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The Relationship Between Boronological Convergence of Net and Topological Convergence of Net

By

Fatma Kamil Majeed Al-Basri

University of Al-Qadisiyah

College Of Education

Department of Mathematics

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Abstract: In this paper, we study the convergence of nets in convex bornological vector space and the related concepts and investigate some properties of them. We study the relationship between the convergence net in bornological space and topological space in which we explain the convergence net in bornological space be convergent in topological space. After that we apply the conditions on the contrary to be valid through the properties which explain that.

Keyword: Boronological Convergence, Boronological Converge Net

Mathematics Subject Classification: 54XX

Introduction:-

The study of bornological spaces was initiated by Hogbe-Neland [5]. Many workers such as Dierolf and Domanski [3], Jan Haluskaand others have studied various bornological properties Patwardhan [6] has successfully studied bornological properties of the spaces of entire functions in terms of the coefficients of Taylor series expansions. In a bornological vector space E one has natural notion of convergence which depends only on the bornology ***ß***. In many applications one uses convex bornological vector spaces (*cbvs*) . After that we studied the concept of converge net in convex bornological space, we discussed in the last section of this study the relationship between converge net in bornological space and converge net in topological space. We have added the conditions in order to make every convergence net in topological space be convergent in bornological space.

2.Preliminary notations:-

In the section, certain definitions and properties concerning convergence net in convex bornological vector space.

Definition 2.1[7]

A net $(x_\gamma)_{\lambda\in\Gamma}$ in a set X is a function $q:\Gamma\to X$ where Γ is some ordered set. The point q(λ) is usually denoted x_λ.

Definition 2.2[5]

Let A and B be two subsets of a vector space E. We say that

i. A is circled if $\lambda A\subset A$ whenever $\lambda\in K$ and $|\lambda|\leq 1$.

ii. A is convex if $\lambda A+\mu A\subset A$ whenever λ and μ are positive real numbers such that $\lambda+\mu=1$;

iii. A is disked, or a disk, if A is both convex and circled;

iv. A absorbs B if there exists $\alpha\in R$, $\alpha>0$, such that $\lambda A\subset B$ whenever $\alpha\leq|\lambda|$;

v. A is an absorbent in E if A absorbs every subset of E consisting of a single point.

Remark 2.3[5]

(i) If A and B are convex and $\lambda,\mu\in K$, then $\lambda A+\mu B$ is convex.

(ii) Every intersection of circled (resp. convex, disked) sets is circled (resp. convex, disked).

(iii) Let E and F be vector spaces and let $u:E\to F$ be a linear map,then the image, direct or inverse, under u of a circled (resp. convex,disked) subset is circled (resp. convex, disked).

Remark 2.4[7]

A subset A is bounded if it is absorbed by any neighborhood of the origin.

Proposition 2.5[5]

If X is an arbitrary topological vector space, the following assertions are true:

(i) Any finite subset of X is bounded;

(ii) If A is bounded and $B\subset A$, then B is also bounded;

(iii) If A and B are bounded, then $A+B$ is bounded;

(iv) If A and B are bounded, then so is $A \cup B$;

(v) If A is bounded, then λA is bounded ($\forall\lambda\in K$);

(vi) The closure of a bounded set is bounded set;

(vii) The circled hull of a bounded set is bounded.

Definition2.6[7]

A metrizable space is a topological space X with the property that there exists at least one metric on the set X whose class of generated open sets is precisely the given topology.

Definition 2.7[4]

A bornology on a set X is a family ***ß*** of subsets of X satisfying the following axioms:

(i) ***ß*** is a covering of X, this mean $X = \bigcup_{B\in\beta} B$;

(ii) ***ß*** is hereditary under inclusion this mean if $A\in$ ***ß*** and B is a subset of X contained in A, then $B\in$ ***ß***;

(iii) ***ß*** is stable under finite union.

