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Nearly Quasi Primary-2-Absorbing Submodules

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ABSTRACT

Let W be a nonzero unital left R -module and R be a commutative ring with nonzero identity. As a generalization of 2-Absorbing submodules, we provide the idea of Nearly Quasi Primary-2-Absorbing submodules in this article. As a proper submodule V of W is called the Nearly Quasi Primary-2-Absorbing submodule of W, if whenever $rsx \in V$ for $r, s \in R$, $x \in W$, implies that either $rx \in W - rad(V) + J(W)$ or $sx \in W - rad(V) + J(W)$ or $rs \in \sqrt{[V + J(W):_R W]}$. Several properties, characterizations and examples concerning this new notion are given.

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1. Introduction

2-Absorbing submodules are well recognized to have a significant impact on the theory of modules over commutative rings. This topic has received a lot of research thus far. One can look at [1, 2, 3] for numerous studies. The idea of generalizing the concept of 2-Absorbing submodule through the use of various methods is one of the key interests of many scholars. For instance, the Semi-2-Absorbing submodule generalization, was proposed and analyzed in [4], followed by a second generalization known as 2-Absorbing primary. In [5], the submodule was examined. It is important to remember that the idea of 2-Absorbing submodule was initially presented in 2011 by Darani A. and Soheilinia F. as a generalization of the 2-Absorbing ideal, whereas the 2-Absorbing ideal was first presented in 2007 by Badawi A, see [6,7]. All rings being considered in this study are commutative with nonzero identity, and all modules are nonzero unitary. Additionally W always signifies such an R -module and R always denotes such a ring. Assume V is a submodule of W and I is an ideal of R. The radical of I, symbolized by $\sqrt{I}$, is then defined as the intersection of all prime ideals containing I and equally consists of all components an of R whose some power in I, that is, "$\{a \in R: a^n \in I \, for \, some \, n \in V\}$". Additionally, the ideal $[V:_R W]$ is defined as "$\{a \in R: aW \subseteq V\}$ and $\{x \in W: ax \in V\}$ for every $a \in R$ in the submodule $[V:_W a]$". "Similar to radical of an ideal, radical of a submodule of R-module W can be define as the intersection of all prime submodules containing V and

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denoted by $W - rad(V)$. If not, that is if there is no prime submodule containing V we say $W - rad(V) \neq W$, where a submodule V of W is a prime submodule if whenever $am \in V$, then either $a \in [V:_R W]$ or $m \in V$ [8]". Primary submodule as a generalization of prime submodule is defined as follow, a proper submodule V of an R-module W is called primary submodule, "if whenever $rx \in V$, for $r \in R, x \in W$, implies that either $x \in V$ or $r^n W \subseteq V$ for some $n \in Z^+$ [9]". "In [10] introduced the concept of 2-Absorbing quasi primary ideal, where a proper ideal I of a ring R is 2-Absorbing quasi-primary ideal if and only if whenever $rst \in I$, then $rs \in \sqrt{I}$ or $rt \in \sqrt{I}$ or $st \in \sqrt{I}$ for each $r, s, t \in R$". Researcher Kos in 2017 popularized the concept of 2-Absorbing quasi primary ideal to 2-Absorbing quasi primary submodule, where a proper submodule V of an R-module W is called 2-Absorbing quasi primary submodule of W, "if whenever $rsx \in V$, where $r, s \in R, x \in W$, implies that either $rx \in W - rad(V)$ or $sx \in W - rad(V)$ or $rs \in \sqrt{[V:_R W]}$ [11]". The concept of a multiplication module is one of the basic concepts in the theory of the module, define as an R-module W is multiplication, if every submodule K of W is of the form $K = IW$ for some ideal I of R. Equivalently W is a multiplication R-module if every submodule K of R of the form $K = [K:_R W]W$ [12]. "Recall that an R-module W is faithful if $ann_R(W) = (0)$, where $ann_R(W) = \{r \in R: rW = (0)\}$ and an R-module W is finitely generated if $W = Rx_1 + Rx_2 + \cdots + Rx_n$ for $x_1, x_2, \ldots, x_n \in W$ [13]. The Jacobsen radical of W is the intersection of all maximal submodule of W. Finally an R-module W is a projective if for any R-epimorphism f from an R-module W on to an R-module $\overline{W}$ and for any homomorphism g from an R-module $\overline{\overline{W}}$ to $\overline{W}$, there exists a homomorphism h from $\overline{\overline{W}}$ to$W$ such that $f \circ h = g$ [13]."

