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On Some Results for M-band Sub Filter Bank

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Abstract: In the present paper, we study aliasing signal, filter banks-band and sub M-band. Weobtain some results, like, the relationship between input and output signal, subsampling, input signal, output signal.

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Keywords: aliasing signal, filter banks, M-band.

1. Introduction:A filter bank is a signal processing device that produces M signals from a single signal by means of filtering by M simultaneous filters. The analysis filter bank splits the input signal $x(n)$ into a number of subband signals$x_k(n)$ **,** At the analysis stage, the input signal $x(n)$ is passed through a bank of M analysis filters$H_i(z)$, At the synthesis stage, the subbands are combined by a set of upsamplers and M synthesis filters $F_i(z)$ to form the reconstructed signal$\hat{x}(n)[2-7]$.

Definition($\mathbf{1.1}$) $[\mathbf{1-5}]$**:**A filter bank is called the perfect reconstruction filter bank if the reconstructed signal$\hat{x}(n)$ is a delayed or possibly scaled version of the original signal $x(n)$, i.e., $\hat{x}(n) = cx(n-d)$, $d \in Z$, $c \neq 0$.

Definition($\mathbf{1.2}$) [4]:The z-transform of a discrete-time signal $x(\mathrm{n})$ is defined as:

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} \qquad (1)$$

or, writing explicitly a few of the terms:

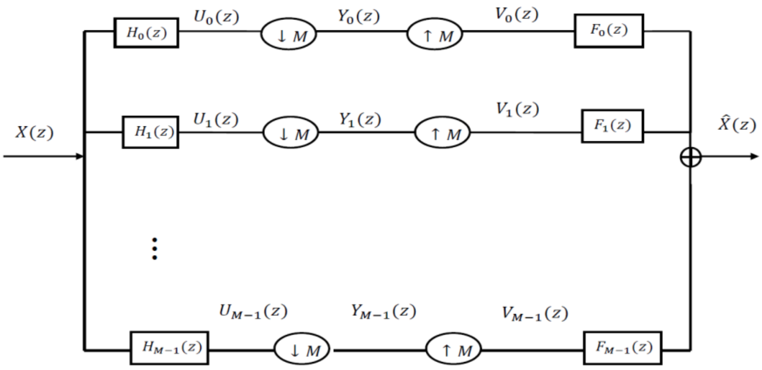
$$X(z) = \cdots + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots,$$

where z is complex variable .

Remark:[6]The z-transform of $h(n)$ is called the transfer function of the filter and is defined by: $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$ (Transfer function) (2)

1.1 Basic Filter Bank

Filter banks appear in two basic parts ,the first one is called analysis filter bank which can divides the signal into Mfiltered and downsampling input signal. Such filter bank is depicted in Fig (1)[6],and the second part is called synthesis filter bank generates a single signal from M upsampled and interpolated signals Fig.(1) shows such a synthesis filter bank[7].



Fig(1) Synthesis and analysis

The main idea of this structure is described as follow : The broadband signal $X(z)$ is split into M uniform sub-signals by analysis filter banks $H_0(z)$, $H_1(z)$, … , $H_{M-1}(z)$, (M is the number of subchannels) .Since bandwidth of sub-signals is narrower than the bandwidth of $X(z)$, the sample rate of sub-signals can be lowered by a factor M[7].

$$Y_i(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_i(W_M^k\, z)X(W_M^k\, z), \tag{3}$$

where$i \in \{0,1,\dots,M-1\}$, $W_M = e^{-\frac{2\pi}{M}j}$

$$\therefore Y_i(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_i\left(e^{-\frac{2k\pi}{M}j}\, z\right) X\left(e^{-\frac{2k\pi}{M}j}\right), \tag{4}$$

where $i = \{0,1 \dots.,M-1\}$.

