \langle\bar{2}\rangle$ or $cu \in X + \big(soc(Z_{12}) + J(Z_{12})\big) = \langle\bar{2}\rangle$. That is if $2.3.\bar{1} \in X$, for $2,3 \in Z, \bar{1} \in Z_{12}$, implies that $2.\bar{1} \in X + \big(soc(Z_{12}) + J(Z_{12})\big) = \langle\bar{2}\rangle$. But X is not nearly quasiprime submodule of Z_{12}, because $2.3.\bar{1} \in X$, but neither $2.\bar{1} \in X + J(Z_{12}) = \langle\bar{6}\rangle + \langle\bar{6}\rangle = \langle\bar{6}\rangle$ nor $3.\bar{1} \in X + J(Z_{12}) = \langle\bar{6}\rangle$.

5. It's obvious that every nearly prime submodule of an R-module U is an Alappnq-prime submodule of U, but contrariwise isn't true as in example:
   Consider the Z-module $Z\oplus Z$, the submodule $X = 3Z\oplus(0)$ is not nearly prime submodule of the Z-module $Z\oplus Z$, since $3(1,0) \in X$, but $(1,0) \notin X + J(Z\oplus Z)$ and $3 \notin [(3Z\oplus(0)) + J(Z\oplus Z){:}_Z Z\oplus Z] = (0)$. But X is an Alappnq-prime submodule of $Z\oplus Z$.

6. It's obvious that every approximaitly quasiprime submodule of an R-module U is an Alappnq-prime submodule of U, but contrariwise isn't true as in example:
   Let $U = Z_{72}, R = Z$ and the submodule $X = \langle\bar{4}\rangle$ is an Alappnq-prime submodule of Z_{72}[see (1)]. But X is not approximaitly prime submodule of Z_{72}, because $2.\bar{2} \in X$, but $\bar{2} \notin X + soc(Z_{72}) = \langle\bar{4}\rangle + \langle\overline{12}\rangle = \langle\bar{4}\rangle$ and $2 \notin [X + soc(Z_{72}){:}_Z Z_{72}] = [\langle\bar{4}\rangle{:}_Z Z_{72}] = 4Z$.

7. It's obvious that every approximaitly prime submodule of an R-module U is an Alappnq-prime submodule of U, but contrariwise isn't true as in example:
   Consider the Z-module $Z\oplus Z$, the submodule $X = 5Z\oplus(0)$ is not an Alappn-prime submodule of the Z-module $Z\oplus Z$, since $5(1,0) \in X$, but $(1,0) \notin X + soc(Z\oplus Z)$ and $5 \notin [(5Z\oplus(0)) + soc(Z\oplus Z){:}_Z Z\oplus Z] = (0)$. But X is an Alappnq-prime submodule of $Z\oplus Z$.

8. The intersection of any two Alappnq-prime submodules of an R-module U need not to be an Alappnq-prime submodule of U, the following example explains that:
Let $U = Z_{72}$, $R = Z$, the submodules $\langle\bar{2}\rangle$ and $\langle\bar{3}\rangle$ are Alappnq-prime submodules of Z_{72} [because $\langle\bar{2}\rangle$ and $\langle\bar{3}\rangle$ are prime]. But $\langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ is not Alappnq-prime submodule of Z_{72}, because $2.3.\bar{1} \in \langle\bar{6}\rangle$, for $2,3 \in Z$, $\bar{1} \in Z_{72}$, but $2.\bar{1} \notin \langle\bar{6}\rangle + \left(soc(Z_{72}) + J(Z_{72})\right) = \langle\bar{6}\rangle + (\langle\overline{12}\rangle + \langle\bar{6}\rangle) = \langle\bar{6}\rangle$ and $3.\bar{1} \notin \langle\bar{6}\rangle + \left(soc(Z_{72}) + J(Z_{72})\right) = \langle\bar{6}\rangle$.

9. The residual of Alappnq-prime submodule of an R-module U need not to be Alappnq-prime ideal of R. The following example explains that:
Let $U = Z_{72}$, $R = Z$, the submodule $X = \langle\bar{4}\rangle$. X is an Alappnq-prime submodule of Z_{72} [see (1)]. But $[X:_Z Z_{72}]$=$[\langle\bar{4}\rangle:_Z Z_{72}] = 4Z$ is not an Alappnq-prime ideal of Z because $2.2.1 \in 4Z$ for $2,1 \in Z$ and $2.1 \notin 4Z + (soc(Z) + J(Z)) = 4Z + (0) = 4Z$.

Now, we introduce many characterizations of Alappnq-prime submodules.

Proposition 2.3 Let U be an R-module, and F be a proper submodule of U. Then F is an Alappnq-prime submodule of U if and only if $[F:_U ac] \subsetneq [F + (soc(U) + J(U)):_U a] \cup [F + (soc(U) + J(U)):_U c]$ for all $a, c \in R$.

