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Strongly regular Palais proper G – space

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Abstract
The main goal of this work is to create a general type of proper *G* – space , namely, strongly regular Palais proper G - space and to explain the relation between st – r – Bourbaki proper and st – r – Palais proper *G* – space and studied some of examples and propositions of strongly regular Palais proper G - space.

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Introduction:-
Let *B* be a subset of a topological space (X,T). We denote the closure of *B* and the interior of *B* by $\overline{B}$ and B^o , respectively. The subset *B* of (X, T) is called regular open (r – open) if $B = \overline{B}^{\,o}$ .The complement of a regular open set is defined to be a regular closed (r – closed) Then the family of all r – open sets in (X,T) forms a base of a smaller topology T^r on *X* ,called the semi – regularization of *T*.

In section one of this work, we include some of results which needed in section two, section two recalls the definition of Palais proper *G* – space, gives a new type of Palais proper *G* – space (to the best of our Knowledge), namely, strongly regular Palais proper *G* – space, studies some of its properties and is given the relation between st – r – Bourbaki proper and st – r – Palais proper *G* – space. (where *G*- space is meant T_2 – space topological *X* on which an r – locally r– compact, non – compact, T_2 – topological group *G* acts continuously on the left).

1. Preliminaries
First ,we present some fundamental definitions and proposition which are needed in the next section.

1.1 Definition [15]:A subset *B* of (X, T) is called regular open (r – open) if $B = \overline{B}^{\circ}$. The complement of regular open set is defined to be a regular closed (r – closed) . If $B = \overline{B^o}$ then the family of all r – open sets in (X,T)forms a base of a smaller topology T^ron *X* ,called the semi – regularization of *T*

1.2 Proposition [2]:
Let *X* and *Y* be two spaces. Then $A_1 \subseteq X$, $A_2 \subseteq Y$ be an r – open (r – closed) sets in *X* and *Y* , respectively if and only if $A_1 \times A_2$ is r – open(r– closed) in $X \times Y$.

1.3 Definition[2]: A subset B of a space X is called regular neighborhood (r – neighborhood) of $x \in X$ if there is an opens subset O of X such that $x \in O \subseteq B$.

1.4 Definition [2]: Let X and Y be spaces and $f: X \to Y$ be a function. Then:

(i) f is called regular continuous (r– continuous) function if $f^{-1}(A)$ is an r – open set in X for every open set A in Y.

(ii) f is called regular irresolute (r – irresolute) function if $f^{-1}(A)$ is an r – open set in X for every r- open set A in Y.

1.5 Proposition [2]:

Let $f: X \to Y$ be a function of spaces. Then f is an r - continuous function if and only if $f^{-1}(A)$ is an r - closed set in X for every closed set A in Y.

1.6 Proposition :

Let X and Y be spaces and let $f: X \to Y$ be a continuous, open function .Then f is r – irresolute function.

Proof:

Let A be an r-open set of Y, then $A = \overline{A}^{o}$ .Since f is continues and open then

$f^{-1}(A) = f^{-1}(\overline{A}^{o}) = \left[f^{-1}(\overline{A})\right]^{\circ} = \left[\overline{f^{-1}(A)}\right]^{\circ}$, $f^{-1}(A)$ is an r-open set of X.

1.7 Definition [2]:

(i) A function $f: X \to Y$ is called strongly regular closed (st – r – closed) function if the image of each r – closed subset of X is an r – closed set in Y.

(ii) A function $f: X \to Y$ is called strongly regular open (st – r – open) function if the image of each r – open subset of X is an r – open set in Y.

1.8 Remark :

(i)A function $f:(X, T) \to (Y,\tau)$ is r–continuous function if and only if $f:(X, T^r) \to (Y,\tau^r)$ is continuous.

(ii)A function $f: (X, T) \to (Y,\tau)$ is r – irresolute function if and only if $f:(X, T^r) \to (Y, \tau^r)$ is continuous.

. 1.9 Definition [2]: Let X and Y be spaces . Then a function $f: X \to Y$ is called a st – r – homeomorphism if:

(i) f is bijective .

(ii) f is continuous .

(iii) f is st – r – closed (st – r – open).