The reconstructed signal $\hat{X}(z)$ is obtained as :

$$\hat{X}(z) = \sum_{i=0}^{M-1} Y_i(z)F_i(z). \tag{5}$$

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Substituting Equation (4) into (5), we can have

$$\hat{X}(z) = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} H_i\left(e^{-\frac{2k\pi}{M}j}\, z\right) X(e^{-\frac{2k\pi}{M}j}\, z)F_i(z)$$

$$= \frac{1}{M}(\textstyle\sum_{i=0}^{M-1}(H_i(\,z)X(\,z)F_i(z) + \; H_i\left(e^{-\frac{2\pi}{M}j}\, z\right) X\left(e^{-\frac{2\pi}{M}j}\, z\right) F_i(z)$$

$$\cdots + H_i\left(e^{-\frac{2(M-1)\pi}{M}j}\, z\right) X(e^{-\frac{2(M-1)\pi}{M}j}\, z)F_i(z)). \qquad (6)$$

In the M-channel filter bank shown in Fig.(1), the reconstructedsignal is given by [4]

$$\hat{X}(z) = \frac{1}{M}\sum_{i=0}^{M-1} H_i(\,z)X(\,z)F_i(z)$$

$$+ \frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=1}^{M-1} H_i\left(e^{-\frac{2k\pi}{M}j}\, z\right) X(e^{-\frac{2k\pi}{M}j}\, z)F_i(z),$$

where $T_k(\mathrm{z}) = \frac{1}{M}\sum_{k=1}^{M-1} X(e^{-\frac{2k\pi}{M}j}\, z)\sum_{i=0}^{M-1} H_i\left(e^{-\frac{2k\pi}{M}j}\, z\right) F_i(z)\,. \quad (7)$

In order to eliminate the aliasing error and guarantee the passband flat, the PR Subband filter banks should meet the condition as follows $[3-8]$:

$$T_k(\mathrm{z}) = 0 \;\; where\; k = 1,2,3, \dots, M-1 \;, \qquad (7a)$$

and$H_0(\,z)F_0(z) + \cdots + H_{M-1}(\,z)F_{M-1}(z) = z^{-k_d}\,,\; k_d \in N\,. \qquad (7b)$

1.2 Modulation Matrices

The input-output relations of M-channel filter bank may also be written in matrix form. For this, we introduce the vector $X(z)$[8]

$X(z) = \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$, where M is the number of sub channels.

And $H_m(z)$ bandpass analysis filters [6]

$$H(z) = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M) & H_1(zW_M) & \cdots & H_{M-1}(zW_M) \\ \vdots & & & \\ H_0(zW_M^{M-1}) & H_1(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix},$$

$F(z) = [F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)]$.

$X(z)$, $H(z)$ and $F(z)$ would be used in next.

2. Main Results:

Theorem (1): The relation between input and output signal can be: If $x(n) = e^{wnj}$is input signal, and $\hat{x}(n)$ is output signal defined in a way shown in Fig.(1), then the output signal will be defined as follows : $\hat{x}(n) = c\, z^{-d}\, x(n)$.

Proof: Since $x(n) = e^{wnj}$ (8) , $x(n)$is original signal ,and$\hat{x}(n)$ is delayed of the original signal $x(n)$.

Then by using Definition (1.1) and by Equation (8), we have $\hat{x}(n) = c\, e^{w(n-d)j}$

$\therefore \hat{x}(n) = c\, z^{-d}\, x(n)$. ∎

Corollary (1): The original signal defined as following $x(n) = z\hat{x}(n)$, where $q \in R\,, d \in Z$.

Theorem (2): If $X(z)$ is z-transform of $x(n)$ and $\hat{X}(z)$ is z-transform of $\hat{x}(n)$ then:

1) The output signal $\hat{X}(z)$ as define $\hat{X}(z) = c\, z^{-d}\, X(z)$.

2) The input signal $X(z)$ as define$X(z) = q\, z^{d} \hat{X}(z)$.

Proof:

1) By Theorem (1), then$\hat{x}(n) = c\, z^{-d}\, x(n)$

$$\sum_{n=-\infty}^{\infty} \hat{x}(n) = cz^{-d} \sum_{n=-\infty}^{\infty} x(n)$$

$\sum_{n=-\infty}^{\infty} \hat{x}(n)\, z^{-n} = cz^{-d} \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \implies \hat{X}(z) = cz^{-d}X(z\,)$.