A pair (X , ***ß***) consisting of a set X and a bornology ***ß*** on X is called a bornological space, and the elements of ***ß*** are called the bounded subsets of X .

Example 2.8[5]

Let R be a field with the absolute value,the collection:

ß = {$A\subseteq$R: A is bounded subset of R in the usual sense for the absolute value},then ***ß*** is a bornology on R called the Canonical Bornology of R.

Example 2.9[5]

The Von–Neumann bornology of a topological vector space, Let E be a topological vector space,the collection ß ={A$\subseteq$E: A is a bounded subset of a topological vector space E}forms a vector bornology on E called the Von–Neumann bornology of E. Let us verify that ß is indeed a vector bornology on E, if $ß_0$ is a base of circled neighborhoods of zero in E, it is clear that a subset A of E is bounded if and only if for every B $\in ß_0$ there exists λ>0 such that A $\subseteq \lambda$ B. Since every neighborhood of zero is absorbent, ß is a covering of E. ß is obviously hereditary and we shall show that its also stable under vector addition. Let A_1, $A_2 \in$ ß and $B_0 \in ß_0$; there exists $B_0^{/}$ such that $B_0^{/} + B_0^{/} \subseteq B_0$ Since A_1 and A_2 are bounded in E, there exists positive scalars λ and μ such that $A_1 \subseteq \lambda B_0^{/}$ and $A_2 \subseteq \mu B_0^{/}$, set α= max(λ, μ), we have $A_1+A_2 \subseteq \lambda B_0^{/} + \mu B_0^{/} \subset \alpha B_0^{/} + \alpha B_0^{/} \subset \alpha(B_0^{/} + B_0^{/}) \subset \alpha B_0$.

Finally, since $ß_0$ is stable under the formation of circled hulls (resp. under homothetic transformations). Then so is ß, and we conclude that ß is a vector bornology on E. If E is locally convex, then clearly ß is a convex bornology. Moreover, since every topological vector space has a base of closed neighborhoods of 0, the closure of each bounded subset of E is again bounded.

Definition 2.10[5]

Let E be a vector space over the field K (the real or complex field). A bornology ***ß*** on E is said to be a bornology compatible with a vector space structure of E or to be a vector bornology on E, if ***ß*** is stable under vector addition, homothetic transformations and the formation of circled hulls, in other words, if the sets A+B, λA and $\bigcup_{|\alpha|\leq 1} \alpha A$ belongs to ***ß*** whenever A and B belong to ***ß*** and $\lambda \in K$.

Definition 2.11[2]

A convex bornological space is a bornological vector space for which the disked hull of every bounded set is bounded this mean it is stable under the formation of disked hull.

Definition 2.12[5]

A separated bornological vector space (E, ***ß***) is one where {0} is the only bounded vector subspace of E.

Proposition 2.13[5]

Let E be a convex bornological vector space and let (x_n) be a sequence in E , there exist a bounded disk $B \subset E$ such that (x_n) is contained in semi normed space E_B and converges to 0 in E_B.

Definition 2.14[6]

Let $(X_i, ß_i)_{i\in I}$ be a family of bornological vector space indexed by a non-empty set I and let $X = \prod_{i\in I} X_i$ be the product of the sets X_i . For every i∈I, let $P_i : X \to X_i$ be the canonical projection then. The product bornology on X is the initial bornology on X for the maps P_i. The set X endowed with the product bornology is called the bornological product of the space $(X_i, ß_i)$.

Definition 2.15[1]

Let X and Y be two bornological spaces and u: $X \to$ Y is a map of X into Y. We say that u is a bounded map if the image under u of every bounded subset of X is bounded in Y i.e. $u(A) \in \mathfrak{B}_y$, $\forall A \in \mathfrak{B}_x$.

1. Bornological convergence net

In every bornological vector space, a notation of convergence of nets can be introduced depending only upon the convex bornology vector space (*cbvs*), and the main results are of considerable interest in many situations.