Proof ($\Longrightarrow$) Let $u \in [F:_U ac]$, implies that $acu \in F$. But F is an Alappnq-prime submodule, then either $au \in F + (soc(U) + J(U))$ or $cu \in F + (soc(U) + J(U))$. It follows that either $u \in [F + (soc(U) + J(U)):_U a]$ or $u \in [F + (soc(U) + J(U)):_U c]$. Thus $[F:_U ac] \subsetneq [F + (soc(U) + J(U)):_U a] \cup [F + (soc(U) + J(U)):_U c]$.

($\Longleftarrow$) Let $acu \in F$, for $a, c \in R$, $u \in U$, then $u \in [F:_U ac] \subsetneq [F + (soc(U) + J(U)):_U a] \cup [F + (soc(U) + J(U)):_U c]$, implies that $u \in [F + (soc(U) + J(U)):_U a]$ or $u \in [F + (soc(U) + J(U)):_U c]$. Hence $au \in F + (soc(U) + J(U))$ or $cu \in F + (soc(U) + J(U))$. Thus F is an Alappnq-prime submodule of U.

Proposition 2.4 Let U be an R-module and F be a proper submodule of U. Then F is an Alappnq-prime submodule of U if and only if for every $a \in R$ and $u \in U$ with $au \notin F + (soc(U) + J(U))$, $[F:_R au] \subseteq [F + (soc(U) + J(U)):_R u]$.

Proof ($\Longrightarrow$) Suppose that F is an Alappnq-prime submodule of U, and let $c \in [F:_R au]$, implies that $acu \in F$. Since F is an Alappnq-prime submodule of U and $au \notin F + (soc(U) + J(U))$ then $cu \in F + (soc(U) + J(U))$.That is $c \in [F + (soc(U) + J(U)):_R u]$. Hence $[F:_R au] \subseteq [F + (soc(U) + J(U)):_R u]$.

($\Longleftarrow$) Let $acu \in F$, for $a, c \in R$, and $u \in U$ with $au \notin F + (soc(U) + J(U))$, then $c \in [F:_R au] \subseteq [F + (soc(U) + J(U)):_R u]$. Hence $c \in [F + (soc(U) + J(U)):_R u]$, that is $cu \in F + (soc(U) + J(U))$. Therefor F is an Alappnq-prime submodule of U.

Proposition 2.5 Let U be a cyclic R-module, and F be a proper submodule of U. Then F is an Alappnq-prime submodule of U if and only if $[F:_R acu] \subsetneq [F + (soc(U) + J(U)):_R au] \cup [F + (soc(U) + J(U)):_R cu]$ for all $a, c \in R$, $u \in U$.

Proof ($\Longrightarrow$) Let $t \in [F:_R acu]$, for $a, c \in R, u \in U$, implies that $ac(tu) \in F$. But F is an Alappnq-prime submodule of U, then either $a(tu) \in F + (soc(U) + J(U))$ or $c(tu) \in F + (soc(U) + J(U))$, it follows that either $t \in [F + (soc(U) + J(U)):_R au]$ or $t \in [F + (soc(U) + J(U)):_R cu]$. Hence $t \in [F + (soc(U) + J(U)):_R au] \cup [F + (soc(U) + J(U)):_R cu]$. Thus $[F:_R acu] \subsetneq [F + (soc(U) + J(U)):_R au] \cup [F + (soc(U) + J(U)):_R cu]$.

($\Longleftarrow$) Let $U = Ru$ for some $u \in U$. Suppose that $acx \in F$ for $a, c \in R, x \in U$. Then there exists an element $a \in R$ such that $x = au$. Therefore $acx = acau \in F$ that is $a \in [F:_R acu] \subsetneq [F + (soc(U) + J(U)):_R au] \cup [F + (soc(U) + J(U)):_R cu]$, it follows that $a \in [F + (soc(U) + J(U)):_U au]$ or $a \in [F + (soc(U) + J(U)):_U cu]$. Hence $aau \in F + (soc(U) + J(U))$ or $cau \in F + (soc(U) + J(U))$, that is $ax \in F + (soc(U) + J(U))$ or $cx \in F + (soc(U) + J(U))$. Thus F is an Alappnq-prime submodule of U.

Proposition 2.6 Let U be an R-module, and F be a submodule of U. Then F is an Alappnq-prime submodule of U if and only if whenever $acL \subseteq F$, for $a, c \in R$ and L is submodule of U, implies that either $aL \subseteq F + (soc(U) + J(U))$ or $cL \subseteq F + (soc(U) + J(U))$.