1.10 Proposition[2]: Every r-homeomorphism is st – r – homeomorphism

1.11 Proposition [2]: Let X, Y be spaces and $f: X \to Y$ be an r - homeomorphism function. Then f is a st – r – closed function

1.12 Definition [2]: Let $(\chi_d)_{d\in D}$ be a net in a space X , $x\in X$. Then :

i) $(\chi_d)_{d\in D}$ is called r – converges to x (written $\chi_d \xrightarrow{r} x$) if $(\chi_d)_{d\in D}$ is eventually in every r – neighborhood of x . The point x is called an r – limit point of $(\chi_d)_{d\in D}$, and the notation "$\chi_d \xrightarrow{r} \infty$" is mean that $(\chi_d)_{d\in D}$ has no r – convergent subnet.

ii) $(\chi_d)_{d\in D}$ is said to have x as an r – cluster point [written $\chi_d \overset{r}{\alpha} x$] if $(\chi_d)_{d\in D}$ is frequently in every r - neighborhood of x .

1.13 Theorem[2]: Let $(\chi_d)_{d\in D}$ be a net in a space (X, T) and x_o in X. Then $\chi_d \overset{r}{\alpha} x_o$ if and only if there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ such that $\chi_{dm} \xrightarrow{r} x_o$.

1.14 Remark:

Let $(\chi_d)_{d\in D}$ be a net in a space (X, T) such that $\chi_d \overset{r}{\alpha} x$, $x\in X$ and let A be an r – open set in X which contains x. Then there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ in A such that $\chi_{dm} \xrightarrow{r} x$.

1.15 Proposition [2]: Let X be a space and $A \subseteq X$, $x\in X$. Then $x\in \overline{A}^r$ if and only if there exists a net $(\chi_d)_{d\in D}$ in A and $\chi_d \xrightarrow{r} x$.

1.16 Remark [1]: Let X be a space, then:

(i) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$ such that $\chi_d \longrightarrow x$ then $\chi_d \xrightarrow{r} x$.

(ii) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$ such that $\chi_d \,\alpha\, x$ then $\chi_d \overset{r}{\alpha} x$.

(iii) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$. Then $\chi_d \xrightarrow{r} x$ in (X, T) if and only if $\chi_d \to x$ in (X, T^r), and $\chi_d \overset{r}{\alpha} x$ in (X, T) if and only if $\chi_d \,\alpha\, x$ in (X, T^r).

1.17 Proposition: Let $f: X\to Y$ be a function, $x\in X$. Then:

(i) f is r – continuous at x if and only if whenever a net $(\chi_d)_{d\in D}$ in X and $\chi_d \xrightarrow{r} x$ then $f(\chi_d) \longrightarrow f(x)$.

(ii) f is r – irresolute at x if and only if whenever a net $(\chi_d)_{d\in D}$ in X and $\chi_d \xrightarrow{r} x$ then $f(\chi_d) \xrightarrow{r} f(x)$.

Proof: **(i)** $\Rightarrow$ Let $x \in X$ and $(\chi_d)_{d\in D}$ be a net in X such that $\chi_d \xrightarrow{r} x$ (To prove that $f(\chi_d) \longrightarrow f(x)$). Let V be a open neighborhood of $f(x)$. Since f is r – continuous, then $f^{-1}(V)$ is r – neighborhood of x, but $\chi_d \xrightarrow{r} x$, then there is $\beta \in D$ such that $\chi_d \in f^{-1}(V)$, $\forall\ d \geq \beta$. Then $f(\chi_d) \in f(f^{-1}(V)) \subseteq V$. Thus $f(\chi_d)$ is eventually in every open neighborhood of $f(x)$, then $f(\chi_d) \longrightarrow f(x)$.

$\Leftarrow$ Suppose that f is not r – continuous. Then there exists $x \in X$ such that f is not r – continuous at x. Then there exists an open set B in Y such that $f(x) \in B$ and $f(A) \not\subset B$ for each A is an r – open in X such that $x \in A$. Thus there exists $\chi_A \in A$ and $f(\chi_A) \notin B$ for each A is r – open in X . Then $\chi_A \xrightarrow{r} x$. But $f(\chi_A) \notin B$ for each $A \in N_r(x)$, then $f(\chi_A)$ is not convergent to $f(x)$ and this is a contradiction. Then f is r – continuous.