2. Nearly Quasi Primary-2-Absorbing Submodules.

Definition 2.1 A proper submodule V of an R-module W is a Nearly Quasi Primary-2-Absorbing submodule of W (simply NQPr-2-ABS), if whenever $rsm \in V$ for $r, s \in R, m \in W$, implies that either $rm \in W - rad(V) + J(W)$ or $sm \in W - rad(V) + J(W)$ or $rs \in \sqrt{[V + J(W):_R W]}$. And a proper ideal I of a ring R is said to be a NQPr-2-ABS ideal of R if I is a NQPr-2-ABS submodule of an R-module R.

Proposition 2.2 Every 2-Absorbing submodule of an R-module W is a NQPr-2-ABS submodule of W.

Proof Clear .

Remark 2.3 The converse of proposition 2.2 it's not true in general, see the following example.

Example 2.4 Let $W = Z_{36}, R = Z$ and the submodule $V = \langle \overline{12} \rangle$. Clear that the submodule $\langle \overline{12} \rangle$ is NQPr-2-ABS submodule of Z_{36}, since $J(Z_{36}) = \langle \overline{6} \rangle$ and $rad_{Z_{36}}(\langle \overline{12} \rangle) = \langle \overline{6} \rangle$. Hence for all $a, b \in R$ and $m \in W$ with $abm \in V$ we gave either $am \in Z_{36} - rad(\langle \overline{12} \rangle) + J(Z_{36}) = \langle \overline{6} \rangle$ or $bm \in Z_{36} - rad(\langle \overline{12} \rangle) + J(Z_{36}) = \langle \overline{6} \rangle$ or $ab \in \sqrt{[V + J(Z_{36}):_Z Z_{36}]}$. But V is not 2-Absorbing submodule of Z_{36}, because $2.3.\overline{2} \in \langle \overline{12} \rangle$ for $2, 3 \in Z$ and $\overline{2} \in Z_{36}$, implies that $2.\overline{2} = \langle \overline{4} \rangle \notin \langle \overline{12} \rangle$ and $3.\overline{2} = \langle \overline{6} \rangle \notin \langle \overline{12} \rangle$. Also $2.3 = 6 \notin [V:_Z W] = 12Z$.

The following propositions are characterizations of NQPr-2-ABS submodule.

Proposition 2.5 For a proper submodule V of W. The sentences that follow are comparable.

1-V is a NQPr-2-ABS submodule of W.

2-For every $r, s \in R$, $[V + J(W):_W r^n s^n] = W$ for some $n \in Z^+$ or $[V:_W rs] \subseteq [W - rad(V) + J(W):_W r] \cup [W - rad(V) + J(W):_W s]$.

3-For every $r,s \in R$, $[V + J(W):_W r^n s^n] = W$ for some $n \in Z^+$or $[V:_W rs] \subseteq [W - rad(V) + J(W):_W r]$ or $[V:_W rs] \subseteq [W - rad(V) + J(W):_W s]$.

Proof (1)⇒(2): Suppose that V is a NQPr-2-ABS submodule of W. Let $r,s \in R$. If $rs \in \sqrt{[V + J(W):_R W]}$, then $(rs)^n = r^n s^n \in [V + J(W):_R W]$ for some $n \in Z^+$ and so $[V + J(W):_W r^n s^n] = W$. Now, assume $rs \notin \sqrt{[V + J(W):_R W]}$. Let $x \in [V:_W rs]$, then we have $rsx \in V$, hence either $rx \in W - rad(V) + J(W)$ or $sx \in W - rad(V) + J(W)$, since V is a NQPr-2-ABS submodule of W. Thus we get the result $[V:_W rs] \subseteq [W - rad(V) + J(W):_W r] \cup [W - rad(V) + J(W):_W s]$.