Proof ($\Longrightarrow$) Suppose that $acL \subseteq F$, $a, c \in R$ and L is submodule of U, with $aL \nsubseteq F + (soc(U) + J(U))$ and $cL \nsubseteq F + (soc(U) + J(U))$. So there exists a nonzero elements $x_1, x_2 \in L$ such that $ax_1 \notin F + (soc(U) + J(U))$ and $cx_2 \notin F + (soc(U) + J(U))$. Now $acx_1 \in F$ and F is an Alappnq-prime submodule of U and $ax_1 \notin F + (soc(U) + J(U))$, implies that $cx_1 \in F + (soc(U) + J(U))$. Also $acx_2 \in F$ and F is an Alappnq-prime submodule of U and $cx_2 \notin F + (soc(U) + J(U))$, implies that $ax_2 \in F + (soc(U) + J(U))$. Again $ac(x_1 + x_2) \in F$ and F is Alappnq-prime submodule of U, implies that either $a(x_1 + x_2) \in F + (soc(U) + J(U))$ or $c(x_1 + x_2) \in F + (soc(U) + J(U))$. If $a(x_1 + x_2) \in F + (soc(U) + J(U))$, that is $ax_1 + ax_2 \in F + (soc(U) + J(U))$ and $ax_2 \in F + (soc(U) + J(U))$, implies that $ax_1 \in F + (soc(U) + J(U))$ which is contradiction. If $c(x_1 + x_2) \in F + (soc(U) + J(U))$, that is $cx_1 + cx_2 \in F + (soc(U) + J(U))$ and $cx_1 \in F + (soc(U) + J(U))$, implies that $cx_2 \in F + (soc(U) + J(U))$ which is contradiction. Hence $aL \subseteq F + (soc(U) + J(U))$ or $cL \subseteq F + (soc(U) + J(U))$.

($\Longleftarrow$) Suppose that $acu \in F$, for $a, c \in R$, and $u \in U$, then $ac(u) \subseteq F$, so by hypothesis either $a(u) \subseteq F + (soc(U) + J(U))$ or $c(u) \subseteq F + (soc(U) + J(U))$. That is either $au \in F + (soc(U) + J(U))$ or $cu \in F + (soc(U) + J(U))$. Hence F is an Alappnq-prime submodule of U.

The following corollaries are direct consequence of above proposition.

Corollary 2.7 Let U be an R-module, and F be a submodule of U. Then F is an Alappnq-prime submodule of U if and only if whenever $IJL \subseteq F$, for I, J are ideals of R and L is submodule of U, implies that either $IL \subseteq F + (soc(U) + J(U))$ or $JL \subseteq F + (soc(U) + J(U))$

Corollary 2.8 Let U be an R-module, and F be a submodule of U. Then F is an Alappnq-prime submodule of U if and only if whenever $aJu \subseteq F$, for $a \in R$, J is an ideal in R and $u \in U$, implies that either $au \in F + (soc(U) + J(U))$ or $Ju \subseteq F + (soc(U) + J(U))$.

Proposition 2.9 Let U be an R-module, and F be a proper submodule of U, with $soc(U) + J(U) \subseteq F$. Then F is an Alappnq-prime submodule of U if and only if $[F:_U I]$ is an Alappnq-prime submodule of U for every ideal I of R.

Proof ($\Longrightarrow$) Assume $acu \in [F:_U I]$, for $a, c \in R, u \in U$, implies that $acuI \subseteq F$, that is $acua \in F$ for each $a \in I$. Since F is an Alappnq-prime submodule of U, it follows that either $aua \in F + (soc(U) + J(U))$ or $cua \in F + (soc(U) + J(U))$, but $soc(U) + J(U) \subseteq F$, implies that $F + (soc(U) + J(U)) = F$. Thus either $aua \in F$ or $cua \in F$ for each $a \in I$. That is either $au \in [F:_U I] \subseteq [F:_U I] + (soc(U) + J(U))$ or $cu \in [F:_U I] \subseteq [F:_U I] + (soc(U) + J(U))$. Hence $[F:_U I]$ is an Alappnq-prime submodule of U.

($\Longleftarrow$) Follows by put $I = R$.

The following propositions give some basic properties of Alappnq-prime submodules.

Proposition 2.10 Let U be an R-module with $J(U)$ or $soc(U)$ is a quasiprime submodule of U, and $F \subset U$ with $F \subseteq J(U)$ or $F \subseteq soc(U)$. Then F is an Alappnq-prime submodule of U.

Proof Suppose that $J(U)$ is quasiprime and $F \subseteq J(U)$. Let $acu \subseteq F$, for $a, c \in R, u \in U$. Since $F \subseteq J(U)$, so $acu \subseteq J(U)$. But $J(U)$ is a quasiprime submodule of U then either $au \in J(U) \subseteq F + (soc(U) + J(U))$ or $cu \in J(U) \subseteq F + (soc(U) + J(U))$. Hence either $au \in F + (soc(U) + J(U))$ or $cu \in F + (soc(U) + J(U))$. Thus F is an Alappnq-prime submodule of U.

Similar arguments follows if $soc(U)$ is quasiprime and $F \subseteq soc(U)$.

The following proposition shows that the intersection of two Alappnq-prime submodules is Alappnq-prime submdule under certain condition.

“It is well known that a submodule F of U is a maximal essential if and only if $soc(U) \subseteq F$ [15, Ex.12(5). p 242]”.

Proposition 2.11 Let U be an R-module with either F or K are maximal essential submodule of U, and $K \nsubseteq F$. If F and K are Alappnq-prime submodules of U then $F \cap K$ is an Alappnq-prime submodule of U.

