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Asymptotic behavior of Eigenvalues and Eigenfunctions of T.Regge Fractional Problem

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ABSTRACT

The asymptotic behavior of eigenvalues and eigenfunctions of T.Regge fractional boundary value problem has been shown, and we state and prove some theorems for many results, also some necessary definitions and results. In this paper, we look into a group of fractional boundary value problem equations involving fractional derivative fractional orders $\alpha \in (1,2]$ and $t \in [0, a]$ with two boundary value conditions. We will which are important to state and prove those theorems; our primary findings are illustrated using examples.

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1. INTRODUCTION

Fractional calculus is a strong tool for describing the memory and inherited features of various materials and processes.[1]–[3], It has applications in biology, chemistry, viscoelasticity, anomalous diffusion, fluid mechanics, acoustics, control theory, and other fields of science and engineering. Fractional differential equations were implicated in a family of integro-differential equations with singularities in these applications.[3-6].The existence and uniqueness theorems for fractional ordinary differential equations were introduced.[4] Several analytical or numerical approaches for solving fractional differential equations were proposed previously, such as [2], [5]and [6], In this paper, we study the asymptotic behaviuor of eigenvalues and eigenfunctions for the fractional boundary value problem, as is known [5] , supported on the interval $[0, a]$ is related to the study of the Regge spectral problem on this interval. This problem has the form

$$-{}_0^C D_x^\alpha y(x) + q(x)y(x) = \lambda^2 p(x)y(x); \qquad x \in [0, a], \qquad 1 < \alpha \leq 2 \tag{1.1}$$

$$y(0) = 0, \qquad y'(a) - i\lambda y(a) = 0, \tag{1.2}$$

such that $q(x), p(x) \in L_+ [0, a]$, where $L_+ [0, a]$ is the set of all integrable function $f(x)$ *on* $[0, a]$ *and* $0 < m \leq f(x) \leq M < \infty$, and $\alpha \in (1,2]$, and λ is a spectral parameter, $q(x), p(x)$ are integrable functions. And $y(x) \in C[0, a]$, ${}_0^C D_x^\alpha y(x) \in C^3[0, a]$.

Let L be a linear operator defined on some set of elements as element $y \neq 0$ is called eigenfunctions of L if $Ly = \lambda y$. , the number λ is called an eigenvalue of operator L.

In another representation the number λ is called an eigenvalue of operator L if there exists in the domain of definition of the operator L a function $y \not\equiv 0$ such that $Ly = \lambda y$.

One of the simplest operators which is frequently encountered in applications is an operator of the form

$$L \equiv -\frac{d^2}{dx^2} + q(x) ,$$

where the function $q(x)$ will be assumed real and, to begin with, continuous on some interval $[a, b]$ For this operater the set of elements (functions) $y(x)$ mentioned above is determined by the obvious differentiability condition and also by certain conditions on the boundary of the interval $[a, b]$ [7] .

1. PRELIMINARIES CONCEPTS

1.1. Preliminaries

In this section, we present some definitions, lemmas and theorems, which are required for our work.

Definition 1.1 [4] The Gamma function is defined by the integral formula

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt\,, \quad \boldsymbol{z} \in \mathbb{C}$$

The integral converges absolutely for $Re(z) > 0$.

Definition 1.2 [8]**(Fractional Integral of Order α)** for every $\alpha > 0$ and a locally integrable function $h(t)$, the right FI of order α is defined:

$$_aI_t^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds, \; -\infty \leq a < t < \infty \tag{1.3}$$

Alternatively, it can be defined also the left FI by:

$$_tI_b^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} h(s) ds, \; -\infty < t \leq b \leq \infty \quad .$$

Properties 1.1 [8].Let $f(x), h(x) \in L^1[0,a]$ are continuous functions $a, b \in R$, and $n, m > 0$, then:

i) $I_a^n\left(I_a^m f(x)\right) = I_a^m\left(I_a^n f(x)\right) = I_a^{n+m} f(x)$

ii) $I_\alpha^n\left(af(x) + bh(x)\right) = aI_\alpha^n f(x) + bI_\alpha^n h(x)$

Definition1.3 [9](Fractional Derivative of Order α) for every α, and $m = [\alpha]$ the Riemann-Liouville derivative of order α can be defined as:

$$_aD_t^\alpha h(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} h(s) ds \tag{1.4}$$

