Exponentially fitted Diagonally Implicit three-stage fifth-order RK Method for Solving ODEs.

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ABSTRACT

The EDITRK5 method. which this paper derives. is an exponentially fitted diagonally implicit RK method for solving ODEs with the equation $y'''(x) = f(x.y)$. With the help of the set functions e^{wx} and e^{-wx} for exponentially fitting problems. this strategy is designed to integrate precise initial value problems (IVPs). The primary frequency of the issue. $w \in R$ is used to increase the method's accuracy. The new approach For the purpose of solving IVPs using exponential functions as solutions. EDITRK5 is a novel three-stage five-order exponentially-fitted diagonally implicit method. When the same issue is reduced to the first-order framework of equations. which can be solved using traditional RK approaches. different forms of third-order ODEs must be constructed using the new system. and numerical comparisons must be made. The numerical results demonstrate that the new strategy is more effective than methods that have already been published.

1-Introduction:

If the continuous vector-valued operation $v \in R^d . f: R \times R^d \rightarrow R^d$. does not directly depend on the second derivatives. This kind of difficulty arises in a variety of physical problems. including thin-film flow. gravity-driven flows. and others [1. 2. 3]. Then. in recent years. several researchers have developed explicit RK techniques for solving 1^{st}-order and 2^{nd}-order ODEs that are fitted exponentially and trigonometrically. Implicit approaches are crucial because they can achieve better levels of accuracy for the same stage number as explicit ones. This makes it simpler to find solutions to difficult issues. On the other hand. implicit (RK) techniques are significant in other problem classes. like solving differential-algebraic equations. Diagonally implicit RK techniques are also often referred to as semi-implicit or semi-explicit RK approaches since they feature a lower triangular A-matrix with at least one non-zero diagonal element. Paternoster [4] created (RKN) methods for trigonometric polynomial periodic solutions to ODEs. Vanden Berghe et al. [5] created exponentially modified RK algorithms. While Simos [6] offers an enhanced RK technique that addresses issues with the Schrödinger equation. Most scientists. engineers. and researchers used to solve (1) by breaking down a three-dimensional system of 1^{st}-order equations into a set of 3^{rd}-order differential equations. It is more effective. nevertheless. to tackle the issue directly using numerical techniques. Examples of this kind of work may be found in [7. 8]. and [9]. In [10]. two explicit two-derivative RKN approaches are built. one with exponential fitting and the other with trigonometric fitting. Demba et al. [11] then devised an explicit trigonometrically fitted RKN approach utilizing the Simos methodology. Additionally. [12. 13] show how certain 3^{rd}- and 4^{th}-order ODEs may be solved using the direct technique.

This study's main objective is to demonstrate how to solve exceptional 3^rd ODEs using a fitted exponential-diagonally implicit RK technique. The algebraic order of the method must also be taken into account while solving (1) numerically since it is crucial to obtaining good accuracy. Section 3 presents the necessary requirements and derivation for exponentially fitted RK-type techniques for resolving 3^{rd}-order ODEs. The efficiency of the new approach is contrasted with that of earlier methods in Section 4.

To address the problems of IVPs (1). the general structure of the EDITRK5 technique with a score of m-stage:

$$v_{n+1} = v_n + \mathrm{h}v_n' + \frac{h^2}{2} v_n'' + h^3 \sum_{i=1}^{m} b_i\, k_i\,. \tag{2}$$
$$v_{n+1}' = v_n' + \mathrm{h}v_n'' + h^2 \sum_{i=1}^{m} b_i'\, k_i\,. \tag{3}$$
$$v_{n+1}'' = v_n'' + \mathrm{h} \sum_{i=1}^{m} b_i''\, k_i\,. \tag{4}$$

Where

$$k_i = f\left(x_n + c_i h.\, v_n + c_i h v_n' + \frac{h^2}{2}\, c_i^2\, v_n'' + h^3\, \sum_{j=1}^{i-1} a_{ij}\, k_j\right) \tag{5}$$

for *i=2.3.….*n

The parameters of diagonal implicit RK type (EDITRK5) strategies are $b_i.b_i'.b_i''.a_{i.j}$ and c_i of where $i = 2.3.\ldots.m.$ are real integers and n is the method's level digit. When $a_{i.j} \neq 0$ for $i \leq j$. techniques. which shows that the decrease the triangle diagonal matric of A has the equal values as $a_{i.j} \neq 0$ wherein $i = j$ at the diagonal