Proof Let $acu \in F \cap K$, for $a, c \in R, u \in U$, then $acu \in F$ and $acu \in K$. But F and K are Alappnq-prime submodules of U, then either $au \in F + (soc(U) + J(U))$ or $cu \in F + (soc(U) + J(U))$ and $au \in K + (soc(U) + J(U))$ or $cu \in K + (soc(U) + J(U))$. Thus $au \in \big(F + (soc(U) + J(U))\big) \cap \big(K + (soc(U) + J(U))\big)$ or $cu \in \big(F + (soc(U) + J(U))\big) \cap \big(K + (soc(U) + J(U))\big)$. Since F or K are maximal essential in U then $soc(U) \subseteq F$ or $soc(U) \subseteq K$ and since F or K are maximal then $J(U) \subseteq F$ or $J(U) \subseteq K$. Suppose F is a maximal essential in U so $soc(U) \subseteq F$ and $J(U) \subseteq F$ and hence $(soc(U) + J(U)) \subseteq F$, it follows that either $au \in F \cap \big(K + (soc(U) + J(U))\big)$ or $cu \in F \cap \big(K + (soc(U) + J(U))\big)$. Therefor by Modular law we have either $au \in (F \cap K) + (soc(U) + J(U))$ or $cu \in (F \cap K) + (soc(U) + J(U))$. Therefore $F \cap K$ is an Alappnq-prime submodule of U.

Similar arguments follows if K is maximal essential.

Proposition 2.12 Let F be a proper submodule of an R-module U. If F is an Alappnq-prime submodule of U then $S^{-1}F$ is an Alappnq-prime submodule of an $S^{-1}R$-module $S^{-1}U$ where S is multiplicatively closed subset of R.

Proof Let $\frac{a_1}{c_1}\frac{a_2}{c_2}\frac{u}{c_3} \in S^{-1}F$, for $\frac{a_1}{c_1}, \frac{a_2}{c_2} \in S^{-1}R, \frac{u}{c_3} \in S^{-1}U$ and $a_1, a_2 \in R, c_1, c_2, c_3 \in S, u \in U$, then there exists a non-zero $t_1 \in S$ such that $a_1a_2(t_1u) \in F$. But F is an Alappnq-prime submodule of U, implies that either $a_1t_1u \in F + soc(U) + J(U)$ or $a_2t_1u \in F + soc(U) + J(U)$. It follows that either $\frac{a_1}{c_1}\frac{u}{c_3} \in S^{-1}(F + soc(U) + J(U)) \subseteq S^{-1}F + soc(S^{-1}U) + J(S^{-1}U)$ or $\frac{a_2}{c_2}\frac{u}{c_3} \in S^{-1}(F + soc(U) + J(U)) \subseteq S^{-1}F + soc(S^{-1}U) + J(S^{-1}U)$. Therefore $S^{-1}F$ is an Alappnq-prime submodule of $S^{-1}U$.

Proposition 2.13 Let $f: U \to U'$ be an R-epimorphism, and $\ker f$ is a small submodule of U. If F is an Alappnq-prime submodule of U' then $f^{-1}(F)$ is an Alappnq-prime submodule of U.

Proof Let $acu \in f^{-1}(F)$, for $a, c \in R, u \in U$, implies that $acf(u) \in F$. But F is an Alappnq-prime submodule of U', so either $af(u) \in F + (soc(U') + J(U'))$ or $cf(u) \in F + (soc(U') + J(U'))$. Hence either $au \in f^{-1}(F) + f^{-1}\Big(\big(soc(U') + J(U')\big)\Big) \subseteq f^{-1}(F) + (soc(U) + J(U))$ or $cu \in f^{-1}(F) + f^{-1}\Big(\big(soc(U') + J(U')\big)\Big) \subseteq f^{-1}(F) + (soc(U) + J(U))$. That is either $au \in f^{-1}(F) + (soc(U) + J(U))$ or $cu \in f^{-1}(F) + (soc(U) + J(U))$. Hence $f^{-1}(F)$ is an Alappnq-prime submodule of U.

Proposition 2.14 Let $f: U \to U'$ be an R-epimorphism, and $\ker f$ is a small submodule of U. If F be an Alappnq-prime submodule of U with $Kea\ f \subseteq F$ then $f(F)$ is an Alappnq-prime submodule of U'.

Proof Let $acu' \in f(F)$, for $a, c \in R, u' \in U'$, and since f is onto, then $f(u) = u'$ for some $u \in U$, thus $acf(u) \in f(F)$, it follows that $acf(u) = f(n)$ for some $n \in F$. That is $f(acu - n) = 0$, so $acu - n \in Kea\ f \subseteq F$, implies that $acu \in F$, implies that $u \in [F :_R ac]$. But F is an Alappnq-prime submodule, then either $au \in F + soc(U) + J(U)$ or $cu \in F + soc(U) + J(U)$. Thus $au' = af(u) \in f(F) + f\Big(\big(soc(U) + J(U)\big)\Big) \subseteq f(F) + (soc(U') + J(U'))$ or $cu' = af(u) \in f(F) + f\Big(\big(soc(U) + J(U)\big)\Big) \subseteq f(F) + (soc(U') + J(U'))$. Hence $au' \in f(F) + (soc(U') + J(U'))$ or $cu' \in f(F) + (soc(U') + J(U'))$. Thus $f(F)$ is an Alappnq-prime submodule of U'.