(ii) $\Rightarrow$ Let $x \in X$ and $(\chi_d)_{d\in D}$ be a net in X such that $\chi_d \xrightarrow{r} x$ .Then by Remark(1.16,iii) $\chi_d \longrightarrow x$ in (X, T^r). Since f: $(X,T) \longrightarrow (Y,\tau)$ is r – irresolute then by Remark(1.8,ii) f: $(X, T^r) \longrightarrow (Y, \tau^r)$ is continuous. Thus $f(\chi_d) \longrightarrow f(x)$ in (Y,τ^r), so by Remark (1.16,iii) $f(\chi_d) \xrightarrow{r} f(x)$.

$\Leftarrow$By Remark (1.16,iii) and Remark (1.8,ii) we have f: $(X, T^r) \longrightarrow (Y, \tau^r)$ is continuous , then f is r – irresolute.

1.18 Definition [2]: A subset A of space X is called r – compact set if every r – open cover of A has a finite sub cover. If $A=X$ then X is called a r – compact space.

1.19 Proposition [2]: A space (X, T) is an r – compact space if and only if every net in X has r – cluster point in X
.

1.20 Proposition [2]: Let X be a space and F be an r – closed subset of X. Then $F\cap K$ is r – compact subset of F, for every r – compact set K in X.

1.21 Definition :

(i) A subset A of space X is called r- relative compact if $\overline{A}$ is r – compact.

(ii) A space X is called r– locally r – compact if every point in X has an r – relative compact r– neighborhood.

1.22 Definition [2] Let f: $X\to Y$ be a function of spaces. Then:

(i) f is called an regular compact (r – compact) function if $f^{-1}(A)$ is a compact set in X for every r – compact set A in Y.

(ii) f is called a strongly regular compact(st – r – compact) function if $f^{-1}(A)$ is an r – compact set in X for every r – compact set A in Y.

1.23 Definition [6]: A topological transformation group is a triple (G,X,φ) where G is a T_2 – topological group, X is a T_2 – topological space and $\varphi : G\times X \to X$ is a continuous function such that:

(i) $\varphi(g_1,\varphi(g_2, x)) = \varphi(g_1g_2, x)$ for all $g_1,g_2\in G$, $x\in X$.

(ii) $\varphi(e, x) = x$ for all $x\in X$, where e is the identity element of G.

We shall often use the notation $g.x$ for $\varphi(g,x)$ $g.(h,x)= (gh).x$ for $\varphi(g, \varphi(h,x))= \varphi(gh,x)$. Similarly for $H \subseteq G$ and $A \subseteq X$ we put $HA = \{ga / a \in H, a \in A\}$ for $\varphi(H, A)$. A set A is said to be invariant under G if $GA = A$.

1.24 Remark [6]:

(i) The function φ is called an action of G on X and the space X together with φ is called a G – space (or more precisely left G – space).

(ii) The subspace $\{g.x / g\in G\}$ is called the orbit (trajectory) of x under G, which denoted by Gx [or $\gamma(x)$], and for every $x\in X$ the stabilizer subgroup G_x of G at x is the set $\{g\in G/ gx = x\}$.

(iii) The continuous function l_g: $G \to G$ defined by $y \to gy$ is called the left translation by g. This function has inverse l_g^{-1} which is also continuous, moreover l_g is a homeomorphism. Similarly all right translation r_g:$G \to G$ are homeomorphism for every $g\in G$.

(ix)$Ag= r_g(A) =\{ag:a\in A\}$;$Ag$ is called the left translate of A by g, where $A \subseteq G$, $g\in G$.

(x)$gA=l_g(A) =\{ga:a\in A\}$;gA is called the right translate of A by g,where $A \subseteq G$, $g\in G$.

2 - *Strongly regular Bourbaki Proper Action*

2.1 Definition [2]: Let X and Y be two spaces. Then f: $X\to Y$ is called a strongly regular proper (st – r - proper) function if :

(i) f is continuous function.

(ii) $f\times I_Z$: $X\times Z\to Y\times Z$ is a st – r – closed function, for every space Z.

2.2 Proposition [2]: Let X, Y and Z be spaces, f: $X\to Y$ and g: $Y\to Z$ be two st – r – proper function. Then $g\circ f$:$X\to Z$ is a st – r – proper function.

2.3 Proposition [2]; Let f_1: $X_1\to Y_1$ and f_2: $X_2\to Y_2$ be two function. Then $f_1\times f_2$: $X_1\times X_2\to Y_1\times Y_2$ is st – r- proper function if and only if f_1 and f_2 are st – r – proper functions.

2.4 Proposition[2]: Let f: $X\to P= \{w\}$ be a function on a space X. Then f is a st – r – proper function if and only if X is an r – compact, where w is any point which dose not belong to X.