TABLE I. Butcher form EDITRK method:

c_1	a_{11}		
c_2	a_{21}	a_{22}	
c_3	a_{31}	a_{32}	a_{33}
	b_1	b_2	b_3

b'_1	b'_2	b'_3
b''_1	b''_2	b''_3

To obtain the parameters of the new method provided by (2)-(5). the EDITRK5 method expression is expanded using Taylor's series expansion. After a few algebraic modifications. this expansion is equivalent to the real answer that Taylor's series expansion yields. The direct extension of the local truncation error was used to create the general order criterion for the new approach. This idea is based on how the order requirements for the RK approach were derived in [8. 14. 15]

The new EDITRK5 technique is written as follows:

$$y_{n+1} = y_n + h\Phi(x_n . y_n).$$
$$y'_{n+1} = y'_n + h\Phi'(x_n . y_n).$$
$$y''_{n+1} = y''_n + h\Phi''(x_n . y_n). \qquad (6)$$

where the functions for increment are

$$\Phi(x_n . y_n) = y'_n + \frac{h}{2} y''_n + h^2 \sum_{i=1}^{m} b_i\, k_i.$$

$$\Phi'(x_n . y_n) = y''_n + h \sum_{i=1}^{m} b'_i\, k_i.$$

$$\Phi''(x_n . y_n) = \sum_{i=1}^{m} b''_i\, k_i. \qquad (7)$$

In which k_i is given in (5). If we suppose that the Taylor series increment characteristic is $\Delta. \Delta'$and Δ''. Thus. with the aid of inserting the exact solution of (1) into (7). the nearby truncation errors of $y(x). y'(x) and\ y''(x)$may be received:

$$\tau_{n+1} = h[\Phi - \Delta].$$
$$\tau'_{n+1} = h[\Phi' - \Delta'].$$
$$\tau''_{n+1} = h[\Phi'' - \Delta'']. \qquad (8)$$

In the phrases of basic differentials. those expressions are satisfactory given and the Taylor series can be expressed as follows:

$$\Delta = y' + \frac{1}{2}hy'' + \frac{1}{6} h^2 F_1^{(3)} + \frac{1}{24} h^3 F_1^{(4)} + O(h^4).$$

$$\Delta' = y'' + \frac{1}{2}\,h\,F_1^{(3)} + \frac{1}{6}\,h^2 F_1^{(4)} + \frac{1}{24}\,h^3 F_1^{(5)} + O(h^4).$$

$$\Delta'' = F_1^{(3)} + \frac{1}{2}\,hF_1^{(4)} + \frac{1}{6}\,h^2 F_1^{(5)} + O(h^3). \tag{9}$$

For the scalar case. the primary few simple differentials are

$$F_1^{(3)} = f.$$

$$F_1^{(4)} = f_x + f_y y'.$$

$$F_1^{(5)} = f_{xx} + 2f_{xy}y' + f_{xy'}y_{xx} + f_y y'' + f_{yy}(y')^2. \tag{10}$$

substituting (10) into (7). for the new method. the increment functions will become Φ. Φ' and Φ''will become

$$\sum_{i=1}^{m} b_i k_i = \sum_{i=1}^{m} b_i\, f + \sum_{i=1}^{m} b_i c_i \left(f_x + f_y y'\right) h + \tfrac{1}{2}\sum_{i=1}^{m} b_i\, c_i^2 \left(f_{xx} + 2f_{xy}y' + f_{xy'}y_{xx} + f_y y'' + f_{yy}(y')^2\right) h^2 + O(h^3).$$

$$\sum_{i=1}^{m} b_i' k_i = \sum_{i=1}^{m} b_i'\, f + \sum_{i=1}^{m} b_i' c_i \left(f_x + f_y y'\right) h + \tfrac{1}{2}\sum_{i=1}^{m} b_i'\, c_i^2 \left(f_{xx} + 2f_{xy}y' + f_{xy'}y_{xx} + f_y y'' + f_{yy}(y')^2\right) h^2 + O(h^3).$$

$$\sum_{i=1}^{m} b_i'' k_i = \sum_{i=1}^{m} b_i''\, f + \sum_{i=1}^{m} b_i'' c_i \left(f_x + f_y y'\right) h + \tfrac{1}{2}\sum_{i=1}^{m} b_i''\, c_i^2 \left(f_{xx} + 2f_{xy}y' + f_{xy'}y_{xx} + f_y y'' + f_{yy}(y')^2\right) h^2 + O(h^3)\,.$$