Proposition 2.15 Let U be an R-module, and F, K are submodules of U such that $K \subseteq F$, and F is a proper submodule of U. If F is an Alappnq-prime submodule of U, then $\frac{F}{K}$ is an Alappnq-prime submodule of $\frac{U}{K}$.

Proof Let $ac(u + K) = acu + K \in \frac{F}{K}$, for $a, c \in R, u + K \in \frac{U}{K}, u \in U$. Then $acu \in F$. But F is an Alappnq-prime submodule, then either $au \in F + (soc(U) + J(U))$ or $cu \in F + (soc(U) + J(U))$. It follows that $au + K \in \frac{F + (soc(U) + J(U))}{K}$ or $cu + K \in \frac{F + (soc(U) + J(U))}{K}$, that is either $au + K \in \frac{F}{K} + \frac{F + soc(U)}{K} + \frac{F + J(U)}{K} \subseteq \frac{F}{K} + soc\left(\frac{U}{K}\right) + J\left(\frac{U}{K}\right)$ or $cu + K \in \frac{F}{K} + \frac{F + soc(U)}{K} + \frac{F + J(U)}{K} \subseteq \frac{F}{K} + soc\left(\frac{U}{K}\right) + J\left(\frac{U}{K}\right)$. Hence $\frac{F}{K}$ is an Alappnq-prime submodule of $\frac{U}{K}$.

Proposition 2.16 Let U be a semi simple R-module, and F, K are submodules of U such that $K \subseteq F$, and F is a proper submodule of U. If K and $\frac{F}{K}$ are Alappnq-prime submodules of U and $\frac{U}{K}$ respectively, then F is an Alappnq-prime submodule of U.

Proof Let $acu \in F$, for $a, c \in R$, $u \in U$. So $(u+K) = acu + acK \in \frac{F}{K}$. If $acu \in K$ and K is an Alappnq-prime submodule, then either $au \in K + (soc(U) + J(U)) \subseteq F + (soc(U) + J(U))$ or $cu \in K + (soc(U) + J(U)) \subseteq F + (soc(U) + J(U))$, it follows that F is an Alappnq-prime submodule of U. So, we may assume that $acu \notin K$. It follows that $(u+K) \in \frac{F}{K}$, but $\frac{F}{K}$ is an Alappnq-prime submodule of $\frac{U}{K}$, then either $a(u+K) \subseteq \frac{F}{K} + \left(soc\left(\frac{U}{K}\right) + J\left(\frac{U}{K}\right)\right)$ or $c(u+K) \subseteq \frac{F}{K} + \left(soc\left(\frac{U}{K}\right) + J\left(\frac{U}{K}\right)\right)$. Since U is a semi simple, hence by [11, Ex.(12)(a), p. 239] $soc\left(\frac{M}{Ð}\right) = \frac{soc(M)+Ð}{Ð}$ and $J\left(\frac{M}{Ð}\right) = \frac{J(M)+Ð}{Ð}$. Thus, either $a(u+K) \in \frac{F}{K} + \frac{K+soc(U)}{K} + \frac{K+J(U)}{K}$ or $c(u+K) \in \frac{F}{K} + \frac{K+soc(U)}{K} + \frac{K+J(U)}{K}$. Since $K \subseteq F$, it follows that $K + soc(U) \subseteq F + soc(U)$ and $K + J(U) \subseteq F + J(U)$, hence $\frac{F}{K} + \frac{K+soc(U)}{K} + \frac{K+J(U)}{K} \subseteq \frac{F}{K} + \frac{F+soc(U)}{K} + \frac{F+J(U)}{K}$. Thus either $a(u+K) \in \frac{F+(soc(U)+J(U))}{K}$ or $c(u+K) \in \frac{F+(soc(U)+J(U))}{K}$, it follows that either $au \in F + (soc(U) + J(U))$ or $cu \in F + (soc(U) + J(U))$. Hence F is an Alappnq-prime submodule of U.

3. Sufficient Conditions Alappnq-prime Submodules to be (Quasiprime, Prime, Nearly quasiprime, Nearly prime, Approximaitly quasiprime, and Approximaitly prime) Submodules.

As we give in Remarks and Examples 2.2 (2)(3)(4)(5)(6)(7), every (quasiprime, prime, nearly quasiprime, nearly prime, approximaitly quasiprime, and approximaitly prime) submodules of an R-module U is Alappnq-prime submodules of U, but contrariwise isn't true. The following results showed that under certain conditions the reverse implication is holds.

Proposition 3.1 Let U be R-module and $F \subset U$ such that $J\left(\frac{U}{F}\right) = (0)$ and $soc(U) \subseteq F$. Then F is quasiprime if and only if F is Alappnq-prime.

Proof ($\Longrightarrow$) Direct.