2.5 Lemma:[2] Every r-irresolute function from an r – compact space into a Hausdorff space is st-r-closed.

2.6 Proposition [2]: Let X and Y be a spaces and $f: X\to Y$ be a continuous function.If Y is T_2-space. Then the following statements are equivalent:
(i) f is a st – r – proper function.
(ii) f is a st – r– closed function and $f^{-1}(\{y\})$ is an r – compact set, for each $y\in Y$. .
(iii)) If $(\chi_d)_{d\in D}$ is a net in X and $y\in Y$ is an r – cluster point of $f(\chi_d)$, then there is an r – cluster point $x\in X$ of $(\chi_d)_{d\in D}$ such that $f(x) = y$.

2.7 Proposition [2]: Let X and Y be a spaces, such that Y is a T_2 – space and $f: X\to Y$ be continuous,r– irresolute function. Then the following statements are equivalent:
(i) f is a st – r– compact function.

(ii) f is a st – r– proper function.

2.8 Proposition:[3]: Let X, Y and Z be spaces, $f: X\to Y$ is an st- r – proper functions and g: $Y\to Z$ is homeomorphism function . Then $g\circ f: X\to Z$ is an st- r – proper function.

2.9 Proposition: Let X be a Hausdorff-spase then the diagonal function $\Delta : X \longrightarrow X\times X$ is st-r-proper function

Proof: Since Δ is continuous and X is T_2 Let $(\chi_d)_{d\in D}$ be a net in X and $y = (x_1, x_2) \in X\times X$ be an r – cluster point of $\Delta(\chi_d)$. Then $\Delta(\chi_d) = (\chi_d, \chi_d) \overset{r}{\alpha} (x_1, x_2)$, so by Proposition (1.13) there exists a subnet of (χ_d, χ_d) ,say itself, such that $(\chi_d, \chi_d) \xrightarrow{r} (x_1, x_2)$,then $\chi_d \xrightarrow{r} x_1$ and $\chi_d \xrightarrow{r} x_2$, since X is a T_2 – space , then $x_1=x_2$. Then there is $x_1\in X$ such that $\chi_d \overset{r}{\alpha} x_1$ and $\Delta(x_1) = y$. Hence by Proposition (2.6.iii) Δ is a st – r – proper function.

2.10 Proposition: Let $f_1: X\to Y_1$ and $f_2: X\to Y_2$ be two st – f – proper functions. If X is a Hausdorff space, then the function $f: X\to Y_1\times Y_2$, $f(x) = (f_1(x), f_2(x))$ is a st – r- proper function

Proof: Since X is Hausdorff, then by Proposition (2.9) Δ is a st – r – proper function. Also by Proposition (2.3) $f_1\times f_2$ is a st–r–proper function. Then by Proposition (2.2) $f=f_1\times f_2 \circ \Delta$ is a st – r – proper function.

2.11 Proposition : Let G be a topological group and $(g_d)_{d\in D}$ be a net in G. Then:

(i)If $g_d \xrightarrow{r} e$, where e is identity element of G, then $gg_d \xrightarrow{r} g$ (or $g_d\, g \xrightarrow{r} g$) for each $g\in G$.

(ii) If $g_d \xrightarrow{r} \infty$, then $gg_d \xrightarrow{r} \infty$ (or $g_d\, g \xrightarrow{r} \infty$) for each $g\in G$.

(iii) If $g_d \xrightarrow{r} \infty$, then $g_d^{-1} \xrightarrow{r} \infty$.

Proof:

i) Since r_g:$G \to G$ is continuous and open , where r_g is right translation by g. then r_g is r – irresolute. Thus by Proposition (1.17,ii) $g_d\, g \xrightarrow{r} g$ for each $g \in G$.

ii) Let $g_d \xrightarrow{r} \infty$ and $g \in G$. suppose that $g_d g \xrightarrow{r} g_1$, for some $g_1 \in G$. Since r_g is r- irresolute, then by Proposition(1.17,ii) $r_g^{-1}(g_d g) \xrightarrow{r} r_g^{-1}(g_1)$ .Then $g_d \xrightarrow{r} g_1 g^{-1}$, a contradiction. Thus $g_d\, g \xrightarrow{r} \infty$.

iii) Let $g_d^{-1} \xrightarrow{r} g$. Since the inversion map of a topological group G, v: $G \to G$ is r – irresolute, then $g_d \xrightarrow{r} g^{-1}$. Thus if $g_d \xrightarrow{r} \infty$, then $g_d^{-1} \xrightarrow{r} \infty$.