From (9) and (11) the local truncation error (8) can be expressed as follows:

$$\tau_{n+1} = h^3 \left[\sum_{i=1}^{m} b_i k_i - \left(\frac{1}{6}F_1^{(3)} + \frac{1}{24}hF_1^{(4)} + \cdots\right)\right].$$

$$\tau_{n+1}' = h^2 \left[\sum_{i=1}^{m} b_i' k_i - \left(\frac{1}{2}F_1^{(3)} + \frac{1}{6}hF_1^{(4)} + \cdots\right)\right].$$

$$\tau_{n+1}'' = h\,\left[\sum_{i=1}^{m} b_i'' k_i - \left(F_1^{(3)} + \tfrac{1}{2}hF_1^{(4)} + \tfrac{1}{6}h^2 F_1^{(5)} \dots\right)\right]. \tag{12}$$

By changing (11) into (12) and expanding as a Taylor expansion with the help of the Maple package. the local truncation mistakes or the order conditions for m-stage up to order six for the new approach may be resolved (see [14] and [12]).

2. EXPONENTIALLY FITTED EDITRK METHOD

Definition 1. 1: The functions e^{w} and e^{-w}must integrate precisely at each stage in order to build the exponentially fitted RK type three-stage 4^{th}-order approach; as a result. the following equations are obtained for $y.y'$. and y".

$$e^{\pm v} = 1 \pm v + \frac{1}{2}\, v^2 \pm v^3 \sum_{i=1}^{m} b_i\, e^{\pm c_i v} \tag{13}$$

$$\mathrm{e}^{\pm \mathrm{v}} = 1 \pm \mathrm{v} + \ \mathrm{v}^2 \sum_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{b}_{\mathrm{i}}'\, \mathrm{e}^{\pm \mathrm{c}_{\mathrm{i}} \mathrm{v}} \tag{14}$$

$$e^{\pm v} = 1 \pm v \textstyle\sum_{i=1}^{m} b_i''\, e^{\pm c_i v} \tag{15}$$

Where $v = wh.\, w \in R$. The relations $\cosh(v) = \frac{e^{v}+e^{-v}}{2}$ and $\sinh(v) = \frac{e^{v}-e^{-v}}{2}$ will be used in the derivation process. The following equations corresponding $y.y'$ and y'' are

$$\cosh(v) = 1 + \frac{1}{2}\, v^2 + v^3 \sum_{i=1}^{m} b_i \sin h(vc_i) \tag{16}$$

$$sin\, h(v) = v + v^3 \sum_{i=1}^{m} b_i\, cosh(vc_i) \tag{17}$$

$$cosh(v) = 1 + v^2 \sum_{i=1}^{m} b_i'\, cosh(vc_i) \tag{18}$$

$$\sinh(v) = v + v^2 \sum_{i=1}^{m} b_i' \sinh(vc_i) \tag{19}$$

$$\cosh(v) = 1 + v \sum_{i=1}^{m} b_i'' \sinh(vc_i) \tag{20}$$

$$sinh(v) = v \sum_{i=1}^{m} b_i''\, cosh(vc_i) \tag{21}$$

The following three-stage. fifth-order diagonally implicit technique was developed in [12].

$$c_1 = \frac{1}{10}\,.c_2 = \frac{1}{2}.c_3 = \frac{4}{5}\ \ .a_{11} = \frac{9}{1000}.a_{21} = 0.a_{22} = \frac{9}{1000}.a_{31} = \frac{1}{10}..a_{32} = 0.a_{33} = \frac{9}{1000}.b_1 = \frac{1}{10}\,.b_2 = \frac{1}{18}\,.b_3 = \frac{1}{1000}\ \ .b_1' = \frac{1}{5}\,.b_2' = \frac{2}{9}\,.b_3' = \frac{3}{100}\,.b_1'' = \frac{5}{18}\,.b_2'' = \frac{4}{9}.b_3'' = \frac{5}{18}.$$