($\Longleftarrow$) Since $J\left(\frac{U}{F}\right) = (0)$, then by [9, Theo. (9.1.4)(b)] we get $J(U) \subseteq F$. Let $acu \in F$ for $a, c \in R$, $u \in U$. Since F is Alappnq-prime, then either $au \in F + soc(U) + J(U)$ or $cu \in F + soc(U) + J(U)$. But $soc(U) \subseteq F$ and $J(U) \subseteq F$, hence $F + soc(U) = F$ and $F + soc(U) + J(U) = F + J(U) = F$. Thus either $au \in F$ or $cu \in F$. Therefore F is quasiprime.

Proposition 3.2 Let U be R-module and F is an essential submodule for U with $J(U) \subseteq F$. Then F is quasiprime if and only if F is Alappnq-prime.

Proof ($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $acu \in F$ for $a, c \in R$, $u \in U$, hence either $au \in F + (soc(U) + J(U))$ or $cu \in F + (soc(U) + J(U))$. Since F is essential submodule of U, then $soc(U) \subseteq F$ and by hypotheses $J(U) \subseteq F$, we get $F + soc(U) = F$ and $F + J(U) = F$, thus $F + soc(U) + J(U) = F$. Hence either $au \in F$ or $cu \in F$.

The following corollaries are direct consequence of proposition (3.1) and proposition (3.2).

Corollary 3.3 Let U be R-module and $F \subset U$ such that $soc(U) + J(U) \subseteq F$. Then F is quasiprime if and only if F is Alappnq-prime submodule.

Corollary 3.4 Let U be R-module and F is maximal submodule for U with $soc(U) \subseteq F$. Then F is quasiprime if and only if F is Alappnq-prime submodule.

Proposition 3.5 Let U be R-module with $soc(U) = (0), J(U) = (0)$ and $F \subset U$. Then F is quasiprime if and only if F is Alappnq-prime submodule of U.

Proof Direct.

Proposition 3.6 Let U be multiplication R-module with F is maximal submodule for U and $soc(U) \subseteq F$. Then F is prime if and only if F is Alappnq-prime.

Proof ($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $au \in F$ for $a \in R, u \in U$, then $a(u) \subseteq F$. Since U is multiplication then $(u) = IU$ for some ideal I for R, it follows that $a(u) = aIU \subseteq F$, that is $aIu \subseteq F$, for all $u \in U$. Since F is an Alappnq-prime submodule of U, hence by corollary 2.8 $au \in F + (soc(U) + J(U))$ or $Iu \subseteq F + (soc(U) + J(U))$. Since F is maximal submodule of U, then $J(U) \subseteq F$ and by hypotheses $soc(U) \subseteq F$, we get $F + (soc(U) + J(U)) = F$. That is either $aU \subseteq F + \left(soc(U) + J(U)\right) = F$ or $(u) \subseteq F + \left(soc(U) + J(U)\right) = F$. Thus either $aU \subseteq F$ or $u \in F$. Hence F is prime.

By the same way we can prove the following propositions.

Proposition 3.7 Let U be multiplication R-module and $F \subset U$ such that $J\left(\frac{U}{F}\right) = (0)$, and $soc(U) \subseteq F$. Then F is prime if and only if F is Alappnq-prime.

Proposition 3.8 Let U be multiplication R-module and $F \subset U$ such that $soc(U) + J(U) \subseteq F$. Then F is prime if and only if F is Alappnq-prime.

Proposition 3.9 Let U be multiplication R-module with F is essential submodule in U and $J(U) \subseteq F$. Then F is prime if and only if F is Alappnq-prime.

Proposition 3.10 Let U be R-module, $soc(U) \subseteq J(U)$ and $F \subset U$. Then F is nearly quasiprime if and only if F is Alappnq-prime.

Proof ($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $acu \in F$ for $a, c \in R, u \in U$. Since F is Alappnq-prime, then either $au \in F + soc(U) + J(U)$ or $cu \in F + soc(U) + J(U)$. Since $soc(U) \subseteq J(U)$, then $soc(U) + J(U) = J(U)$, thus either $au \in F + J(U)$ or $cu \in F + J(U)$. Hence F is nearly quasiprime submodule of U.

Proposition 3.11 Let U be R-module with $F \subset U$ and $soc(U) \subseteq F$. Then F is nearly quasiprime if and only if F is Alappnq-prime.

Proof ($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $acu \in F$ for $a, c \in R, u \in U$. Since F is Alappnq-prime, then either $au \in F + soc(U) + J(U)$ or $cu \in F + soc(U) + J(U)$. Since $soc(U) \subseteq F$, then $F + soc(U) = F$, so $F + soc(U) + J(U) = F + J(U)$. Thus either $au \in F + J(U)$ or $cu \in F + J(U)$. Hence F is nearly quasiprime submodule of U.

The proofs of the following results are direct.

Proposition 3.12 Let U be R-module with F is proper of U, and $soc(U) = (0)$. Then F is nearly quasiprime if and only if F is Alappnq-prime.

Proposition 3.13 Let U be R-module and F is an $e$ssential submodule for U. Then F is nearly quasiprime if and only if F is Alappnq-prime.