Definition 3.1[4]

Let (x_γ) be a net in a *cbvs* E. We say that (x_γ) converges bornologically to 0 ($(x_\gamma) \to 0$) if there exists a bounded and absolutely convex set $B \subset E$ and a net (λ_γ) in K converging to 0, such that $x_\gamma \in \lambda_\gamma B$, for every $\gamma \in \Gamma$. Then we say that a net (x_γ) converges bornologically to a point $x \in E$ and $x_\gamma \to x$ when $(x_\gamma - x) \to 0$.

Remark 3.2

Let E be a *cbvs*. Anet (x_γ) in E is bornologically convergent to a point $x \in E$ if there exists a decreasing net (λ_γ) of positive real numbers tending to zero such that the net $(\frac{x_\gamma - x}{\lambda_\gamma})$ is bounded.

Proposition 3.3

Every bornologically convergent net is bounded.

Proof

Let E be a *cbvs* and a net (x_γ) converges bornologically to a point $x \in E$ this mean $(x_\gamma - x) \longrightarrow 0$, there exists a bounded and absolutely convex subset B of E and a net (λ_γ) of scalars tend to 0, such that $x_\gamma \in B$ and $(x_\gamma - x) \in \lambda_\gamma B$ for every $\gamma \in \Gamma$. Since (λ_γ) is a net of scalars tend to 0, and E is a convex bornological vector space then $\lambda_\gamma B$ is a bounded subset of E,
This mean $\{(x_\gamma - x)_{\gamma \in \Gamma}\} \subseteq \lambda_\gamma B$ (every subset of bounded is bounded) this implies $(x_\gamma - x)$ is a bounded subset of E, then (x_γ) is bounded.

Proposition 3.4

Suppose that (x_γ) and (y_γ) are bornologically convergent nets in a *cbvs* E, (λ_γ) is a convergent net in K such that $x_\gamma \longrightarrow x$, $y_\gamma \longrightarrow y$ and $\lambda_\gamma \to \lambda$ then

(i) $x_\gamma + y_\gamma \longrightarrow x+y$;

(ii) $c\,x_\gamma \longrightarrow c\,x$, for any number $c \in K$;

(iii) $\lambda_\gamma x_\gamma \longrightarrow \lambda\, x$.

Proof

since $x_\gamma \longrightarrow x$, $y_\gamma \longrightarrow y$ in E this mean $x_\gamma - x \longrightarrow 0$, $y_\gamma - y \longrightarrow 0$ in E ,then there exist bounded and absolutely convex sets B_1, B_2 of E and nets $(\alpha_\gamma),(\beta_\gamma)$ of scalars tending to 0, such that

$(x_\gamma - x) \in \alpha_\gamma B_1$ and $(y_\gamma - y) \in \beta_\gamma B_2$ for every $\gamma \in \Gamma$.

Then $x_\gamma - x + y_\gamma - y \in \alpha_\gamma B_1 + \beta_\gamma B_2 = (\alpha_\gamma + \beta_\gamma)(B_1 + B_2) - \alpha_\gamma B_2 - \beta_\gamma B_1$.

Since $\alpha_\gamma \to 0$ and $\beta_\gamma \to 0$

Then $((x_\gamma + y_\gamma) - (x+y)) \in \alpha_\gamma B_1 + \beta_\gamma B_2 \subseteq (\alpha_\gamma + \beta_\gamma)(B_1 + B_2)$

Now, if $\alpha_\gamma \to 0$ and $\beta_\gamma \to 0$ in K , then $\alpha_\gamma + \beta_\gamma \to 0$ in K and $B_1 + B_2$ is bounded and absolutely convex set when B_1 and B_2 are bounded and absolutely convex sets and since E is a bornological vector space this

implies $((x_\gamma + y_\gamma) - (x+y)) \longrightarrow 0$, then $(x_\gamma + y_\gamma) \longrightarrow (x+y)$.