Next. we solve (16 – (21)) and use of the coefficients listed above to find $b_1 . b_2 . b_1' . b_2' . b_1''$ and b_2''

$$b_1 = \frac{1}{1000} . \frac{(\cosh\left(\frac{1}{2}V\right)\sinh\left(\frac{4}{5}V\right) - \cosh\left(\frac{4}{5}V\right)\sinh(\frac{1}{2}V)}{v(\cosh(\frac{1}{10}V)\sinh\left(\frac{1}{2}V\right) - \cosh(\frac{1}{2}V)\sinh(\frac{1}{10}V)} + \frac{1}{2}\frac{\cosh\left(\frac{1}{2}V\right)V^2 + 2\sinh(V)\sinh\left(\frac{1}{2}V\right) - 2\cosh(\frac{1}{2}V)\cosh(V) - 2\sinh\left(\frac{1}{2}V\right)V + 2\cosh(\frac{1}{2}V)}{V^3(\cosh(\frac{1}{10}V)\sinh\left(\frac{1}{2}V\right) - \cosh(\frac{1}{2}V)\sinh(\frac{1}{10}))}$$

$$b_2 = -\frac{1}{1000} . \frac{(\cosh\left(\frac{1}{2}V\right)\sinh\left(\frac{4}{5}V\right) - \cosh\left(\frac{4}{5}V\right)\sinh\left(\frac{1}{2}V\right)}{(\cosh\left(\frac{1}{10}V\right)\sinh\left(\frac{1}{2}V\right) - \cosh\left(\frac{1}{2}V\right)\sinh\left(\frac{1}{10}V\right)} - \frac{1}{2}\frac{V^2\cosh\left(\frac{1}{10}V\right) + 2\sinh(\frac{1}{10}V)\sinh(V) - 2\cosh(V)\cosh\left(\frac{1}{10}V\right) - 2\sinh\left(\frac{1}{10}V\right)V + 2\cosh(\frac{1}{10}V)}{V^3(\cosh(\frac{1}{10}V)\sinh\left(\frac{1}{2}V\right) - \cosh(\frac{1}{2}V)\sinh(\frac{1}{10}))}$$

$$b_1' = \frac{5}{18} . \frac{(\cosh\left(\frac{1}{2}V\right)\sinh\left(\frac{4}{5}V\right)V - \frac{5}{18}\cosh\left(\frac{4}{5}V\right)\sinh\left(\frac{1}{2}V\right)V - cosh\left(\frac{1}{2}V\right)cosh(V) + sinh(V)\, sinh\left(\frac{1}{2}V\right) + cosh(\frac{1}{2}V))}{v(\cosh(\frac{1}{10}V)\sinh\left(\frac{1}{2}V\right) - \cosh(\frac{1}{2}V)\sinh(\frac{1}{10}V))}$$

$$b_2' = -\frac{5}{18} . \frac{(\cosh\left(\frac{1}{10}V\right)\sinh\left(\frac{4}{5}V\right)V - \frac{5}{18}\cosh\left(\frac{4}{5}V\right)\sinh\left(\frac{1}{102}V\right)V - cosh\left(\frac{1}{10}V\right)cosh(V) + sinh(V)\, sinh\left(\frac{1}{10}V\right) + cosh(\cdots}{v(\cosh(\frac{1}{10}V)\sinh\left(\frac{1}{2}V\right) - \cosh(\frac{1}{2}V)\sinh(\frac{1}{10}V))}$$

$$b_1'' = \frac{3}{100} . \frac{(\cosh\left(\frac{1}{2}V\right)\sinh\left(\frac{4}{5}V\right) - \cosh\left(\frac{4}{5}V\right)\sinh(\frac{1}{2}V)}{(\cosh(\frac{1}{10}V)\sinh\left(\frac{1}{2}V\right) - \cosh(\frac{1}{2}V)\sinh(\frac{1}{10}V)} - \frac{\cosh\left(\frac{1}{2}V\right)\sinh(V) - \cosh\left(\frac{1}{2}V\right)V - \sinh(\frac{1}{2}V)\cosh(V) + \sinh(\frac{1}{2}V)}{V^2(\cosh(\frac{1}{10}V)\sinh\left(\frac{1}{2}V\right) - \cosh(\frac{1}{2}V)\sinh(\frac{1}{10}))}$$