2.12 Proposition: If (G, X, φ) is a topological transformation group, then φ is r – irresolute.

Proof: Let $A \times B$ is an open set in $G \times X$, then $\varphi\,(A \times B) = AB$. Since $AB = \{\, x \in X \,/\, x = ab,\ a \in A,\ b \in B\} = \bigcup_{a \in A} aB = \bigcup_{a \in A} \varphi(B)$. Since φ_a: $X \to X$ is homeomorphism from X on itself such that $a \in G$. Then aB is an open set in X, so $\bigcup_{a \in A} aB = AB$ is open. Since φ is continuous and open function, then it's clear that the action φ is an r – irresolute

2.13 Definition : A G – space X is called a strongly regular Bourbaki proper G – space (st – r – proper G – space) if the function θ: $G \times X \to X \times X$ which is defined by $\theta\,(g, x) = (x, g.x)$ is a st – r – proper function.

2.14 Example:

The topological group $Z_2 = \{-1, 1\}$ (as Z_2 with discrete topology) acts on the topological spaceS^n (as a subspace of R^{n+1} with usual topology) as follows:

1. $(x_1, x_2, \ldots, x_{n+1}) = (x_1, x_2, \ldots, x_{n+1})$

-1. $(x_1, x_2, \ldots, x_{n+1}) = (-x_1, -x_2, \ldots, -x_{n+1})$

Since Z_2 is an compact, then by Proposition (2.4) the constant function $Z_2 \to P$ is an r – proper. Also the identity function is an r – proper, then by Proposition (2.3) the function of $Z_2 \times S^n$ into $P \times S^n$ is an r – proper.

Since $P \times S^n$ is homeomorphic to S^n, then by Proposition (2.8) , the composition $Z_2 \times S^n \to S^n$ is an r – proper function , hence $Z_2 \times S^n \to S^n$ is a st- r – proper function . Let φ be the action of Z_2 on S^n. Thenφ continuous,. Since S^n is T_2 – space. Then by Proposition (2.4) φ is st- r – proper function . Thus by Proposition (2.9) $Z_2 \times S^n \to S^n \times S^n$ is a st – r – proper function ,thus $S^n$is st-r - proper Z_2- space.

2.15 Proposition [6]: Let X be a G – space then the function $\theta: G\times X\to X\times X$ which is defined by $\theta(g, x) = (x, g.x)$ is continuous function and $\theta^{-1}(\{(x, y)\})$ is closed in $G\times X$ for every $(x, y)\in X\times X$.

3 – Strongly regular Palais proper action:

in this section by G – space we mean a topological T_2 – space X on which an r – locally r – compact, non – compact, T_2 – topological group G continuously on the left (always in the sense of st – r - Palais proper G – space) , definitions, propositions, theorems and Example of a strongly regular Palais proper G - space (st – r - Palais proper G – space) is given and the relation between st – r – Bourbaki proper and st – r – Palais proper G – space is studied.

3.1 Definition:

Let X be a G – space .A subset A of X is said to be regular thin (r – thin) relative to a subset B of X if the set $((A, B)) = \{g\in G \ / \ gA\cap B\neq\phi\}$ has an r – neighborhood whose closure is r – compact in G. If A is r – thin relative to itself, then it is called r – thin.

3.2 Remark: The r – thin sets have the following properties:

(i) Since $(gA\cap B) = g(A\cap g^{-1}B)$ it follows that if A is r – thin relative to B, then B is r – thin relative to A.

(ii) Since $(gg_1A\cap g_2B) = g_2(g_2^{-1}g\, g_1A\cap B)$ it follows that if A is r – thin relative to B, then so are any translates gA and gB.

(iii) If A and B are r – relative thin and $K_1\subseteq A$ and $K_2\subseteq B$, then K_1 and K_2 are r –relatively thin.

(iv) Let X be a G – space and K_1, K_2 be r – compact subset of X, then $((K_1, K_2))$ is r – closed in G.

(v) If K_1 and K_2 are r – compact subset of G – space X such that K_1 and K_2 are r – relatively thin, then $((K_1, K_2))$ is an r – compact subset of G.

Proof: The prove of (i), (ii), (iii) and (v) are obvious.

