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Principally ⊕-G-$\mathbf{Rad_g}$-supplemented modules

Thaar Younis Ghawi[a], Narjis Mujtabah Kamil[b]

[a]Department of Mathematics, College of Education, University of Al-Qadisiyah, Iraq, thar.younis@qu.edu.iq

[b]Al-Samawah Education Directorate, Iraqi Ministry of Education, Iraq, edu-math.post18@qu.edu.iq

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ABSTRACT

In this paper, a new concept has been presented that is a stronger than a previous concept called ⊕-G-$\mathrm{Rad_g}$-supplemented. The basic definition is in an R ring where an R-module M is said to be principally ⊕-G-Rad_g-supplemented (shortly, ⊕-PG-$\mathrm{Rad_g}$-supplemented) if any cyclic submodule mR of M with $Rad_g(M) \subseteq mR$, there exists a direct summand A of M such that $M = mR + A$ and $mR \cap A \subseteq Rad_g(A)$. A set of properties and relations between previous modules and the given module has been dealt with simple examples illustrating those relations.

MSC..

1. Introduction

Throughout this paper, we will use $N \subseteq M$, $N \leq M$ and $N \leq^{\oplus} M$ to signify that N is a subset a submodule, or a direct summand of a module M. Let N a submodule of M then N is called essential in M denoted by $N \trianglelefteq M$ if, for any submodule L in M with $N \cap L = 0$ implies $L = 0$ [7]. Dually, for any submodule L of M, if $N + L = M$ implies $L = M$, then the proper submodule $N \leq M$ is called to be small in M and denoted as $N \ll M$. Recall [7] where the author Kasch named a submodule L of a right R-module M as a maximal submodule of M, for short $L \leq^{max} M$ if, $L \neq M$ and for every right submodule B of M with $L \subset B \subseteq M$, then $B = M$. We called the intersection of all maximal submodules of M, the radical of M, and denoted it by $Rad(M)$ or, as in alternative, the sum of all small submodules of M. If M does not contained any maximal submodules, then it is show as $Rad(M) = M$. A submodule L of M is called generalized small, denoted by $L \ll_g M$, if for essential submodule K of M with the property $M = L + K$ implies $K = M$, in fact, the authors Zhou and Zhang [15] calls a g-small submodule as an e-small submodule. In [15], the authors defined the generalized radical of a module M (or $Rad_g(M)$) as the intersection of all generalized maximal submodules of M, equivalently, the sum of all g-small submodules of M. Assume L and V are two submodules of a

*Corresponding author: Narjis Mujtabah Kamil

Email addresses: edu-math.post18@qu.edu.iq

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module M. Recall [14] that L is a supplement of V in M if it is minimal with respect to property $M = V + L$. Equivalently, L is known as a supplement of V in M if $M = V + L$ and $V \cap L \ll L$. If every submodule of M has a supplement inside M, then M is known as a supplemented module. in M. Moreover, M is called a principally $\oplus$-supplemented module if every cyclic submodule of M has a supplement in the form of a direct summand of M. Clearly, the principally $\oplus$-supplemented modules are principally supplemented. Moreover, the module M called (principally) lifting if, for all (cyclic) submodule N of M, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is small in M ([14]). A module M is said to be g-lifting if it has the decomposition $M = T \oplus \grave{T}$ such that $T \leq A$ and $A \cap \grave{T} \ll_g M$, if for any submodule $A \leq M$ [13]. Ghawi [3] recall that M is a principally g-lifting module if, for each $m \in M$, M has a decomposition $M = A \oplus B$ such that $A \leq mR$ and $mR \cap B$ is g-small in B. Recall ([9] and [13]) the authors defined a submodule V of M as a g-supplement of L in M if, $M = V + L$ and $V \cap L \ll_g L$. A module M is called to be g-supplemented if every submodule of M has a g-supplement A module M is called principally g-supplemented if, every cyclic submodule of M has a g-supplement in M, see ([10]). Furthermore, a module M is called principally $\oplus$-g-supplemented if every cyclic submodule of M has a g-supplement that is a direct summand of M [11]. Clearly, principally $\oplus$-g-supplemented modules are g-supplemented. Recall that a module is called (principally) strongly generalized $\oplus$-radical supplemented modules or briefly $\mathrm{sgrs}^{\oplus}$-module (principally $\mathrm{sgrs}^{\oplus}$-module) if, any (cyclic) submodule N of M with $Rad_g(M) \subseteq N$ has a g-supplement which is a direct summand of M [5],[6]. Recall [8] that an R-module M is called $\oplus$-G-$\mathrm{Rad_g}$-supplemented if any submodule N of M with $Rad_g(M) \subseteq N$ there is a direct summand A of M such that $M = N + A$ and $N \cap A \subseteq Rad_g(M)$. Motivated by the above concepts it was natural to introduce a new definition of modules known as principally $\oplus$-G-$\mathrm{Rad_g}$-supplemented (briefly $\oplus$-PG-$\mathrm{Rad_g}$-supplemented modules) as generalization of $\oplus$-G-$\mathrm{Rad_g}$-supplemented modules. This paper is diviced into two sections that intersect by the ring that is associative with identity 1, and all modules into the both sections are unital right R-modules. Many basic properties and examples of $\oplus$-PG-$\mathrm{Rad_g}$-supplemented modules are investigated and discuss in section 2, In addition, we proved some connections between our concept and other kinds of modules in the same section.