Proposition 3.14 Let U be multiplication R-module with $F \subset U$ and $soc(U) \subseteq F$. Then F is nearly prime if and only if F is Alappnq-prime.

Proof ($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $au \in F$ for $a \in R, u \in U$. Since U is multiplication then $(u) = IU$ for some ideal I for R, it follows that $a(u) = aIU \subseteq F$, that is $aIu \subseteq F$, for all $u \in U$. Since F is an Alappnq-prime submodule of U, hence by corollary 2.8 $au \in F + (soc(U) + J(U))$ or $Iu \subseteq F + (soc(U) + J(U))$. But $soc(U) \subseteq F$, then $F + soc(U) + J(U) = F + J(U)$, thus either $aU \subseteq F + J(U)$ or $u \in F + J(U)$. Hence F is nearly prime submodule of U.

The proofs of the following results are direct.

Proposition 3.15 Let U be multiplication R-module with F is proper of U, and $soc(U) = (0)$. Then F is nearly prime if and only if F is Alappnq-prime.

Proposition 3.16 Let U be multiplication R-module and F is an $e$ssential submodule of U. Then F is nearly prime if and only if F is Alappnq-prime.

Proposition 3.17 Let U be R-module with $F \subset U$ and $J(U) \subseteq F$. Then F is approximaitly quasiprime if and only if F is Alappnq-prime.

Proof ($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $acu \in F$ for $a, c \in R, u \in U$. Since F is Alappnq-prime, then either $au \in F + soc(U) + J(U)$ or $cu \in F + soc(U) + J(U)$. Since $J(U) \subseteq F$, then $F + soc(U) + J(U) = F + soc(U)$. Thus either $au \in F + soc(U)$ or $cu \in F + soc(U)$. Hence F is approximaitly quasiprime.

Proposition 3.18 Let U be R-module with $J(U) \subseteq soc(U)$, and $F \subset U$. Then F is approximaitly quasiprime if and only if F is Alappnq-prime.

Proof ($\Longrightarrow$) Direct.

($\Longleftarrow$) Since $J(U) \subseteq soc(U)$, then $F + J(U) + soc(U) = F + soc(U)$. Let $acu \in F$ for $a, c \in R$, $u \in U$. Since F is Alappnq-prime, then either $au \in F + soc(U) + J(U) = F + soc(U)$ or $cu \in F + soc(U) + J(U) = F + soc(U)$. Thus F is approximaitly quasiprime.

The proofs of the following results are direct.

Proposition 3.19 Let U be R-module with $J(U) = soc(U) = (0)$, and $F \subset U$. Then F is approximaitly quasiprime if and only if F is Alappnq-prime.

Proposition 3.20 Let U be R-module and F is maximal submodule for U. Then F is approximaitly quasiprime if and only if F is Alappnq-prime.

Proposition 3.21 Let U be R-module with $soc(U) \subseteq F, J(U) \subseteq F$ and $F \subset U$. Then the following concepts are equivalent:

1. F is quasiprime submodule of U.

2. F is approximaitly quasiprime submodule of U.

3. F is Alappnq-prime submodule of U.

4. F is nearly quasiprime submodule of U.

Proof (1) $\Longrightarrow$ (2) Let F be a quasiprime submodule of an R-module U and $acu \in F$, for $a, c \in R, u \in U$. Since F is quasiprime submodule, then either $au \in F \subseteq F + soc(U)$ or $cu \in F \subseteq F + soc(U)$. Thus either $au \in F + soc(U)$ or $cu \in F + soc(U)$. Hence F is an approximaitly quasiprime submodule of U.

(2) $\Leftrightarrow$ (3) It follows by proposition 3.17

(3) ⇔ (4) It follows by proposition 3.11

(4) ⟹ (1) Since $J(U) \subseteq F$, then $F + J(U) = F$, Let $acu \in F$ for $a, c \in R, u \in U$. Since F is nearly quasiprime, then either $au \in F + J(U) = F$ or $cu \in F + J(U) = F$. Thus either $au \in F$ or $cu \in F$. Therefore F is quasiprime.

Proposition 3.22 Let U be multiplication R-module such that $J(U) \subseteq soc(U)$, and $F \subset U$. Then F is approximaitly prime if and only if F is Alappnq-prime.

Proof (⟸) Direct.

(⟸) Since $J(U) \subseteq soc(U)$, then $F + soc(U) + J(U) = F + soc(U)$. Let $au \in F$, for $a \in R$, $u \in U$. Since U is multiplication and F is Alappnq-prime, then by proposition (3.2.9) F is Alappn-prime implies that either $\in F + soc(U) + J(U) = F + soc(U)$ or $aU \subseteq F + soc(U) + J(U) = F + soc(U)$. Thus either $u \in F + soc(U)$ or $aU \subseteq F + soc(U)$. Hence F is approximaitly prime submodule of U.

The proofs of the following results are direct.

Proposition 3.23 Let U be multiplication R-module such that $J\left(\frac{U}{F}\right) = (0)$, and $F \subset U$. Then F is approximaitly prime if and only if F is Alappnq-prime.