$$b_2'' = -\frac{3}{100}.\frac{(\cosh\left(\frac{1}{10}V\right)\sinh\left(\frac{4}{5}V\right) - \cosh\left(\frac{4}{5}V\right)\sinh\left(\frac{1}{10}V\right)}{(\cosh\left(\frac{1}{10}V\right)\sinh\left(\frac{1}{2}V\right) - \cosh\left(\frac{1}{2}V\right)\sinh\left(\frac{1}{10}V\right)} + \frac{\cosh\left(\frac{1}{10}V\right)\sinh(V) - \cosh\left(\frac{1}{10}V\right)V - \sinh(\frac{1}{10}V)\cosh(V) + \sinh(\frac{1}{10}V)}{V^2(\cosh(\frac{1}{10}V)\sinh\left(\frac{1}{2}V\right) - \cosh(\frac{1}{2}V)\sinh(\frac{1}{10}))}$$

EDITRK5. a three-stage fifth-order diagonally implicit exponentially fitted RK type technique. was created as a consequence. The corresponding Taylor series extension of the answer is provided by

$$b_1 = \frac{1259}{12000} - \frac{1313}{7200000}V^2 + \frac{349747}{60480000000}V^4 - \frac{5613577}{60480000000000}V^6 + \frac{7214336873}{479001600000000000}V^8 - \frac{31933381356353}{1307674368000000000000000}V^{10}$$

$$b_2 = \frac{243}{4000} + \frac{557}{7200000}V^2 + \frac{493793}{60480000000}V^4 - \frac{15282961}{181440000000000}V^6 + \frac{7318459747}{479001600000000000}V^8 - \frac{31889153986003}{1307674368000000000000000}V^{10}$$

$$b_1' = \frac{277}{1200} - \frac{439}{720000}V^2 + \frac{90941}{6048000000}V^4 - \frac{1451231}{6048000000000}V^6 + \frac{1861044919}{47900160000000000}V^8 - \frac{8233503141559}{1307674368000000000000000}V^{10}$$

$$b_2' = \frac{287}{1200} + \frac{1171}{720000}V^2 + \frac{228679}{6048000000}V^4 - \frac{6414611}{6048000000000}V^6 + \frac{2070291941}{47900160000000000}V^8 - \frac{8110978170509}{1307674368000000000000000}V^{10}$$

$$b_1'' = \frac{5}{24} - \frac{7}{2880}V^2 + \frac{119}{3456000}V^4 - \frac{5587}{10368000000}V^6 + \frac{71837}{8294400000000}V^8 - \frac{10459081}{74649600000000000}V^{10} + \frac{15658809599}{6897623040000000000000}V^{12}$$

$$b_2'' = \frac{37}{72} + \frac{43}{2880}V^2 + \frac{1261}{3456000}V^4 + \frac{145921}{72576000000}V^6 + \frac{1072601}{58060800000000}V^8 - \frac{668809087}{5748019200000000000}V^{10} + \frac{20695460803}{6897623040000000000000}V^{12}$$

3. NUMERICAL EXPERIMENTS

The concepts discussed in this section were tested against five different problems. The numerical outcomes of the suggested approaches are compared to those of existing implicit RK algorithms of the same order. The following equipment was used to conduct the numerical experiments:

Problem 1: (Non-homogeneous Linear Problem)

$$v'''(t) = v(t) + cos(v). \quad v(0) = 0. \quad v'(0) = 0. \ v''(0) = 1$$

Theoretical solution :

$$v(t) = (e^t - \cos(t) - \sin(t)) .$$

Problem 2: (Non-homogeneous Nonlinear Problem)

$$v'''(t) = (v(t))^2 + cos^2(v) - cos(t) - 1.$$
$$v(0) = 0. v'(0) = 1. v''(0) = 1.$$