(ii) $(x_\gamma - x) \longrightarrow 0$ there exists a bounded and absolutely convex set B of E and a net (α_γ) of scalars tends to 0, such that $x_\gamma \in B$ and $(x_\gamma - x) \in \alpha_\gamma B$ for every $\gamma \in \Gamma$. $c(x_\gamma - x) \in c\,\alpha_\gamma B$ then $c\,x_\gamma - c\,x \in \alpha_\gamma (c\,B)$

Since $c \in K$ and E is a convex bornological vector space then $c\,B$ is a bounded and absolutely convex set of E when B is a bounded and absolutely convex set of E. by Definition 3.1 $c\,x_\gamma - c\,x \longrightarrow 0$, then $c\,x_\gamma \longrightarrow c\,x$.

(iii) If (x_γ) converges bornologically to x in E, then there exists a bounded and absolutely convex subset B of E and a net (α_γ) of scalars tends to 0, such that

$x_\gamma \in B$ and $(x_\gamma - x) \in \alpha_\gamma B$ for every $\gamma \in \Gamma$.

$\lambda_\gamma x_\gamma - \lambda x = (x_\gamma - x)(\lambda_\gamma - \lambda) + x(\lambda_\gamma - \lambda) + \lambda(x_\gamma - x)$

now $\lambda_\gamma \to \lambda$, then $\lambda_\gamma - \lambda \to 0$ and $x_\gamma \longrightarrow x$

if $(x_\gamma - x) \longrightarrow 0$,then $\lambda_\gamma x_\gamma - \lambda x = (x_\gamma - x)(\lambda_\gamma - \lambda) \in \alpha_\gamma((\lambda_\gamma - \lambda)B)$

this mean $\lambda_\gamma x_\gamma - \lambda x \in \alpha_\gamma((\lambda_\gamma - \lambda)B)$

since E is a convex bornological vector space, then $(\lambda_\gamma - \lambda)B$ is bounded and absolutely convex set when B is a bounded and absolutely convex subset of *E*. by Definition 3.1 $(\lambda_\gamma x_\gamma - \lambda x) \longrightarrow 0$ then $\lambda_\gamma x_\gamma \longrightarrow \lambda x$.

Proposition 3.5

Let *E* be a *cbvs* and let (x_γ) be a net in *E*. the following statements are equivalent:

(i)The net (x_γ) converges bornologically to 0;

(ii) There exists a bounded and absolutely convex set $B \subset E$ and a decreasing net (α_γ) of positive real numbers, tends to 0, such that $x_\gamma \in \alpha_\gamma B$ for every $\gamma \in \Gamma$;

(iii) There exists a bounded and absolutely convex set $B \subset E$ such that, given any $\epsilon > 0$, we can find an integer $\Gamma(\epsilon)$ for which $x_\gamma \in \epsilon B$ whenever $\gamma \geq \Gamma(\epsilon)$.

(iv) There exists a bounded disk $B \subset E$ such that (x_γ) belongs to the semi-normed E_B and converges to 0 in E_B.

Proof

(i)$\Rightarrow$(ii): For any integer $p \in \Gamma$ there exists $\Gamma_p \in \Gamma$ such that if $\gamma \geq \Gamma_p$ then $\lambda\gamma \leq \frac{1}{p}$; hence $\lambda_\gamma B \subset (\frac{1}{p})B$, since *B* is disk. We may assume that the net Γ_p is strictly increasing, and for $\Gamma_p \leq K \leq \Gamma_{p+1}$,let $\alpha_K = \frac{1}{p}$. Then the net $\{\alpha_K\}$ satisfies the conditions of assertion (ii). Clearly (ii) $\Rightarrow$(iii).