2. $\oplus$-PG-$\mathbf{Rad_g}$-supplemented modules and some basic properties

Our main definition, principally $\oplus$-G-$\mathrm{Rad_g}$-supplemented modules, as well as several features about this concept are presented in this section.

Definition 2.1. Let R be a ring. An R-module M is said to be principally $\oplus$-G-Rad_g-supplemented (briefly, $\oplus$-PG-$\mathrm{Rad_g}$-supplemented) if any cyclic submodule mR of M with $Rad_g(M) \subseteq mR$, there exists a direct summand A of M such that $M = mR + A$ and $mR \cap A \subseteq Rad_g(A)$. A ring R is called $\oplus$-PG-Rad_g-supplemented if, R_R is $\oplus$-PG-$\mathrm{Rad_g}$-supplemented.

Remarks and Examples 2.2.

(1) By definitions, it is clear to see the following:

(a) Each principally $\oplus$-g-supplemented module and hence every principally g-lifting module is a $\oplus$-PG-$\mathrm{Rad_g}$-supplemented module.

(b) Each principally $\mathrm{sgrs}^{\oplus}$-module and hence any $\mathrm{sgrs}^{\oplus}$-module is a $\oplus$-PG-$\mathrm{Rad_g}$-supplemented module.

(2) Every $\oplus$-G-$\mathrm{Rad_g}$-supplemented module is $\oplus$-PG-$\mathrm{Rad_g}$-supplemented.

Proof. Let M be a $\oplus$-G-$\mathrm{Rad_g}$-supplemented module such that $m \in M$ and $Rad_g(M) \subseteq mR$. There is a direct summand X of M such that $M = mR + X$ and $mR \cap X \subseteq Rad_g(M)$. As $mR \cap X \leq X \leq^{\oplus} M$, so [3, Lemma 2.5] implies $mR \cap X \subseteq Rad_g(X)$, and this end the proof. ∎

(3) In $\mathbb{Z}$-module $\mathbb{Z}$, for any nonzero cyclic submodule $n\mathbb{Z}$ of $\mathbb{Z}$, $n \in \mathbb{Z}$ and $Rad_g(\mathbb{Z}) = 0 \subseteq n\mathbb{Z}$, the only direct summand H of $\mathbb{Z}$ that satisfy property $n\mathbb{Z} + H = \mathbb{Z}$ is $H = \mathbb{Z}$, but $n\mathbb{Z} \cap \mathbb{Z} = n\mathbb{Z} \nsubseteq 0 = Rad_g(\mathbb{Z})$, that conclude $\mathbb{Z}_{\mathbb{Z}}$ is not $\oplus$-PG-$\mathrm{Rad_g}$-supplemented.

A module M is called indecomposable if the only direct summands of M are 0 and M [7]. A module M is said to be uniform if all nonzero submodules of M are essential [4]. Under indecomposable modules, the next consequence provides an equivalent condition for the notion of $\oplus$-PG-$\mathrm{Rad_g}$-supplemented modules.