(iv) Let $g\in\overline{((K_1,K_2))}^{r}$. Then there is a net $(g_d)_{d\in D}$ in $((K_1, K_2))$ such that $g_d\xrightarrow{r}g$. Then we have net $(k_d^1)_{d\in D}$ in K_1, such that $g_d k_d^1\in K_2$, since K_2 is r – compact, then by Theorem (1.10) there exists a subnet $(g_{d_m}k_{d_m}^1)$ of $(g_d k_d^1)$ such that $g_{d_m}k_{d_m}^1\xrightarrow{r}k_o^2$, where $k_o^2\in K_2$. But $(k_{d_m}^1)$ in K_1 and K_1 is r – compact , thus there is a point $k_o^1\in K_1$ and a subnet of $k_{d_m}^1$ say itself such that $k_{d_m}^1\xrightarrow{r}k_o^1$. Then $g_{d_m}k_{d_m}^1\xrightarrow{r}gk_o^1=k_o^2$, which mean that $g\in((K_1, K_2))$, therefore $((K_1, K_2))$ is r – closed in G.

3.3 Definition:

A subset S of a G – space X is an regular small (r – small) subset of X if each point of X has r – neighborhood which r – thin relative to S.

3.4 Theorem:

Let X be a G – space. Then:

(i) Each r – small neighborhood of a point x contains an r – thin neighborhood of x.

(ii) A subset of an r – small set is r – small.

(iii) A finite union of an r – small sets is r – small.

(iv) If S is an r – small subset of X and K is an r– compact subset of X then K is r – thin relative to S.

Proof:

i) Let S is an r – small neighborhood of x. Then there is an r – neighborhood U of x which is r – thin relative to S. Then $((U, S))$ has r – neighborhood whose closure is r – compact. Let $V= U\cap S$, then V is r – neighborhood of x and $((V, V)) \subseteq ((U, S))$, therefore V is r – thin neighborhood of x.

ii) Let S be an r – small set and $K \subseteq S$. Let $x\in X$, then there exists an r – neighborhood U of x, which is r – thin relative to S. Then $((U, K)) \subseteq ((U, S))$, thus $((U, K))$ has r – neighborhood whose closure is r – compact. Then K is r – small.

iii) Let $\{S_i\}_{i=1}^{n}$ be a finite collection of r – small sets and $y\in X$. Then for each i there is r – neighborhood K_i of y such that the set $((S_i , K_i))$ has r – neighborhood whose closure is r – compact. Then $\bigcup_{i=1}^{n}((S_i , K_i))$ has r– neighborhood whose closure is r – compact. But $((\bigcup_{i=1}^{n} S_i, \bigcap_{i=1}^{n} K_i)) \subseteq \bigcup_{i=1}^{n}((S_i , K_i))$, thus $\bigcup_{i=1}^{n} S_i$ is an r – small set.

iv) Let S be an r – small set and K be r – compact. Then there is an r – neighborhood U_k of K, $\forall\, k\in K$, such that U_k is r – thin relative to S. Since $K \subseteq \bigcup_{k\in K} U_k$ .i.e., $\{U_k\}_{k\in K}$ is r – open cover of K , which is r – compact, so there is a finite sub cover $\{U_{k_i}\}_{i=i}^{n}$ of $\{U_k\}_{k\in K}$, since $((U_{k_i}, S))$ has r– neighborhood whose closure is r – compact , thus $((\bigcup_{i=1}^{n} U_{k_i}, S))$ so is . But $((K,S)) \subseteq ((\bigcup_{i=1}^{n} U_{k_i}, S))$ therefore K is r – thin relative to S.

3.5 Definition:

A G – space X is said to be a strongly regular Palais proper G - space (st – r – Palais proper G – space) if every point x in X has an r – neighborhood which is r – small set.

3.6 Examples:

(i) The topological group $Z_2=\{-1, 1\}$ act on itself (as Z_2 with discrete topology) as follows:

$r_1.r_2 = r_1 r_2 \quad \forall r_1.r_2 \in Z_2$.

for each point $x \in Z_2$, there is an r – neighborhood which is r – small U of x where $U=\{x\}$, i.e., for any point y of Z_2, there exists an r – neighborhood V of y such that $V=\{y\}$ and $((U, V)) = \{r\in Z_2 \ / \ rU\cap V\neq\phi\} = Z_2$, then $((U, V))$ has r – neighborhood whose closure is compact.