To show that (iii)$\Rightarrow$(i), let for every$\gamma \in \Gamma$, $\epsilon_\gamma = \inf\{\epsilon > 0; x_\gamma \in \epsilon B\}$ and $\lambda_\gamma = \epsilon_\gamma + \frac{1}{\gamma}$,

then the net (λ_γ) converges to 0 and $x_\gamma \in \lambda_\gamma B$ for every $\gamma \in \Gamma$.

Thus the assertions (i, ii, iii) are equivalent. Suppose that the bornology of *E* is convex.

Clearly (iv) implies (i) with $\lambda_\gamma = p_B(x_\gamma)$ and p_B the gauge of B, while (ii) implies that $x_\gamma \in E_B$ and $p_B(x_\gamma) \le \alpha_\gamma \to 0$.

Remark 3.6

It is clear that (x_γ) for every $i_\gamma \in \Gamma$ converges to 0 if and only if every subnet of (x_γ) converges to 0.

Proposition 3.7

A net x_γ in a product *cbvs* $\prod_{i\in I} E_i$ converges bornologically to y if and only if the net $\{x^i_\gamma\}_{i\in I}$ converges bornologically to y_i in cbvsE_i.

Proof

Let $(x_\gamma) \to y = (y_1, y_2, \dots, y_n, \dots)$ in $\prod_{i\in I} E_i$, since p_i is bounded linear map, then by Theorem 3.4 $p_i(x_\gamma) \to p_i(y)$, for each $i\in I$, then $x \to y_i$ in E_i. Suppose that $x^i_\gamma \to y_i$ for each $i\in I$, then there exists a bounded and absolutely convex sets B_i of E_i , $i\in I$ and a net (λ^i_γ) of scalars tending to 0, such that $x^i_\gamma \in B_i$ and for each $i\in I$ $x^i_\gamma - y_i \in \lambda^i_\gamma B_i$, $\gamma\in\Gamma$ since a net $\lambda^i_\gamma \to 0$, $i\in I$ for each $\gamma\in\Gamma$ then there is a diagonal net converging to 0 such that when $i=\gamma$ then $\lambda^i_\gamma \to 0$

since $x^i_\gamma - y_i \in \lambda^i_\gamma B_i$ whenever $i=\gamma$ and $i=1,2,\dots,n, \dots$

If $\gamma \ne i$, since a disk B_i and $|\lambda^i_\gamma| \to 0$ and if $a\in B_i$ then $\lambda\gamma a \in B_i$ then $x^i_\gamma - y_i \in \lambda^n_\gamma B_i$

. Therefore $x_\gamma = (x^1_\gamma, x^2_\gamma, \dots) - y \in \lambda^n_\gamma \prod_{i\in I} B_i$ this mean $x_\gamma \longrightarrow y$.

3.Convergence between bornological and topological spaces.

In this section, the convergence net between bornology and topology has been studied with investigation to the relations between them in case of convergence with establishing the necessary properties which describe these relations.

Definition 4.1[5]

Let E be a topological vector space. A net (x_γ) of points of E converges bornologically to x in E if it converges bornologically to x when E is endowed with its Von neumannbornology .

proposition 4.2

Every bornologically converge net is topologically converge but the converse is not true.

Proof

Let (x_γ) converge bornologically to x then there exist a bounded and absolutely convex set $B \subset E$ and a net (λ_γ) in $\mathbb{K}$ converging to 0 such that $x_\gamma - x \in \lambda_\gamma B$ for every $\gamma \in \Gamma$. Since every bounded sub set of E is absorbed by each neighbourhoodof 0, then (x_γ) converge topologically to x. The converse is not true . The following example explain that.

Example 4.3

Let $\mathbb{R}^I$ be the product topological vector space such that I $= [0,1]$, $\mathbb{R}^I$ locally convex space, if (λ_n) the set of all nets of positive real numbers sending to ∞ when $n \in \Gamma$ then the net $(x_n) \subset \mathbb{R}^I$ defined by $(x_n)(i) = \frac{1}{\lambda_n}$ converges to 0 topologically but not bornologically .