(2)⇒(3): It is common knowledge that if a submodule contains two other submodules, it must contain at least one of them.

(3)⇒(1): Let $rsx \in V$ for $r,s \in R$ and $x \in W$ with $rs \notin \sqrt{[V + J(W):_R W]}$. Then we have $[V + J(W):_W r^n s^n] \neq W$ for every $n \in Z^+$. So by (3) we obtain that $x \in [V:_W rs] \subseteq [W - rad(V) + J(W):_W r]$or $x \in [V:_W rs] \subseteq [W - rad(V) +$

$J(W):_W \mathrm{s}]$, hence either $\mathrm{r}x \in W - rad(V) + J(W)$ or $\mathrm{s}x \in W - rad(V) + J(W)$. Therefore V is a NQPr-2-ABS submodule of W.

Proposition 2.6 Let V be a proper submodule of an R-module W. Then V is a NQPr-2-ABS submodule of W if and only if $\mathrm{rs}E \subseteq V$, for $\mathrm{r}, \mathrm{s} \in R$ and E is a submodule of W with $\mathrm{rs} \notin \sqrt{[V + J(W):_R W]}$ implies that either $\mathrm{r}E \subseteq W - rad(V) + J(W)$ or $\mathrm{s}E \subseteq W - rad(V) + J(W)$.

Proof ($\Longrightarrow$) Since $\mathrm{rs}E \subseteq V$, for $\mathrm{r}, \mathrm{s} \in R$ and E is a submodule of W, then $E \subseteq [V:_W \mathrm{rs}]$ and $[V + J(W):_W \mathrm{r}^n \mathrm{s}^n] \neq W$ for every $n \in Z^+$, then by proposition 2.5 we gave $E \subseteq [V:_W \mathrm{rs}] \subseteq [W - rad(V) + J(W):_W \mathrm{r}]$ or $E \subseteq [V:_W \mathrm{rs}] \subseteq [W - rad(V) + J(W):_W \mathrm{s}]$. Hence either $\mathrm{r}E \subseteq W - rad(V) + J(W)$ or $\mathrm{s}E \subseteq W - rad(V) + J(W)$.

($\Longleftarrow$) Direct.

Proposition 2.7 For a proper submodule V of W. The sentences that follow are comparable.

1- V is a NQPr-2-ABS submodule of W.

2- For every $a \in R$, an ideal I of R and submodule E of W with $aIE \subseteq V$, then either $aI \subseteq \sqrt{[V + J(W):_R W]}$ or $aE \subseteq W - rad(V) + J(W)$ or $IE \subseteq W - rad(V) + J(W)$.

3- For ideals I, J of R and submodule E of W with $IJE \subseteq V$, then either $IJ \subseteq \sqrt{[V + J(W):_R W]}$ or $IE \subseteq W - rad(V) + J(W)$ or $JE \subseteq W - rad(V) + J(W)$.