2) By proof (1)

$$\hat{X}(z) = cz^{-d}X(z\,) \quad \implies \quad X(z) = qz^{d}\hat{X}(z)\,. \blacksquare$$

Remark (1) If $x(n)$ is input signal, $\hat{x}(n)$ is output signal, then the aliasing in $x(n)$, $\hat{x}(n)$is cz^{-d}, such that z is complex number .

Proof: Let aliasing is $\rho$and$\hat{x}(n) = \rho x(n)$.(9)

By Theorem (1), we have $\hat{x}(n) = cz^{-d}x(n)$. (10)

Form Equation (9) in Equation (10),we obtain

$$\rho x(n) = cz^{-d}x(n) \quad \Rightarrow \rho = cz^{-d}$$

$\therefore$ The aliasing is cz^{-d} .

Theorem (3): If H (z) is analysis filter and $X(z)$ is input signal,then the subsampling by M is $Y(z) = \frac{1}{M} H^{T}(z)X(z)$ if and only if $Y(z) = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} H_i\left(zW_M^k\right)X(zW_M^k)$.

Proof: $(1 \Longrightarrow 2)$Let $Y(z) = \frac{1}{M} H^T(z)X(z)$

$$= \frac{1}{M}\begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M^1) & H_1(zW_M^1) & \cdots & H_{M-1}(zW_M^1) \\ \vdots & & & \\ H_0(zW_M^{M-1}) & H_1(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix}\begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$= \frac{1}{M}\left[\sum_{k=0}^{M-1} H_0\left(zW_M^k\right)X\left(zW_M^k\right) + \sum_{k=0}^{M-1} H_1\left(zW_M^k\right)X\left(zW_M^k\right)\right.$$

$$\left. + \cdots + \sum_{k=0}^{M-1} H_0(zW_M^k)X(zW_M^k)\right]$$

$\therefore Y(z) = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} H_i(zW_M^k)X(zW_M^k)$.

$(2 \Longrightarrow 1)$Let $Y(z) = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} H_i\left(zW_M^k\right)X(zW_M^k)$.

$$= \frac{1}{M}(H_0(z)X(z) + H_0\left(zW_M\right)X\left(zW_M\right) + \cdots + H_0(zW_M^{M-1})$$

$$X(zW_M^{M-1}) + \cdots + H_{M-1}(z)X(z) + H_{M-1}\left(zW_M\right)X\left(zW_M\right)$$

$$+ \cdots + H_{M-1}(zW_M^{M-1})X(zW_M^{M-1}))$$

$$= \frac{1}{M}\begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M) & H_1(zW_M) & & H_{M-1}(zW_M) \\ \vdots & & & \\ H_0(zW_M^{M-1}) & H_1(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix}^T \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$= \frac{1}{M} H^T(z)X(z). \blacksquare$$

Theorem (4): If $H(z)$ is analysis filter, $F(z)$ is synthesis filter and $X(z)$ is input signal,then the output signal is $\hat{X}(z) = \frac{1}{M} F(z)H^T(z)X(z)$ if and only if

$$\hat{X}(z) = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\left(zW_M^k\right)X(zW_M^k)$$

Proof: $(1 \Longrightarrow 2)$.Let $\hat{X}(z) = \frac{1}{M} F(z)H^T(z)X(z)$. Then

$$\hat{X}(z) = \frac{1}{M}[F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)]$$

$$\cdot\begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M) & H_1(zW_M) & \cdots & H_{M-1}(zW_M) \\ \vdots & & & \\ H_0(zW_M^{M-1}) & H_1(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix}\begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$= \left[\sum_{i=0}^{M-1} F_i(z)H_i(z) \quad \sum_{i=0}^{M-1} F_i(z)H_i(zW_M) \quad \cdots \right.$$

$$\left. \sum_{i=0}^{M-1} F_i(z)H_i(zW_M^{M-1})\right] \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)X(z)$$

$$+ \left[\sum_{i=0}^{M-1} F_i(z)H_i(zW_M)X(zW_M) + \cdots + \sum_{i=0}^{M-1} F_i(z)H_i(zW_M^{M-1})X(zW_M^{M-1})\right]$$

$$= \frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1} F_i(z)H_i(zW_M^k)\, X(zW_M^k) \qquad (11)$$