(ii) R – {0} be r – locally r – compact topological group (as R – {0} with discrete topology) acts on the completely regular Hausdorff space R^2 as follows:

$r.(x_1, x_2) = (rx_1, rx_2)$, for every $r\in R - \{0\}$ and $(x_1, x_2)\in R^2$.

Clear R^2 is (R – {0}) – space . But $(0,0)\in R^2$ has no r – neighborhood which is an r – small. Since for any two r- neighborhood U, V of (0,0) then $((U,V)) = R - \{0\}$. Since R is not r – compact . Thus R^2 is not a st – r – Palais proper (R – {0}) – space.

3.7 Proposition:

Let X be a G – space . Then:

(i) If X is st –r– Palais proper G – space , then every compact subset of X is an r – small set.

(ii) If X is a st – r – Palais proper G – space and K is a compact subset of X , then $((K,K))$ is an r – compact subset of G.

Proof:

i) Let A be a subset of X such that A is an compact. Let $x\in X$, since X is a st – r – proper G – space then there is an r – neighborhood U which is r – small of x. Then for every $a\in A$ there exist an r- neighborhood U_a which is r – small , then $A \subseteq \bigcup_{a\in A} U_a$, since A is compact then A is r – compact , then there exists $a_1, a_2, \ldots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$, Thus by Theorem (3.4.iii.ii) A is an r – small set in X.

ii) Let X be a st – r – proper G – space and K is compact , then by (i) K is an r – small subset of X , and by Theorem (3.4.iv) K is r – thin , so $((K,K))$ has r – neighborhood whose closure is r – compact . Then by Remark (3.2.iv) $((K,K))$ is r – closed in G. Thus $((K,K))$ is r – compact .

***3.8 Definition*:** Let X be a G – space and $x\in X$. Then $J^r(x)=\{y\in X$: there is a net $(g_d)_{d\in D}$ in G and there is a net $(\chi_d)_{d\in D}$ in X with $g_d \xrightarrow{r} \infty$ and $\chi_d \xrightarrow{r} x$ such that $g_d x \xrightarrow{r} y\}$ is called regular first prolongation limit set of x.

3.9 Proposition: Let X be a G – space. Then X is a st – r – Bourbaki proper G – space if and only if $J^r(x)=\phi$ for each $x\in X$.

Proof: ⇒ Suppose that $y \in J^r(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{r} \infty$ and there is a net $(\chi_d)_{d\in D}$ in X with $\chi_d \xrightarrow{r} x$ such that $g_d\chi_d \xrightarrow{r} y$, so $\theta((g_d,\chi_d))=(x_d, g_d\chi_d) \xrightarrow{r} (x, y)$. But X is a st – r – Bourbaki proper, then by Proposition (2.6) there is $(g, x_1)\in G\times X$ such that $(g_d, x_d) \overset{r}{\propto} (g, x_1)$. Thus $(g_d)_{d\in D}$ has a sub net (say itself). such that $g_d \xrightarrow{r} g$, which is contradiction, thus $J^r(x)=\phi$.

⇐ Let $(g_d, \chi_d)_{d\in D}$ be a net in $G\times X$ and $(x, y)\in X\times X$ such that $\theta((g_d, \chi_d))=(\chi_d, g_d\chi_d) \overset{r}{\propto} (x, y)$, so $(\chi_d, g_d\chi_d)_{d\in D}$ has a sub net, say itself, such that $(\chi_d, g_d\chi_d) \xrightarrow{r} (x, y)$, then $\chi_d \xrightarrow{r} x$ and $g_d\chi_d \xrightarrow{r} y$. Suppose that $g_d \xrightarrow{r} \infty$ then $y\in J^r(x)$, which is contradiction . Then there is $g\in G$ such that $g_d \xrightarrow{r} g$, then $(g_d, \chi_d) \xrightarrow{r} (g, x)$,since θ is an r-irresolute then $\theta((g_d, \chi_d)) \xrightarrow{r} \theta(g, x)$,i.e $(\chi_d, g_d\chi_d) \xrightarrow{r} (x,g.x)$,but $(\chi_d, g_d\chi_d) \xrightarrow{r} (x,y)$,since $X\times X$ isT_2 space then $(x,g.x)=(x,y)$ i.e $\theta(g, x) = (x, y)$. Thus by Proposition (2.6) X is a st – r – Bourbaki proper G – space.