Definition 1.4 [8]Let $\alpha > 0,\ m = [\alpha]$. The Caputo derivative operator of order α and $f(t)$ be n −times differentiable function, $t > a$ is defined as

$$_a^CD_t^\alpha h(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \left(\frac{d}{ds}\right)^m h(s) ds \tag{1.5}$$

Or $$_a^CD_t^\alpha h(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{h^m(s)}{(t-s)^{\alpha-m+1}} ds\ .$$

Remark1.1 [4], [10]The Relation between integration and differentiation of the Caputo operator of order α are given as shown:

- The Caputo derivative of fractional integral is

$$ {}^{C}D_a^{\alpha}(I_a^{\alpha} f(t)) = f(t) \tag{1.6}$$

- The fractional integral of Caputo derivative is

$$ I_a^{\alpha}({}^{C}D_a^{\alpha} f(t)) = f(t) - \sum_{n=0}^{m-1} \frac{(t-a)^n}{n!} f^{(n)}(a), \tag{1.7}$$

From the above we got ${}^{C}D_a^{\alpha}(I_a^{\alpha} f(t)) \neq I_a^{\alpha}({}^{C}D_a^{\alpha} f(t))$

2. From the above definitions and properties, we have the Caputo fractional derivative is not equivalent with (Riemann-Liouville) fractional derivative but their fractional integral are equivalent.

1.2. Relation between Caputo α order derivative and Riemann-Liouville α order derivative. [4], [8]

Let $n \in \mathbb{N}, \alpha \in [n-1, n)$. And let $f(x)$ be a function such that ${}^{c}D_a^{\alpha} f(x)$ and $D_a^{\alpha} f(x)$ exist. Then the relation between the (R-L) and the Caputo derivatives is given by:

$$ {}^{c}D_a^{\alpha} f(x) = D_a^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(a) \tag{1.8}$$

Definition 1.5 [7] **(**EIGENVALUES AND EIGENFUNCTIONS)

Let L be a linear operator defined on some set of elements as element $y \neq 0$ is called eigenfunctions of L if $Ly = \lambda y$, the number λ is called an eigenvalue of operator L.

In another representation the number λ is called an eigenvalue of operator L if there exists in the domain of definition of the operator L a function $y \not\equiv 0$, such that $Ly = \lambda y$.

Definition1.6[11] A normed space X is a vector space with a norm defined on it, A Banach space is a complete normed space.

Definition 1.7 [11]The vector space $C[e,d]$ of the complex-valued continuous functions defined on a closed interval $[e,d]$ is Banach space with respect to the following norm $\|v\|_{C[e,d]} = max_{x\in[e,d]} |v(x)| \quad , v \in C[e,d]$.

Lemma 1.1 [12]**:**

Let $\beta > 0$ and $n = [\beta] + 1$, then the solutions to the equation ${}_0^C D_t^{\beta} h(t) = 0$ is given by $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_i \in R$, $i = 0,1,2, \dots , n-1$ are some constants, If assume that $h \in C^n[0,a]$,then $I^{\beta}\ {}_0^C D_t^{\beta} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ for some constants $c_i \in R$, $i = 0,1,2, \dots , n-1$.

Lemma 1.2 [13] Let $y(x) \in C(0,a]$ with $1 < \alpha \leq 2$ Then the solution of the boundary value problem (1.1)-(1.2) is

$$y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left(q(t) - \lambda^2 p(t)\right) y(t) dt + x \frac{1}{\Gamma(\alpha)(ai\lambda - 1)} \int_0^a (\alpha - 1 - i\lambda a + i\lambda t)\, (a-t)^{\alpha-2} \left(q(t) - \lambda^2 p(t)\right) y(t) dt$$

Or $\quad y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left(q(t) - \lambda^2 p(t)\right) y(t) dt + x\omega$

Where $\quad \omega = \frac{1}{\Gamma(\alpha)(ai\lambda - 1)} \int_0^a (\alpha - 1 - i\lambda a + i\lambda t)\, (a-t)^{\alpha-2} \left(q(t) - \lambda^2 p(t)\right) y(t) dt \quad .$

2. ASYMPTOTIC BEHAVIORS OF EIGENVALUES AND EIGENFUNCTIONS

In this section we will define the fractional operator related to theT.Regge differential equation with fractional boundary conditions and the weight function $p(x) \equiv 1$ and show the asymptotic behavior for Eigenvalues and Eigenfunctions.