$(2 \Longrightarrow 1)$Let $\hat{X}(z) = \frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1} F_i(z)H_i(zW_M^k)\, X(zW_M^k)$

$$= \frac{1}{M}\left(\sum_{k=0}^{M-1} F_0(z)H_0(zW_M^k)X(zW_M^k) + \sum_{k=0}^{M-1} F_1(z)H_1(zW_M^k)\right.$$

$$\left. X(zW_M^k) + \cdots + \sum_{k=0}^{M-1} F_0(z)H_0(zW_M^k)X(zW_M^k)\right)$$

$$= \frac{1}{M}[F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)] \begin{bmatrix} \sum_{k=0}^{M-1} H_0(zW_M^k)X(zW_M^k) \\ \sum_{k=0}^{M-1} H_1(zW_M^k)X(zW_M^k) \\ \vdots \\ \sum_{k=0}^{M-1} H_{M-1}(zW_M^k)X(zW_M^k) \end{bmatrix}$$

$$= \frac{1}{M}[F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)]$$

$$\begin{bmatrix} H_0(z)X(z) + H_0(zW_M)X(zW_M) + \cdots + H_0(zW_M^{M-1})X(zW_M^{M-1}) \\ H_1(z)X(z) + H_1(zW_M)X(zW_M) + \cdots + H_1(zW_M^{M-1})X(zW_M^{M-1}) \\ \vdots \\ H_{M-1}(z)X(z) + H_{M-1}(zW_M)X(zW_M) + \cdots + H_{M-1}(zW_M^{M-1})X(zW_M^{M-1}) \end{bmatrix}$$

$$= \frac{1}{M}[F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)]$$

$$\begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M) & H_1(zW_M) & \cdots & H_{M-1}(zW_M) \\ \vdots & & & \\ H_0(zW_M^{M-1}) & H_1(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix}^T \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$= \frac{1}{M} F(z) H^T(z) X(z). \blacksquare$$

Theorem (5): If $X(z)$ is input signal and reconstructed without distortions, $H(z)$ is analysis filter and $F(z)$ is synthesis filter, then$\frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z) = cz^{-d}$, such that z is complex number, $\neq 0$.

Proof: Let $X(z)$ is reconstructed without distortions

By Theorem (2) we get$\hat{X}(z) = c\, z^{-d}\, X(z)$, (12)

fromEquation (6) inEquation (12) we get

$$\frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)X(z) + \frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=1}^{M-1} F_i(z)H_i\left(zW_M^k\right) X\left(zW_M^k\right) = c\, z^{-d}\, X(z)$$

$$\therefore\ \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z) = c\, z^{-d}. \blacksquare$$

Theorem (6): If $H(z)$ is analysis filter, $F(z)$ is synthesis filter, z is complex number and $\neq 0$, then $c\, z^{-d} = \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)$ if and only if

$c\, z^{-d}X(z) = F(z)Y(z) - \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\left(zW_M^k\right) X(z\, W_M^k)$.

Proof: Let $z^{-d} = \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)$,

$$c\, z^{-d}X(z) = \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)X(z)$$

$$= (\frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)X(z) + \frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1} F_i(z)H_i\left(zW_M^k\right)X\left(zW_M^k\right))$$

$$-\frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1} F_i(z)H_i\left(zW_M^k\right)X\left(zW_M^k\right).$$

By Theorem (3) we get

$c\, z^{-d} X(z) = F(z)Y(z) - \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\big(zW_M^k\big)\, X(z\, W_M^k)$.

Now, let $c\, z^{-d} X(z) = F(z)Y(z) - \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\big(zW_M^k\big)\, X(z\, W_M^k)$and

let$\lambda = F(z)Y(z)$

By Theorem (3) we get

$\lambda = \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)X(z) + \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\big(zW_M^k\big)\, X(z\, W_M^k)$.

Now

$$c\, z^{-d} X(z) = \lambda - \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\big(zW_M^k\big)\, X(z\, W_M^k)$$

$\therefore\ c\, z^{-d} = \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)$.■

Conclusion:
In this paper we find a relation between input signal and output signal and from that we can find the aliasing part in the system.

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