Proof (1)⇒(2): Let $aIE \subseteq V$ for $a \in R$, an ideal I of R and submodule E of W with $aI \nsubseteq \sqrt{[V + J(W):_R W]}$ and $IE \nsubseteq W - rad(V) + J(W)$. Then there exists $\mathrm{r}, \mathrm{s} \in I$ such that $a\mathrm{r} \notin \sqrt{[V + J(W):_R W]}$ and $\mathrm{s}E \nsubseteq W - rad(V) + J(W)$. Now, to prove that $aE \subseteq W - rad(V) + J(W)$. Assume that $aE \nsubseteq W - rad(V) + J(W)$, since $a\mathrm{r}E \subseteq V$ and V is a NQPr-2-ABS submodule of W with $a\mathrm{r} \notin \sqrt{[V + J(W):_R W]}$, hence by proposition 2.6 we get $\mathrm{r}E \subseteq W - rad(V) + J(W)$ and so $(\mathrm{r} + \mathrm{s})E \nsubseteq W - rad(V) + J(W)$, then by proposition 2.6 we have $a(\mathrm{r} + \mathrm{s}) = a\mathrm{r} + a\mathrm{s} \in \sqrt{[V + J(W):_R W]}$, because $a(\mathrm{r} + \mathrm{s})E \subseteq V$. Since $a\mathrm{r} + a\mathrm{s} \in \sqrt{[V + J(W):_R W]}$ and $a\mathrm{r} \notin \sqrt{[V + J(W):_R W]}$, then $a\mathrm{s} \notin \sqrt{[V + J(W):_R W]}$. As $a\mathrm{s}E \subseteq V$, by proposition 2.6 $\mathrm{s}E \subseteq W - rad(V) + J(W)$ or $aE \subseteq W - rad(V) + J(W)$, which is contradiction.

(2)⇒(3): Let $IJE \subseteq V$, with $IJ \nsubseteq \sqrt{[V + J(W):_R W]}$ for an ideals I and J of R, then there exist $\mathrm{r} \in I$ such that $\mathrm{r}J \nsubseteq \sqrt{[V + J(W):_R W]}$. Assume that $IE \nsubseteq W - rad(V) + J(W)$ and $JE \nsubseteq W - rad + J(W)$, then by (2) we get $\mathrm{r}E \subseteq W - rad(V) + J(W)$ for some $\mathrm{r} \in I$. Also there exist an element $\mathrm{r} \in I$ such that $\mathrm{s}E \nsubseteq W - rad(V) + J(W)$, by hypotheses $IE \nsubseteq W - rad(V) + J(W)$. As $\mathrm{s}JE \subseteq V$, we get the result that $\mathrm{s}J \subseteq \sqrt{[V + J(W):_R W]}$ and $(\mathrm{r} + \mathrm{s})J \nsubseteq \sqrt{[V + J(W):_R W]}$, since $(\mathrm{r} + \mathrm{s})JE \subseteq V$, we have $(\mathrm{r} + \mathrm{s})E \subseteq W - rad(V) + J(W)$ and hence $\mathrm{s}E \subseteq W - rad(V) + J(W)$, which is a contradiction.

(3)⇒(1): Let $\mathrm{rs}x \in V$ for $\mathrm{r}, \mathrm{s} \in R$ and $x \in W$. Then $\langle \mathrm{r} \rangle \langle \mathrm{s} \rangle x \subseteq V$. By (3) either $\langle \mathrm{r} \rangle \langle \mathrm{s} \rangle \subseteq \sqrt{[V + J(W):_R W]}$ or $\langle \mathrm{r} \rangle x \subseteq W - rad(V) + J(W)$ or $\langle \mathrm{s} \rangle x \subseteq W - rad(V) + J(W)$.Thus either $\mathrm{rs} \in \sqrt{[V + J(W):_R W]}$ or $\mathrm{r}x \in W - rad(V) + J(W)$ or $\mathrm{s}x \in W - rad + J(W)$.

Remark 2.8 If the submodule V of W is a NQPr-2-ABS submodule of W, then the residual of W ($[V:_R W]$) should not be NQPr-2-ABS ideal of R. For example:

Consider the Z-module $W = Z \oplus Z$, the submodule $V = 5Z \oplus 6Z$ is a NQPr-2-ABS of W, since $6.2.(0,1) \in V$, implies that $6.(0,1) \in W - rad(V) + J(W) = V$, but $[5Z \oplus 6Z :_R Z \oplus Z] = 30Z$ will not occur NQPr-2-ABS ideal of Z, because $5.3.2 \in 30$, but $5.2 \notin W - rad(30Z) + J(W) = 30Z$ and $3.2 \notin 30Z$ and $5.3 = 15 \notin \sqrt{[30Z + J(Z):_Z Z]} = \sqrt{30Z} = 30Z$.