Definition 2.1: [14] Denoting the fractional T.Regge operator as

$$\mathcal{L}_{\alpha,x} := -{}_0^C D_x^{\alpha} + q(x)$$

Remember the T.Regge problem is

$$-y''(x) + q(x) y(x) = \lambda^2 \rho(x) y(x), \qquad 0 \leq x \leq a, \qquad (2.1)$$

$$y(0) = 0, \qquad y'(a) - i\lambda y(a) = 0, \qquad (2.2)$$

Consider the fractional T.Regge equation $\mathcal{L}_{\alpha,x} y(x) = -{}_0^C D_x^{\alpha} y(x) + q(x) y(x)$

And $\quad \mathcal{L}_{\alpha,x} y(x) + \lambda^2 p(x) y(x) = 0 \,. \qquad (2.3)$

Where $p(x) > 0$ and $p(x)$, $q(x)$ are real valued continuous functions in the interval $[0, a]$, $\alpha \in (1,2]$ with the following boundary conditions

$$D_0^\alpha y_\lambda(a) - i\lambda I_0^{2-\alpha} y_\lambda(a) = 0 \tag{2.4}$$

$$D_0^\alpha y_\lambda(a) - i\lambda D_0^{\alpha-1} y_\lambda(a) = 0 \tag{2.5}$$

Property 2.1. [14]Operators$I_{a^+}^\alpha, I_{b^-}^\alpha, D_{a^+}^\alpha, D_{b^-}^\alpha$, ${}^cD_{a^+}^\alpha$, and ${}^cD_{b^-}^\alpha$ are satisfy the following

1. $\int_a^b f(x) I_{a^+}^\alpha g(x)dx = \int_a^b g(x) I_{b^-}^\alpha f(x)dx$,
2. $\int_a^b f(x) D_{b^-}^\alpha g(x)dx = \int_a^b g(x) {}^cD_{a^+}^\alpha f(x)dx + \sum_{k=0}^{m-1} (-1)^{m-k} f^{(k)}(x) D^{m-k-1} I_{b^-}^{m-\alpha} g(x)\Big|_{x=a}^{b}$,
3. $\int_a^b f(x) D_{a^+}^\alpha g(x)dx = \int_a^b g(x) {}^cD_{b^-}^\alpha f(x)dx + \sum_{k=0}^{m-1} (-1)^{k} f^{(k)}(x) D^{m-k-1} I_{a^+}^{m-\alpha} g(x)\Big|_{x=a}^{b}$

Theorem 2.1. If $I_0^{2-\alpha} y_\lambda(0) = 0, D_0^{\alpha-1} y_\lambda(0) = 0$ then Eigenvalues of Fractional Boundary Value Problem (FBVP) given by 2.3, 2.4 and 2.5 are only real or imaginary parts.

Proof: Assume that λ , $\bar{\lambda}$ ($\bar{\lambda}$ *is Lambda cojugate*) are an eigenvalue for problem 2.3, 2.4 and 2.5 and $y(x), \bar{y}(x)$ are corresponding eignfunctions.

So we have $\mathcal{L}_{\alpha,x} y(x) + \lambda^2 p(x) y(x) = 0$

And $\quad I_0^{2-\alpha} y_\lambda(0) = 0$, $\quad {}_0^CD_x^{\alpha-1} y_\lambda(0) = 0$,

${}_0^CD_x^\alpha y_\lambda(a) - i\lambda I_0^{2-\alpha} y_\lambda(a) = 0$, $\quad {}_0^CD_x^\alpha y_\lambda(a) - i\lambda {}_0^CD_x^{\alpha-1} y_\lambda(a) = 0$,