Proposition 4.4

Let E be separated topological vector space for every compact set $K \subset E$, there exists a bounded disk $B \subset E$ such that K is compact in E_B then every topologically convergent net in E is also bornological convergent.

Proof

If (x_γ) converges topologically to 0 in E, then the set $A = (x_\gamma) \cup \{0\}$ is compact in E. Since for every compact set $K \subset E$ there exist a bounded disk $B \subset E$ such that K is compact in E_B , since the canonical embedding $E_B \to F$ is continuous the topologies of F and E_B correspond on A and therefore (x_γ) converge to 0 in E_B.

Now let E be separated locally convex space if every bounded sub set of E is relatively compact in a space E_B with B a bounded disk in E, then every topologically convergent net in E is bornologically convergent.

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Proposition 4.5

In a metrizable topological vector space every topologically convergent net is bornologically convergent.

Proof

Let (V_n) be a countable base of neighbour hoods of 0 in E such that $V_n \supset V_{n+1}$ for every $n \in$ N and let $A = (x_\gamma)$ be a net in E converges to 0 topologically then the net (x_γ) converges bornologically to 0 from proposition 2.13. If $N(\epsilon)$ is a positive integer such that $x_\gamma \in V_m$when $\gamma \geq N(\epsilon)$ then $x_\gamma \in A \cap V_m \subset \in B$ for every $\gamma \geq N(\epsilon)$. That is mean there exist a bounded disk set B such that for every $\in > 0$ there is an integer $m \in N$,then $A \cap V_m \subset \in B$…(1) Since the net A converges topologically to 0 it is absorbed by every neighbourhood of 0 , then for every $n \in$ N there exist a positive real number λ_n such that $A \subset \lambda_n V_n$ then $A \subset \bigcap_{n=1}^{\infty} \lambda_n V_n$.

Let (α_n) be a net of positive real numbers converging to 0 , let $M_n = \frac{\lambda_n}{\alpha_n}$ such that $B = \bigcap_{n=1}^{\infty} \lambda_n V_n$,then B is bounded in E and B is the set in eqution …(1).

Let $\in > 0$ since the sequence $\frac{M_n}{\lambda_n} = \frac{1}{\alpha_n} \to \infty$ there is an integer $l \in$ N such that for $n \geq l, \frac{M_n}{\lambda_n} > \frac{1}{\epsilon}$,then $\lambda_n < \epsilon M_n$. Since $A \subset \bigcap_{n=1}^{\infty} \lambda_n V_n$. $A \subset \epsilon\, M_n V_n$ for $n \geq l$. But the set $\bigcap_{n<l} \in M_n V_n$ is a neighbourhood of 0.

So, there exist an integer $m \in$ N such that $V_m \subset \bigcap_{n<l} \in M_n V_n$ thus $A \cap V_m \subset\in M_n V_n$ for every $n \in$ N then $A \cap V_m \subset\in \bigcap_{n=1}^{\infty} M_n V_n = \epsilon B$.

Remark 4.6

In a metrizable topological vector space(locally convex) every topologically convergent net is bornologically convergent.

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دراسة العلاقة بين تقارب الشبكات في الفضاء البرنولوجي وتقارب الشبكات في الفضاء التبولوجي

فاطمة كامل مجيد البصري

جامعة القادسية

كلية التربية

قسم الرياضيات

المستخلص:-

في هذا البحث درسنا تقارب الشبكات في فضاء المتجهات البرنولوجي المحدب والمفاهيم المرتبطة به و أعطينا بعض الخصائص لهذه المفاهيم. بعد ذلك درسنا علاقة تقارب الشبكه بين الفضاء البرنولوجي والفضاء التبولوجي حيث بينا ان كل تقارب شبكه في الفضاء البرنولوجي يكون تقارب في الفضاء التبولوجي ثم وضعنا الشروط على العكس لكي يكون صحيح من خلال الخواص التي توضح ذلك.