Proposition 2.3. The following are equivalent for an indecomposable module M.

(1) M is $\oplus$-PG-$\mathrm{Rad_g}$-supplemented.

(2) For $m \in M$ with $Rad_g(M) \subseteq mR \neq M$, we have $Rad_g(M) = mR$.

Proof. $(1) \Rightarrow (2)$ Assume that $Rad_g(M) \subseteq mR \neq M$ and $m \in M$. Since M is a $\oplus$-PG-$\mathrm{Rad_g}$-supplemented module, there is a direct summand H of M such that $M = mR + H$ and $mR \cap H \subseteq Rad_g(H)$. If $H = 0$ then $mR = M$, a contradiction. By assumption, $H = M$. Thus, $mR = mR \cap H \subseteq Rad_g(M)$. Hence $mR = Rad_g(M)$.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$ Let $m \in M$ with $Rad_g(M) \subseteq mR$. If $mR = M$, then there exists a direct summand $\mathbf{0}$ such that trivially $M = mR + 0$ and $mR \cap 0 \subseteq Rad_g(0)$. Let $mR \neq M$, so by $(\mathbf{2})$, $mR \subseteq Rad_g(M)$. It follows that $M = mR + M$ and $mR \cap M = mR \subseteq Rad_g(M)$. Hence M is $\oplus$-PG-$\mathrm{Rad_g}$-supplemented. □

Corollary 2.4. The following are equivalent for a uniform R-module M.

(1) M is $\oplus$-PG-$\mathrm{Rad_g}$-supplemented.

(2) For $m \in M$ with $Rad_g(M) \subseteq mR \neq M$, we have $Rad_g(M) = mR$.

Proof. By [10, Lemma 2.11] every uniform module is indecomposable. The result is followed by Proposition 2.3. □

Proposition 2.5. Consider the following assertions for an R-module M:

(1) M is a principally $\mathrm{sgrs}^{\oplus}$-module.

(2) M is a $\oplus$-PG-$\mathrm{Rad_g}$-supplemented module.

Then $(1) \Longrightarrow (2)$. If $Rad_g(M) \ll_g M$, $(2) \Longrightarrow (1)$.

Proof. $(1) \Longrightarrow (2)$ By Remarks and Examples 2.2(1-b).

$(2) \Longrightarrow (1)$ Let $m \in M$ and $Rad_g(M) \subseteq mR$. By (2), there exists a direct summand K of M such that $M = mR + K$ and $mR \cap K \subseteq Rad_g(K)$. Since $Rad_g(K) \subseteq Rad_g(M)$, then $mR \cap K \ll_g M$, by assumption. As K is a direct summand of M, [3, Lemma 2.12] implies $mR \cap K \ll_g K$. Thus, (1) holds. □

Corollary 2.6. Let M be a finitely generated module. Then M is a principally $\mathrm{sgrs}^{\oplus}$-module if and only if M is $\oplus$-PG-$\mathrm{Rad_g}$-supplemented.

Proof. If M is a finitely generated module, [3, Lemma 5.4] implies that $Rad_g(M) \ll_g M$. Hence the result is obtained by Proposition 2.5. □

A module M is said to be Noetherian if and only if every nonempty set of submodules possesses a maximal element [7].

Corollary 2.7. Let R be a commutative ring and let M be a Noetherian R-module. Then M is a principally $\mathrm{sgrs}^{\oplus}$-module if and only if its $\oplus$-PG-$\mathrm{Rad_g}$-supplemented.

Corollary 2.8. Let R be a ring. Then R is a principally $\mathrm{sgrs}^{\oplus}$-ring if and only if R is $\oplus$-PG-$\mathrm{Rad_g}$-supplemented.

Proof. Since $R = \langle 1 \rangle$ is finitely generated, hence the result is obtained by Corollary 2.6. ☐

Proposition 2.9. Let M be a $\oplus$-PG-$\mathrm{Rad_g}$-supplemented module such that every essential submodule of M contains a maximal submodule. Then M is a principally $\mathrm{sgrs}^{\oplus}$-module.