On W ,we obtain the subsequent characterizations under specific circumstances.
But before that we need to recall the following lemmas.

Lemma 2.9 [13, Theo. (9. 2. 1)(g)] "For any projective R-module W, we have $J(W) = J(R)W$."

Lemma 2.10 [17] "Let W be faithful multiplication R-module, then $J(W) = J(R)W$."

Lemma 2.11 [[15, Prop. (2. 12)]] "Let R be a commutative ring with identity, V be a proper submodule of a multiplication R-module W and $I = [V:_R W]$. Then $W - rad(V) = \sqrt{I}W = \sqrt{[V:_R W]}W$."

Proposition 2.12 Let W be a multiplication projective R-module and V be a proper submodule of W. Then V is a NQPr-2-ABS submodule of W if and only if $[V:_R W]$ is NQPr-2-ABS ideal of R.

Proof $(\Longrightarrow)$Let $rst \in [V:_R W]$, where $r, s, t \in R$ and $rs \notin \sqrt{[[V:_R W] + J(R):_R R]} = \sqrt{[V:_R W] + J(R)}$,sine $rs(tW) \subseteq V$ and V is NQPr-2-ABS submodule of W with $rs \notin \sqrt{[V:_R W] + J(R)}$, then either $r(tW) \subseteq W - rad(V) + J(W)$ or $s(tW) \subseteq W - rad(V) + J(W)$. But W is multiplication then by lemma 2.11 $rad_W(V) = \sqrt{[V:_R W]}W$. Thus either $rtW \subseteq \sqrt{[V:_R W]}W + J(W)$ or $stW \subseteq \sqrt{[V:_R W]}W + J(W)$. Now, the module W be a projective R-module, then by lemma 2.9 we have either $rtW \subseteq \sqrt{[V:_R W]}W + J(R)W$ or $stW \subseteq \sqrt{[V:_R W]}W + J(R)W$, hence either $rt \in \sqrt{[V:_R W]} + J(R)$ or $st \in \sqrt{[V:_R W]} + J(R)$. Therefore $[V:_R W]$ is NQPr-2-ABS ideal of R.

$(\Longleftarrow)$Assume that $[V:_R W]$ is NQPr-2-ABS ideal of R, and $rsx \in V$, for $r,s \in R$, $x \in W$ with $rs \notin \sqrt{[V + J(W):_R W]}$, it follows that $(rs)^n W \nsubseteq V + J(W)$ for some $n \in Z^+$. But W is projective multiplication, then by lemma 2.9 $J(R)W = J(W)$. Hence $(rs)^n W \nsubseteq [V:_R W]W + J(R)W$ for some $n \in Z^+$. It follows that $(rs)^n \notin [V:_R W] + J(R) = [[V:_R W] + J(R): R]$, hence $rs \notin \sqrt{[[V:_R W] + J(R): R]}$. Now, $rsx \in V$, that is $rs(x) \subseteq V$ and W is a multiplication, then $(x) = JW$ for some ideal J of R, that is $rsJW \subseteq V$, it follows that $rsJ \subseteq [V:_R W]$. Since $[V:_R W]$ is NQPr-2-Absorbing ideal of R and $rs \notin \sqrt{[[V:_R W] + J(R): R]}$, then either $rJ \subseteq \sqrt{[V:_R W]} + J(R)$ or $sJ \subseteq \sqrt{[V:_R W]} + J(R)$. That is $rJW \subseteq \sqrt{[V:_R W]}W + J(R)W$ or $sJW \subseteq \sqrt{[V:_R W]}W + J(R)W$. Thus by lemma 2.9 and lemma 2.11 we get $r(x) \subseteq W - rad(V) + J(W)$ or $s(x) \subseteq W - rad(V) + J(W)$. Hence either $rx \in W - rad(V) + J(W)$ or $sx \in W - rad(V) + J(W)$. Thus V is NQPr-2-ABS submodule of W.