And we have

$$\mathcal{L}_{\alpha,x} \bar{y}(x) + \bar{\lambda}^2 p(x) \bar{y}(x) = 0 \ , \tag{2.6}$$

$$\text{And} \quad I_0^{2-\alpha} \bar{y}_{\bar{\lambda}}(0) = 0 \ , \quad {}_0^CD_x^{\alpha-1} \bar{y}_{\bar{\lambda}}(0) = 0 \ , \tag{2.7}$$

$${}_0^CD_x^\alpha \bar{y}_\lambda(a) + i\bar{\lambda} I_0^{2-\alpha} \bar{y}_{\bar{\lambda}}(a) = 0 \ , \quad {}_0^CD_x^\alpha \bar{y}_{\bar{\lambda}}(a) + i\bar{\lambda} {}_0^CD_x^{\alpha-1} \bar{y}_{\bar{\lambda}}(a) = 0 \tag{2.8}$$

We multiply equation 2.3 by $\bar{y}(x)$ and equation 2. 6 by $y(x)$ we get

$\bar{y}(x) \mathcal{L}_{\alpha,x} y(x) + \lambda^2 p(x) y(x) \bar{y}(x) = 0$ Implies that $\bar{y}(x) \mathcal{L}_{\alpha,x} y(x) + \lambda^2 p(x) |y(x)|^2 = 0$ (2.9)

And $y(x) \mathcal{L}_{\alpha,x} \bar{y}(x) + \bar{\lambda}^2 p(x) y(x) \bar{y}(x) = 0$

$$y(x) \mathcal{L}_{\alpha,x} \bar{y}(x) + \bar{\lambda}^2 p(x) \, |y(x)|^2 = 0 \tag{2.10}$$

Subtract equation 2.9 *and* 2.10 to obtain

$$y(x)\mathcal{L}_{\alpha,x}\bar{y}(x)-\bar{y}(x)\mathcal{L}_{\alpha,x}y(x)=\left(\lambda^2-\bar{\lambda}^2\right)p(x)|y(x)|^2$$

From this equality we obtain $\left(\lambda^2-\bar{\lambda}^2\right)p(x)|y(x)|^2=\bar{y}(x)\,{}_0^CD_x^\alpha y(x)-y(x)\,{}_0^CD_x^\alpha \bar{y}(x).$

Integrating both sides of the above equation from $0\ to\ a$, we have

$$\left(\lambda^2-\bar{\lambda}^2\right)\int_0^a p(x)|y(x)|^2\,dx=\int_0^a \bar{y}(x)\,{}_0^CD_x^\alpha y(x)-y(x)\,{}_0^CD_x^\alpha \bar{y}(x)\,dx$$

Use properties 2.1 and note that the right-hand side of the integrated equality contains only boundary terms, we obtain the above equation is equal to

$$\left(\lambda^2-\bar{\lambda}^2\right)\int_0^a p(x)|y(x)|^2\,dx=-[{}_0^CD_x^{\alpha-1}y(x)\,{}_0^CD_x^\alpha\bar{y}(x)-{}_0^CD_x^{\alpha-1}\bar{y}(x)\,{}_0^CD_x^\alpha y(x)]+\left[{}_0^CD_x^\alpha\bar{y}(x)I_{0^+}^{2-\alpha}y(x)-{}_0^CD_x^\alpha y(x)I_{0^+}^{2-\alpha}\bar{y}(x)\right]\Big|_0^a,$$

$$\left(\lambda^2-\bar{\lambda}^2\right)\int_0^a p(x)|y(x)|^2\,dx=-[{}_0^CD_x^{\alpha-1}y(a)\,{}_0^CD_x^\alpha\bar{y}(a)-{}_0^CD_x^{\alpha-1}\bar{y}(a)\,{}_0^CD_x^\alpha y(a)]+[{}_0^CD_x^\alpha\bar{y}(a)I_0^{2-\alpha}y(a)-{}_0^CD_x^\alpha y(a)I_0^{2-\alpha}\bar{y}(a)]+[{}_0^CD_x^{\alpha-1}y(0)\,{}_0^CD_x^\alpha\bar{y}(0)-{}_0^CD_x^{\alpha-1}\bar{y}(0)\,{}_0^CD_x^\alpha y(0)]-[{}_0^CD_x^\alpha\bar{y}(0)I_0^{2-\alpha}y(0)-{}_0^CD_x^\alpha y(0)I_0^{2-\alpha}\bar{y}(0)],$$