Theoretical solution:

$$v(t) = sin(t).$$

Problem 3: (Non-homogeneous Nonlinear Problem)

$$v'''(\mathrm{t}) = 8\left(\frac{v^2(t)}{e^{2t}}\right) .$$

$$v(0) = 1 . \quad v'(0) = 2. \quad v''(0) = 4 .$$

Theoretical solution :

$$v(t) = e^{2t}$$

Problem 4: (linear System)

$$y_1'''(\mathrm{t}) = y_2(\mathrm{t}).$$
$$y_2'''(\mathrm{t}) = -y_1(t) - 2y_2(t) + 2y_3(t).$$
$$y_3'''(\mathrm{t}) = y_1(t) + y_2(t)$$

The exact solution is given by :

$$y_1(t) = cosh(t).$$
$$y_2(t) = sinh(t).$$
$$y_3(t) = e^t.$$

Problem 5: (linear System)

$$y_1'''(\mathrm{t}) = y_2(\mathrm{t}).$$

$$y_2'''(\mathrm{t}) = y_1(\mathrm{t}).$$

$$y_3'''(\mathrm{t}) = y_1(t) + y_2(t) - \sinh(t)$$

$$y_1(0) = 1\,.\,y_1'(0) = 0\,.\,y_1''(0) = 1\,.$$

$$y_2(0) = 0\,.\,y_2'(0) = 1\,.\,y_2''(0) = 0\,.$$

$$y_3(0) = 1\,.\,y_3'(0) = 0\,.\,y_3''(0) = 1\,.$$

The exact solution is given by

$$y_1(t) = \cosh(t).$$

$$y_2(t) = \sinh(t).$$

$$y_3(t) = e^t + 1 - \cosh(t) + \frac{t^2}{2} - t.$$

The decimal logarithm of the largest global error and the logarithm of function evaluations are displayed in Figures 1–5. respectively. to demonstrate the effectiveness of the EDITRK5 techniques. The EDITRK5 technique needs fewer function evaluations than other implicit RK methods of the same order. This is due to the fact that when the issues were transformed into a system of 1^{st}-order ODEs. the number of equations quadrupled. The EDITRK5 approaches. as shown in Figures 1–5. have the smallest maximum global error and the fewest number of function evaluations each step when compared to other implicit RK methods of the same order. Figures 1–5 demonstrate that the EDITRK5 delivers results that are more accurate than those from other studies in the literature (Radau IA. Radau II). In this study. the decimal logarithm of the largest global mistake for 5 test problems is used to calculate the logarithm of function evaluations with various step sizes. $h\ =\ 0.1.\,0.05.\,0.025.\,0.00125.$ and 0.00625 respectively.

FIGURE 1. Accuracy curve for EDITRK5. Radau IA. Radau II. with $h\ =\ 0.1.\,0.05\ 0.025.\,0.00125.\,0.00625$ for the problem 1.

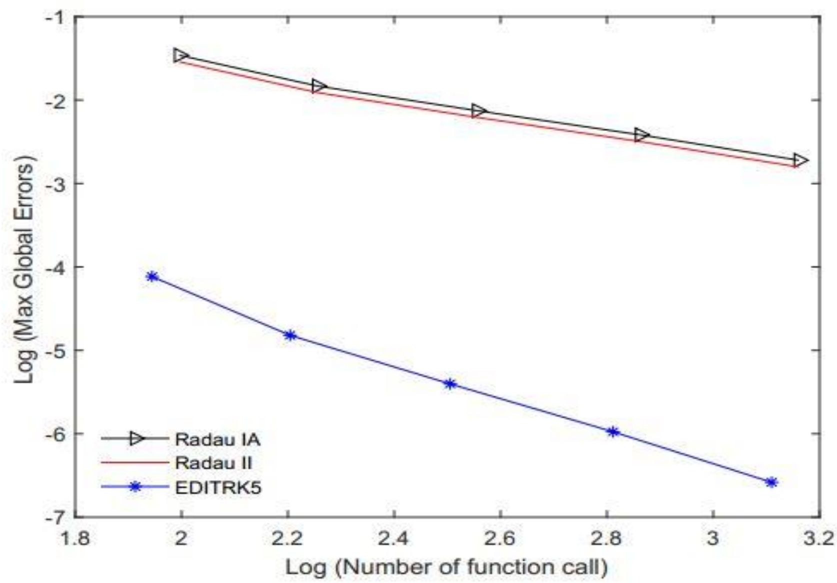


FIGURE 2. Accuracy curve EDITRK5. Radau IA. Radau II. with h = 0.1. 0.05 0.025. 0.00125. 0.00625 for the problem 2.