By lemmas 2.10-11 and by proof of proposition 2.12 we get the result.

Proposition 2.13 Let W be a faithful multiplication R-*module* and V be a proper submodule of W. Then V is NQPr-2-ABS submodule of W if and only if $[V:_R W]$ is NQPr-2-ABS ideal of R.

Lemma 2.14 [14, Coro. of Theo. 9] "Let I and J are ideals of ring R, and W be a finitely generated multiplication R-module. Then $IW \subseteq JW$ if and only if $I \subseteq J + ann_R(W)$."

Lemma 2.15 [15, Prop. (2.4)] "Let W be a multiplication R-module and I is an ideal of R such that $ann(W) \subseteq I$, then $rad_W(IW) = \sqrt{I}W$."

Proposition 2.16 Let W be a finitely generated faithful multiplication R-module and I is NQPr-2-ABS ideal of R. Then IW is NQPr-2-ABS submodule of W.

Proof Let $rsD \subseteq IW$ for $r, s \in R$, and D is a submodule of W with $rs \notin \sqrt{[IW + J(W):_R W]}$, that is $(rs)^n W \nsubseteq IW + J(W)$ for some $n \in Z^+$. Since W is faithful R-module then by lemma 2.10 $J(W) = J(R)W$, that is $(rs)^n W \nsubseteq IW + J(R)W$ for some $n \in Z^+$, it follows that $(rs)^n \notin I + J(R) = [I + J(R):_R R]$ implies that $rs \notin \sqrt{[I + J(R):_R R]}$, Now, since $rsD \subseteq IW$ and W is a multiplication then $D = JW$ for some ideal J of R, thus $rsJW \subseteq IW$. Hence by lemma 2.14 $rsJ \subseteq I + ann_R(W)$, but W is a faithful, thus $ann_R(W)=(0)$, hence $abJ \subseteq I$. But I is NQPr-2-ABS ideal of R and $rs \notin \sqrt{[I + J(R):_R R]}$ then by proposition 2.6 either $rJ \subseteq \sqrt{I} + J(R)$ or $sJ \subseteq \sqrt{I} + J(R)$, hence either $rJW \subseteq \sqrt{I}W + J(R)W$ or $sJW \subseteq \sqrt{I}W + J(R)W$. It follows by lemma 2.10 and lemma 2.15, $rJW \subseteq W - rad(IW) + J(W)$ or $sJW \subseteq W - rad(IW) + J(W)$. That is $rD \subseteq W - rad(IW) + J(W)$ or $sV \subseteq W - rad(IW) + J(W)$. Hence by proposition 2.6 IW is NQPr-2-ABS submodule of W.

Lemma 2.17 [18, Prop. (3. 1)] "If W ia a multiplication R-module, then W is concellation if and only if W is faithful finitely generated."

Proposition 2.18 Let W be a faithful finitely generated multiplication R-*module* and V be *a proper* submodule of W. The sentences that follow are comparable.

1- V is NQPr-2-ABS submodule of W.

2- $[V:_R W]$ is NQPr-2-ABS ideal of R.

3- $V = \mathcal{B}W$ for some NQPr-2-ABS ideal $\mathcal{B}$ of R.

Proof (1) $\Longleftrightarrow$ (2) By proposition 2.13.

(2) $\Longrightarrow$ (3) Since $[V:_R W]$ is NQPr-2-ABS ideal of R and W be a faithful, hence $(0) = ann_R(W) = [(0):_R W] \subseteq [V:_R W]$ and W be multiplication R-module, so $V = [V:_R W]W$, implies that $V = \mathcal{B}W$ for some NQPr-2-ABS ideal $\mathcal{B} = [V:_R W]$ of R.

(3) $\Longrightarrow$ (2) Let $V = \mathcal{B}W$ for some NQPr-2-ABS ideal $\mathcal{B}$ of R. Since W is multiplication, then $V = [V:_R W]W = \mathcal{B}W$. But W is faithful finitely generated multiplication, then by lemma 2.17 $\mathcal{B} = [V:_R W]$, it follows that $[V:_R W]$ is NQPr-2-ABS ideal $\mathcal{B}$ of R.