From boundary condition we get

$$I_0^{2-\alpha}y_\lambda(0)=0\ ,\quad {}_0^CD_x^{\alpha-1}y_\lambda(0)=0\ ,I_0^{2-\alpha}\bar{y}_{\bar{\lambda}}(0)=0\ ,\quad {}_0^CD_x^{\alpha-1}\bar{y}_{\bar{\lambda}}(0)=0$$

Now

$$\left(\lambda^2-\bar{\lambda}^2\right)\int_0^a p(x)|y(x)|^2\,dx=-[{}_0^CD_x^{\alpha-1}y(a)\cdot{}_0^CD_x^\alpha\bar{y}(a)-{}_0^CD_x^{\alpha-1}\bar{y}(a)\cdot{}_0^CD_x^\alpha y(a)]+[{}_0^CD_x^\alpha\bar{y}(a).I_0^{2-\alpha}y(a)-{}_0^CD_x^\alpha y(a).I_0^{2-\alpha}\bar{y}(a)],$$

Finally, from the boundary conditions we have

$${}_0^CD_x^{\alpha-1}y_\lambda(a)=I_0^{2-\alpha}y_\lambda(a)\ ,\qquad and\quad {}_0^CD_x^{\alpha-1}\bar{y}_{\bar{\lambda}}(a)=I_0^{2-\alpha}\bar{y}_{\bar{\lambda}}(a)\ ,$$

using the boundary conditions (2.5), and (2.8) we obtain

$$\left(\lambda^2-\bar{\lambda}^2\right)\int_0^a p(x)|y(x)|^2\,dx=-[{}_0^CD_x^{\alpha-1}y(a)\cdot{}_0^CD_x^\alpha\bar{y}(a)-{}_0^CD_x^{\alpha-1}\bar{y}(a)\cdot{}_0^CD_x^\alpha y(a)-{}_0^CD_x^\alpha\bar{y}(a).\,{}_0^CD_x^{\alpha-1}y(a)+{}_0^CD_x^\alpha y(a).\,{}_0^CD_x^{\alpha-1}\bar{y}(a)]\ ,$$

$$\left(\lambda^2-\bar{\lambda}^2\right)\int_0^a p(x)|y(x)|^2\,dx=0.$$

And since $y(x)$ is a non-trivial solution and $p(x)>0$

so we infer that $\left(\lambda^2 - \bar{\lambda}^2\right) = 0 \rightarrow \left(\lambda - \bar{\lambda}\right)\left(\lambda + \bar{\lambda}\right) = 0$

If $\lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow$ λ is real number

and if $\lambda + \bar{\lambda} = 0 \Rightarrow \lambda = -\bar{\lambda} \Rightarrow \lambda$ is complex number ■

Theorem 2.2. If $I_0^{2-\alpha} y_\lambda(0) = 0, D_0^{\alpha-1} y_\lambda(0) = 0$ then eigenfunctions, corresponding to distinct eigenvalues of Fractional Boundary Value Problem (FBVP) given by2.3, 2.4 and 2. 5 are orthogonal w.r.t. weight function $p(x)$ on $[0, a]$ that is

$$\int_a^b p(x) y_{\lambda_1}(x) y_{\lambda_2}(x) dx = 0 \qquad , \lambda_1 \neq \lambda_2$$

When functions y_{λ_j}correspond to eigenvalues λ_j .

Proof: The proof is similar to theorem 2.1.

Theorem 2.3 if the Eigenvalues of Fractional Boundary Value Problem (FBVP) given by2.3, 2.4 and 2. 5 are only real or imaginary part and $I_0^{2-\alpha} y_\lambda(0) = 0, D_0^{\alpha-1} y_\lambda(0) = 0$then $\frac{\mathcal{L}_{\alpha,x}\bar{y}(x)}{\bar{y}(x)} = \frac{\mathcal{L}_{\alpha,x} y(x)}{y(x)}$.

Proof: The proof is similar to theorem 2.1.