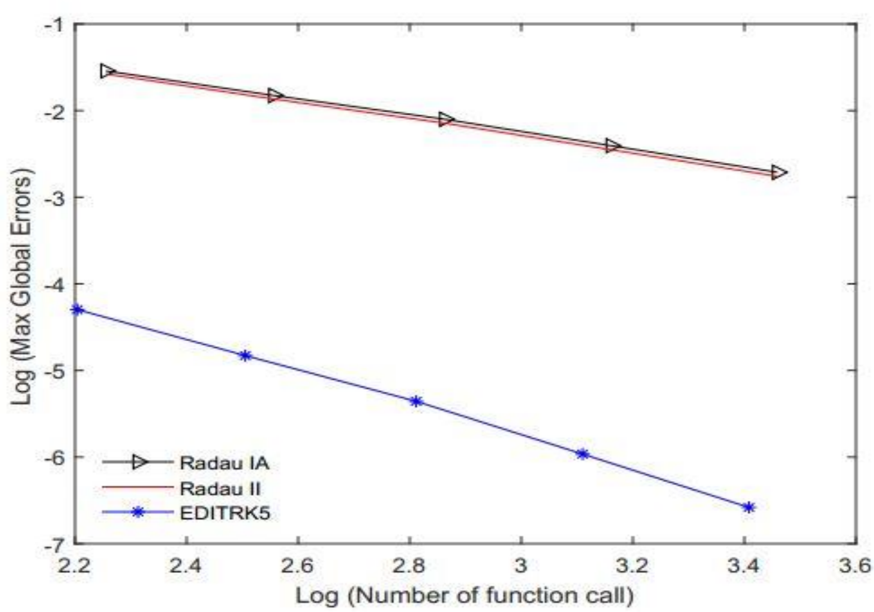


FIGURE 3. Accuracy curve for EDITRK5. Radau IA. Radau II. with h = 0.1. 0.05 0.025. 0.00125. 0.00625 for the problem 3.

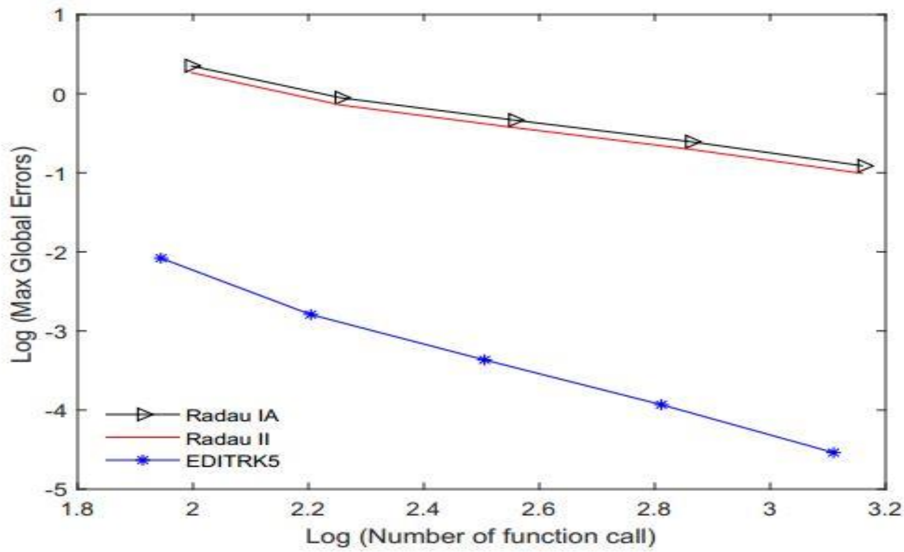


FIGURE 4. Accuracy curve for EDITRK5. Radau IA. Radau II. with h = 0.1. 0.05 0.025. 0.00125. 0.00625 for the problem 4.