3.10 Proposition: Let X be a G – space and y be a point in X . Then y has no r – small whenever $y\in J^r(x)$ for some point $x\in X$.

Proof:

Let $y\in J^r(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{r} \infty$ and a net $(\chi_d)_{d\in D}$ in X with $\chi_d \xrightarrow{r} x$ such that $g_d\chi_d \xrightarrow{r} y$. Now, for each r- neighborhood S of y and every r – neighborhood U of x there is $d_o\in D$ such that $\chi_d \in U$ and $g_d\chi_d \in S$ for each $d \geq d_o$, thus $g_d\in((U,S))$, but $g_d \xrightarrow{r} \infty$, thus $((U,S))$ has no r – compact closure . i.e., S is not an r – small neighborhood.

3.11 Proposition: Let X be a st – r – Palais proper G – space. Then $J^r(x) = \phi$ for each $x\in X$.

Proof:

Suppose that there exists $x\in X$ such that $J^r(x)\neq\phi$, then there exists $y\in J^r(x)$. Thus there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{r} \infty$ and a net $(\chi_d)_{d\in D}$ in X with $\chi_d \xrightarrow{r} x$ such that $g_d\chi_d \xrightarrow{r} y$. Since X be a st – r– Palais proper G – space, then there is an r – small (r – thin) r – neighborhood U of x. Thus there is $d_o \in D$ such that $g_d\chi_d\in U$ and $\chi_d\in U$ for each $d \geq d_o$, so $g_d \in ((U,U))$, which has an r – compact closure , therefore $(g_d)_{d\in D}$ must have an r– convergent subnet , which is a contradiction. Thus $J^r(x) = \phi$ for each $x\in X$.

In general, the definition of a st – r – Palais proper G – space implies that st – r– Bourbaki proper G – space, which is review in following proposition.

3.12 Proposition: Every st – r – Palais proper G – space is st – r – Bourbaki proper G – space.

Proof:

By Propositions (3.11) and (3.9).

The converse of Propositions (3.12), is not true in general as the following example shows.

3.13 Example:

Let G be a topological group where G is not r – locally r – compact, then G is acts on itself translation. The map θ : $G\times G\rightarrow G\times G$, which is defined by θ $(g_1, g_2) = (g_2, g_1g_2)$, $\forall$ $(g_1, g_2)\in G\times G$ is a st – r – homeomorphism , hence it is st – r – Bourbaki proper G – space . But it is not st – r – Palais proper G – space, because G is not r – locally r– compact.

3.14 Lemma[2]: Let X be an r – locally r – compact G – space . Then $J^r(x)$ $=\phi$ for each $x\in X$ if and only if every pair of point of X has r– relatively thin r– neighborhood.

3.15 Proposition Let X be an r – locally r – compact G – space. Then the definition of st – r – Palais proper G – space and the definition st – r – Bourbaki proper G – space are equivalent.

Proof:

The definition of st – r – Palais proper G – space implies to the definition st – r – Bourbaki proper G – space. by Propositions (3.12).

Conversely, let X be a st – r – Bourbaki proper G – space, then by Proposition (3.9) $J^r(x)=\phi$ for each $x\in X$. Let $x\in X$, we will how that x has a r – small r – neighborhood. Since X is r – locally r – compact, then there is a r – compact r – neighborhood U_x of X, we claim that U_x is an r – small r – neighborhood of x. Let $y\in X$, we may assume without loss of generality , that U_y is an r – compact r – neighborhood of y such that U_x and U_y are r – relative thin i.e., $((U_x, U_y))$ has r – compact closure, therefore U_x is an r – small r – neighborhood of x. Thus X is st – r – Palais proper G – space.

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المستخلص

أن الهدف الرئيسي من هذا البحث هو تقديم نوع جديد(حسب علمنا) من فضاءات – G سمي فضاء – G باليه و أعطينا خصائص وبعض المبرهنات الخاصة بهذا الفضاء ثم بينا العلاقة بين فضاء –G باليه و بين المجموعة $J^r(x)$ و كذلك العلاقة بينه وبين فضاء -G السديد المنتظم القوة لبورباكي.

Strongly regular Palais proper G – space

Abstract

The main goal of this work is to create a general type of proper *G* – space , namely, strongly regular Palais proper G - space and to explain the relation between st – r – Bourbaki proper and st – r – Palais proper *G* – space and is studied some of examples and propositions of strongly regular Palais proper G - space.