Proof. By Proposition 2.5, it is enough to prove $Rad_g(M) \ll_g M$. Assume $Rad_g(M) + E = M$ for some $E \trianglelefteq M$. If $E \neq M$, so by hypothesis, there exists a $L \leq^{max} M$ such that $L \leq E$; that means $L \leq E \subset M$, a contradiction with maximality for L. So, $E = M$ and hence $Rad_g(M) \ll_g M$. Therefore M is a principally $\mathrm{sgrs}^{\oplus}$-module. ☐

Proposition 2.10. Let $M = \bigoplus_{i\in I} M_i$ be an infinite direct sum of $\oplus$-PG-$\mathrm{Rad_g}$-supplemented $\{M_i\}_{i\in I}$ has SSP. If any cyclic submodule of M is fully invariant, then M is a $\oplus$-PG-$\mathrm{Rad_g}$-supplemented module.

Proof. It is enough to prove when $I = \{1,2\}$. If $M = M_1 \oplus M_2$ be a module such that $m \in M$ and $Rad_g(M) \subseteq mR$. So, $mR = (mR \cap M_1) \oplus (mR \cap M_2)$, by hypothesis and [11, Lemma 2.1]. We have that $mR \cap M_i$ is cyclic in M_i for $i \in \{1,2\}$. As $Rad_g(M_i) \subseteq mR \cap M_i$ for $i \in \{1,2\}$, there is a direct summand submodule L_i of M_i such that $M_i = (mR \cap M_i) + L_i$ and $(mR \cap M_i) \cap L_i = mR \cap L_i \subseteq Rad_g(L_i)$ for $i \in \{1,2\}$. Thus, $M = M_1 + M_2 = mR + (L_1 + L_2)$. We will prove $mR \cap (L_1 + L_2) = (mR \cap L_1) + (mR \cap L_2)$. The inclusion $(mR \cap L_1) + (mR \cap L_2) \subseteq mR \cap (L_1 + L_2)$ always holds. For the inverse inclusion, by [10, Lemma 2.11] we conclude that $mR \cap (L_1 + L_2) \leq L_1 \cap (mR + X_2) + L_2 \cap (L_1 + mR) = L_1 \cap [(mR \cap M_1) + (mR \cap M_2) + L_2] + L_2 \cap [L_1 + (mR \cap M_1) + (mR \cap M_2)] = L_1 \cap [(mR \cap M_1) + M_2] + L_2 \cap [M_1 + (mR \cap M_2)]$. Again, by [10, Lemma 2.11], we have that $L_1 \cap [(mR \cap M_1) + M_2] \leq (mR \cap M_1) \cap (L_1 + M_2) + M_2 \cap ((mR \cap M_1) + L_1) = mR \cap X_1$. Similarly, $L_2 \cap [M_1 + (mR \cap M_2)] \leq mR \cap L_2$.

Thus, $mR \cap (L_1 + L_2) \leq (mR \cap L_1) + (mR \cap L_2)$. From two inclusions, $mR \cap (L_1 + L_2) = (mR \cap L_1) + (mR \cap L_2)$. Since $mR \cap L_i \subseteq Rad_g(L_i)$ for $i \in \{1,2\}$, then $mR \cap (L_1 + L_2) \subseteq Rad_g(L_1) + Rad_g(L_2) \subseteq Rad_g(L_1 + L_2)$. By SSP for M, we have that $L_1 + L_2$ is a direct summand in M. Therefore M is $\oplus$-PG-$\mathrm{Rad_g}$-supplemented. ☐

We said that a submodule A of a module M is weak distributive if $A = (A \cap X) + (A \cap Y)$ for all submodules X, Y of M with $X + Y = M$. A module M is said to be weakly distributive if every submodule of M is a weak distributive submodule of M [2]. A module M is said to have the summand sum property (SSP) if the sum of any two direct summands of M is again a direct summand of M [1].

Proposition 2.11. Let $\{M_1, M_2, \dots, M_n\}$ be a finite family of $\oplus$-PG-$\mathrm{Rad_g}$-supplemented modules with $\bigoplus_{i=1}^{n} M_i$ has SSP. If any cyclic submodule of M is weak distributive, then $\bigoplus_{i=1}^{n} M_i$ is $\oplus$-PG-$\mathrm{Rad_g}$-supplemented.

Proof. Straight line of Proposition 2.10. ☐

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