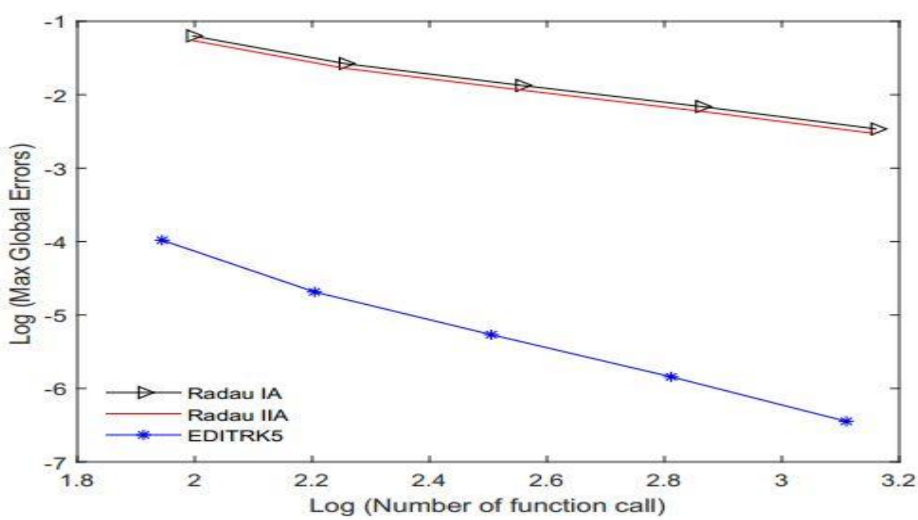
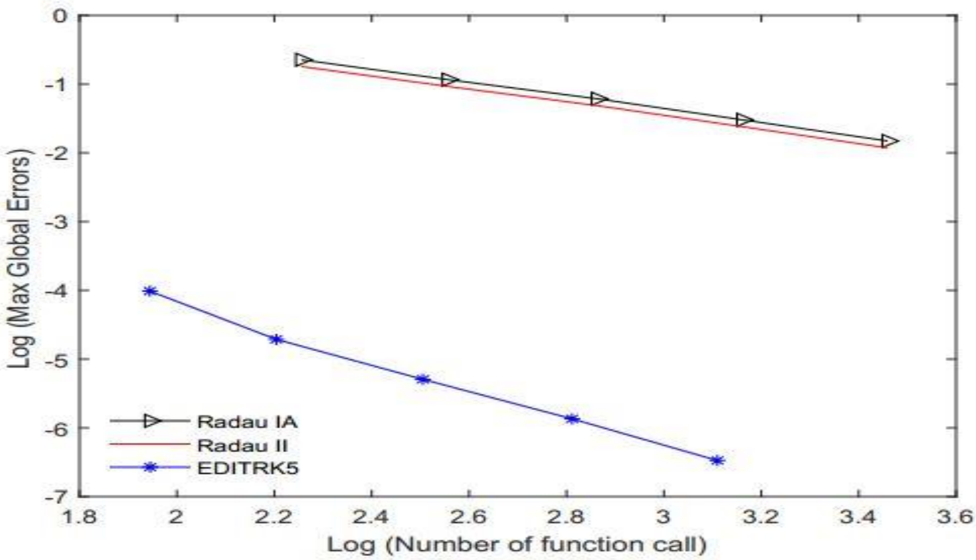


FIGURE 5. Accuracy curve for EDITRK5. Radau IA. Radau II. with h = 0.1. 0.05 0.025. 0.00125. 0.00625 for the problem 5.



4. CONCLUSION

In this research. we addressed the $y'''(x) = f(x.y)$ problem using an exponentially fitted diagonally implicit RK type approach. The EDITRK5 method. a three-stage fifth-order exponentially-fitted diagonally implicit method based on calculating the maximum error in the solution $(max(|\, y(tn) - yn|))$. which is equal to the maximum difference between absolute errors of actual and computed solutions. was created as a result and used in the numerical comparison of criteria. Figures 1–5 display the numerical outcomes. Additionally. compared to the (Radau IA. Radau II) methods. the EDITRK5 technique needs less capacity evaluations. The figures demonstrate how the common logarithm of the greatest global error during integration and computation cost was determined using the number of function evaluations. The numerical results made it abundantly evident that the unique exponentially fitted technique RK type approach has a smaller global error than the other existing approaches for a brief time of integration. When solving 3^{rd}-order ODEs of the kind $y''' = f(x.y)$ directly. the ground-breaking EDITRK5 methodology is substantially more effective than the other current methods.

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