Proposition 3.24 Let U be multiplication R-module and F is a maximal submodule of U. Then F is approximaitly prime if and only if F is Alappnq-prime.

Proposition 3.25 Let U be a multiplication R-module such that $soc(U) \subseteq F$, $J(U) \subseteq F$ and $F \subset U$. Then the following concepts are equivalent:

1. F is prime submodule of U.

2. F is quasiprime submodule of U.

3. F is approximaitly quasiprime submodule of U.

4. F is nearly quasiprime submodule of U.

5. F is Alappnq-prime submodule of U.

6. F is nearly prime submodule of U.

7. F is approximaitly prime submodule of U.

Proof (1) ⟹ (2) It follows by [1, Rem. and Exam. (2.1.2)(1)(7)]

(2) ⟹ (3) Let F be quasiprime submodule of an R-module U with $acu \in F$, for $a, c \in R, u \in U$. Since F is quasiprime submodule, then either $au \in F \subseteq F + soc(U)$ or $cu \in F \subseteq F + soc(U)$. Thus either $au \in F + soc(U)$ or $cu \in F + soc(U)$. Hence F is an approximaitly quasiprime submodule of U.

(3) ⟹ (4) Let F be approximaitly quasiprime submodule of an R-module U and $acu \in F$, for $a, c \in R, u \in U$. Since F is approximaitly quasiprime submodule, then either $au \in F + soc(U)$ or $cu \in F + soc(U)$. But $soc(U) \subseteq F$, then $F + soc(U) = F$. Thus $au \in F \subseteq F + J(U)$ or $cu \in F \subseteq F + J(U)$. Hence F is nearly quasiprime submodule of U.

(4) ⇔ (5) It follows by proposition 3.11

(5) ⇔ (6) It follows by proposition 3.14

$(6) \Longrightarrow (7)$ Let F be nearly prime submodule of an R-module U and $au \in F$, for $a \in R, u \in U$. Since F is nearly prime submodule, then either $u \in F + J(U)$ or $aU \subseteq F + J(U)$. But $J(U) \subseteq F$, then $F + J(U) = F$. Thus either $u \in F \subseteq F + soc(U)$ or $aU \subseteq F \subseteq F + soc(U)$. Hence F is approximaitly prime submodule of U.

$(7) \Longrightarrow (1)$ Since $soc(U) \subseteq F$, then $F + soc(U) = F$, Let $au \in F$ for $a \in R, u \in U$. But F is approximaitly prime, then either $u \in F + soc(U) = F$ or $aU \subseteq F + soc(U) = F$. Thus either $u \in F$ or $aU \subseteq F$. Therefore F is prime submodule of U.

References

[1] Abdul-Razak H. M. Quasi-Prime Modules and Quasi-Prime submodules, M.Sc. Thesis, University of Baghdad, 1999.

[2] Dauns, J. Prime Modules, Journal Reine Angew, Math., 2 (1978), 156-181.

[3] Nuhad S. and Adwia J. Nearly Quasi-prime Submodules, Journal Adv. Res., 5 (1) (2015), 170-180.

[4] Haibat K. M. and Ali Sh. A. Approximaitly Quasi-prime Submodules and Some Related Concepts, Journal of AL-Qadisiyah for computer science and mathematics. 11 (2) (2019), 54-62.

[5] Ali Sh. A. and Omar A. A. and Haibat K. M. Approximaitly Quasi primary Submodules, Ibn Al-Haitham Journal for Pure and Applied Sci., 33 (4) (2020), 92-101.

[6] Haibat K. M. and Shahad J. M. Weakly Approximaitly Quasi-Prime Submodules And Related Concepts. Ibn AL-Haitham Journal For Pure and Applied Sciences, 34 (2) (2021), 56-69.

[7] Hero J. H. and Haibat K. M. Weakly Nearly Quasi Prime Submodules, Tikrit Journal of Pure Science. (2022) to appear.

[8] Anderson, F. W. and Fuller, K. F. Rings and Categories of Modules, *Springer Verlag New Yourk*, 1992.

[9] Nuhad S. and Adwia J. Nearly prime Submodules, International Journal of Advanced Sci. and Technical Research online, (2015), 166-173.

[10] Ali Sh. A. and Haibat K. M. Approximaitly Prime Submodules and Some Related Concepts, Ibn Al-Haitham Journal for Pure and Applied Sci., 32 (2) (2019), 103-113.

[11] Kasch, F. Modules and Rings, *London Math. Soc. Monographs, New York, Academic press*, 1982.

[12] Dung, N.V., Huynh, D.V., Smith, P.I., and Wishbauer, R. Extending Modules, Pitman Research Notes in Math. Series. Longman, Harlow, (1994).

[13] Larsen, M. D. and McCarthy, P. J. Multiplicative Theory of Ideals, *Academic press*, *New York and London*, 1971.

[14] El-Bast, Z.A. and Smith, P.F. Multiplication modules, Comm. In Algebra 16(4) (1988), 755–779.

[15] Lam, T. Y. Lectures on Modules and Rings, *GTN, 189, Springer Verlage, Berlin New York*, 1999.