Theorem 2.4 if the Eigenvalues of Fractional Boundary Value Problem (FBVP) given by2.3, 2.4 and 2. 5 are only real or imaginary part and $I_0^{2-\alpha} y_\lambda(0) = 0, D_0^{\alpha-1} y_\lambda(0) = 0$ then

$\frac{\mathcal{L}_{\alpha,x}(u)}{u} = \frac{\mathcal{L}_{\alpha,x}(v)}{v}$, where $y(x) = u(x) + iv(x)$.

Proof: The proof is similar to theorem 2.1.

Corollary 2.1 If $\frac{\mathcal{L}_{\alpha,x}(u)}{u} = \frac{\mathcal{L}_{\alpha,x}(v)}{v}$ then the eigenvalues of fractional boundary value problem given by 2.3, 2.4 and 2.5 are only real or imaginary part.

Proof: The same as theorem 2.4

Corollary 2.2 If the eigenvalues of fractional boundary value problem (FBVP) given 2.3, 2.4 and 2.5, are only real or imaginary parts then $Im\left(y(x)\right)\mathcal{L}_{\alpha,x} Re\left(y(x)\right) = Re\left(y(x)\right)\mathcal{L}_{\alpha,x} Im\left(y(x)\right)$

Proof: Similar to theorem 2.4 assume $Re\left(y(x)\right) = u(x)$ and $Im\left(y(x)\right) = v(x)$

Corollary 2.3 If $Im\big(y(x)\big)\,{}_0^CD_x^\alpha Re\big(y(x)\big) = Re\big(y(x)\big)\,{}_0^CD_x^\alpha Im\big(y(x)\big)$ then the eigenvalues of Fractional Boundary Value Problem (FBVP) given by 2.3, 2.4 and 2.5 are only real or imaginary parts.

Proof: Similar to corollary 2.2

Proposition 2.5: The eigenvalues of fractional boundary value problem (FBVP) given by2.3, 2.4 and 2.5 are only real or imaginary parts if and only if

$Im\big(y(x)\big)\mathcal{L}_{\alpha,x}Re\big(y(x)\big) = Re\big(y(x)\big)\mathcal{L}_{\alpha,x}Im\big(y(x)\big).$

Proof: Adding corollary 2.2 and 2.3 obtain the proof.

Example 2.1: consider the Boundary value problem

$$\begin{cases} D_x^\alpha y(x) + \lambda y(x) = 0\,, \quad 1 < \alpha \le 2 \\ py(0) - ry'(0) = 0\,, \;\; qy(1) + sy'(1) = 0 \end{cases}$$

Solution: In our problem $p = 1\,, r = 0\,, s = 1\,, q = i\lambda$ and let $\lambda \in R$

If $= 0$, letting the general solution $u(x) = A + Bx$ satisfy the boundary conditions we obtain $A = B = 0$ i.e the problem only has zero solution.

If $\neq 0$, letting the general solution is

$y(x) = BxE_{\alpha,2}(-\lambda^2x^\alpha)$ and $\big[i\lambda E_{\alpha,2}(-\lambda) - \lambda E_{\alpha,1}(-\lambda)B\big] = 0$

The coefficient determined is

$i\lambda E_{\alpha,2}(-\lambda) = \lambda E_{\alpha,1}(-\lambda)B \rightarrow BE_{\alpha,1}(-\lambda) = iE_{\alpha,2}(-\lambda) \rightarrow B = i\frac{E_{\alpha,2}(-\lambda)}{E_{\alpha,1}(-\lambda)}$

So the solution is $y(x) = BxE_{\alpha,2}(-\lambda^2x^\alpha) = ix\frac{E_{\alpha,2}(-\lambda)}{E_{\alpha,1}(-\lambda)}E_{\alpha,2}(-\lambda^2x^\alpha)$.

CONCLUSIONS

In this work, fractional boundary value problem (T.Regge problem for fractional order) has been studied. The T.Regge Problem for fractional order has some behavior of eigenvalues and eigenfunctions. The asymptotic behavior for eigenvalues and eigenfunctions for fractional order T.Regge problem have been investigated. We got some result for eigenvalues that are the results prove the accuracy of the work.

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