Proposition 2.19 Let W be a finitely generated multiplication projective R-module and I is NQPr-2-ABS ideal of R with $ann_R(W) \subseteq I$. Then IW is NQPr-2-ABS submodule of W.

Proof Clear.

Remark 2.20 The intersection of two NQPr-2-ABS submodules of W need not to be NQPr-2-ABS submodule of W. The example below clarifies that:
Consider the Z-$module$ Z and the submodules $5Z, 6Z$ are NQPr-2-ABS submodules of Z-$module$ Z, but $5Z \cap 6Z = 30Z$ is not NQPr-2-ABS submodule of Z-$module$ Z (because if $2.3.5 \in 30Z$, but $2.5 \notin Z - rad(30Z) + J(Z) = 30Z$ and $3.5 = 15 \notin Z - rad(30Z) + J(Z) = 30Z$. Also $2.3 = 6 \notin \sqrt{[30Z:_Z Z]} = \sqrt{[30Z + J(Z):_Z Z]} = \sqrt{30Z} = 30Z$.

Under the certain condition the intersection of two NQPr-2-ABS submodules is NQPr-2-ABS submodule.

Lemma 2.21[13, lemma (2.3.15)] "Let A, B and C are submodules of an R-module W with $B \subseteq C$, then $(A + B) \cap C = (A \cap C) + B = (A \cap C) + (B \cap C)$."

Lemma 2.22[16, Theo. 15(3)] "Let W be a multiplication R-module and D, V be a submodules of W. Then $W - rad(D \cap V) = W - rad(D) \cap W - rad(D)$."

Proposition 2.23 Let V and D be a proper submodules of multiplication R-module W with $J(W) \subseteq V$ or $J(W) \subseteq D$. If V and D are NQPr-2-ABS submodules of W, then $V \cap D$ is NQPr-2-ABS submodule of W.

Proof Suppose that V and D are NQPr-2-ABS submodules of W, and let $rst \in V \cap D$ for $r, s \in R, t \in W$, then $rst \in V$ and $rst \in D$. But both V and D are NQPr-2-ABS submodules of W, then either $rt \in W - rad(V) + J(W)$ or $st \in W - rad(V) + J(W)$ or $rs \in \sqrt{[V + J(W):_R W]}$ and either $rt \in W - rad(D) + J(W)$ or $st \in W - rad(D) + J(W)$ or $rs \in \sqrt{[D + J(W):_R W]}$. Hence either $rt \in (W - rad(V) + J(W)) \cap (W - rad(D) + J(W))$ or $st \in (W - rad(V) + J(W)) \cap (W - rad(D) + J(W))$ or $(rs)^n W \subseteq (V + J(W)) \cap (D + J(W))$. If $J(W) \subseteq D \subseteq W - rad(D)$, then $D + J(W) = D$ and $J(W) + W - rad(D) = W - rad(D)$. Thus either $rt \in (W - rad(V) + J(W)) \cap W - rad(D)$ or $st \in (W - rad(V) + J(W)) \cap W - rad(D)$ or $(rs)^n W \subseteq (V + J(W)) \cap D$. It follows that by lemma 2.21 either $rt \in \left(W - rad(V) \cap W - rad(D)\right) + J(W)$ or $st \in \left(W - rad(V) \cap W - rad(D)\right) + J(W)$ or $(rs)^n W \subseteq (V \cap D) + J(W)$. Hence by lemma 2.22 we obtain that either $rt \in W - rad(V \cap D) + J(W)$ or $st \in W - rad(V \cap D) + J(W)$ or $rs \in \sqrt{[(V \cap D) + J(W):_R W]}$. That is $V \cap D$ is NQPr-2-ABS submodule of W. Similarly if $J(W) \subseteq V$, we get $V \cap D$ is NQPr-2-ABS submodule of W.

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