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Some New result of Compact sets in fuzzy metric space

Noori F. AL-Mayahi , Sarim H . Hadi

(Department of Mathematics ,College of Computer Science and Mathematics ,University of AL –Qadissiya)

afm2005@yahoo.com & sarim.hazim@yahoo.com

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Abstract:

In this paper, we have introduce definition of compact set in fuzzy metric space,also we have to prove some result in compact fuzzy metric space and we have to proved equivalent between compact and sequentially compact .

Mathematics Subject Classification:46A50

I. Introduction:

The theory of fuzzy sets was introduction by Zadeh [9] in 1965. After the pioneer work of Zadeh , many researchers have extended this concept in various branches of mathematics, after Zadeh Tarapada bag and Syamal samanta are study compact in fuzzy normed space , also P. Tirado is study compact and G-compact in fuzzy metric space and Aphan introduction some of theorem on compact set .

This paper is study some new result of compact fuzzy metric space and we have to prove equivalent between compact and sequentially compact in fuzzy metric space.

Keyword : t-norm ,fuzzy metric space ,compact set in fuzzy metric space.

Definition(1.1)[1]: A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous (t-norm) on the set $[0,1]$,if $*$ is satisfying the following conditions :

(TN-1) $a * b = b * a \ for\ all\ a, b \in [0,1]$ (i.e. $*$ is commutative).

(TN-2) $a * (b * c) = (a * b) * c \ for\ all\ a, b, c \in [0,1]$,(i.e. $*$ is associative).

(TN-3) $a * 1 = a \ for\ all\ a \in [0,1]$.

(TN-4) If $b, c \in [0,1]$ such that $b \leq c$, then $a * b \leq a * c \ for\ all\ a \in [0,1]$, (i.e. $*$ is monotone).

Definition(1.2)[1]: Let X be a non-empty set, $*$ be a continuous t-norm on[0,1].
A function $M: X \times X \times (0, \infty) \to [0, 1]$ is called a fuzzy metric function on X if it satisfies the following axioms: $for\ all\ x, y, z \in\ X\ and\ for\ all\ t, s > 0$

(FM-1) $M\ (x, y, t) > 0$.

(FM-2) $M\ (x, y, t) = 1 \iff x = y$.

(FM-3) $M\ (x, y, t) = M\ (y, x, t)$.

(FM-4) $M\ (x, y, t + s) \geq M\ (x, z, t) * M\ (z, y, s)$.

(FM-5) $M\ (x, y, \circ): (0, \infty) \to [0, 1]$ is continuous.

Definition(1.3)[2]: Let $(X, M, *)$ be a fuzzy metric space. Then

(a) A sequence $\{x_n\}$ in X is said to be convergent to x in X if for each $r \in (0, 1)$and each $t > 0$, there exist $k \in Z^+$ such that $M\ (x_n, x, t) > 1 - r$ for all $n \geq k$ (or equivalent $\lim_{n\to\infty} M\ (x_n, x, t) = 1$).

(b) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $r \in (0, 1)$ and each $t > 0$, there exist $k \in Z^+$ such that $M\ (x_n, x_m, t) > 1 - r$ for all $n, m \geq k$ (or equivalent $\lim_{n,m\to\infty} M\ (x_n, x_m t) = 1$).

Definition(1.4),[3] : Let $(X, M, *)$ be a fuzzy metric space . A subset A of X is said to be bounded if there exist $t > 0$ and $0 < r < 1$ such that

$$M(x, y, t) > 1 - r \qquad \text{for all } x, y \in A.$$

Definition(1.5),[2]: Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to be closed if for any sequence $\{x_n\}$ in A converge to $x \Rightarrow x \in A$.

Definition(1.6),[2] : Let $(X, M, *)$ and $(Y, M, *)$ be two fuzzy metric space . The function $f: X \to Y$ is said to be continuous at $x_0 \in X$ if for all $r \in (0,1)$ and $t > 0$ there exist $r_1 \in (0,1)$ and $s > 0$ such that for all $x \in X$

$$M(x, x_0, s) > 1 - r_1 \quad \text{implies} \quad M(f(x), f(x_0), t) > 1 - r$$

The function f is called a continuous function if it is fuzzy continuous at every point of X.

Definition(1.7),[4] : Let $(X, M, *)$ be a fuzzy metric space and $A \subset X$ .We say A is compact set if every open cover has a finite subcover .

Definition(1.8),[4]: $(X, M, *)$ is said to be sequentially compact fuzzy metric space if

every sequence in X has a convergent subsequence in it.

Example(1.9)[5] : Let $X = [-1, 1]$. Obviously (X, d) is a compact metric space, where d is the Euclidean metric. Therefore $(X, M_d, *)$ is a compact fuzzy metric space.

Example(1.10)[5]: Let (X, d) be the metric space where $X = [0, 1]$ and d the Euclidean metric on X. Let $*$ be the continuous t-norm. We define a fuzzy set M in $X \times X \times [0, \infty)$ given by the following condition:

$$M\ (x, y, t) = 0, \ \ if\ t = 0$$

$$M\ (x, y, t) = 1 - d(x, y), \ \ if\ 0 < t \leq 1,$$

$$M\ (x, y, t) = 1, \qquad if\ t > 1.$$

It is clear that $(X, M, *)$ is a compact fuzzy metric space.

Remark(1.11)[2]: In fuzzy metric space every open ball is open set .

Theorem(1.12) : Every finite set in fuzzy metric space is compact .

Proof : Let $A = \{a_1, a_2, \dots \dots, a_n\}$ be finite set

For every $0 < r < 1$, $t > 0$

suppose $\{B(a, r, t) : a \in A\}$ is an open cover of A such that

$A \subseteq \bigcup_{a \in A} B(a, r, t)$

For all $a_i \in A$, $i = 1, 2, \dots, n$.

$\Rightarrow a_i \in \bigcup_{a_i \in A} B(a_i, r, t) \Rightarrow a_i \in B(a_i, r, t)$ for some i

Hence $B(a_i, r, t)$ is finite sub cover such that $A \subseteq \bigcup_{a_i \in A} B(a_i, r, t)$

Hence A is compact .

Proposition(1.13) : Let A, B are compact set in fuzzy metric space , then $A \cup B$ is compact

Corollary(1.14) : Let A finite set and B be a compact set in fuzzy metric space then $A \cup B$ is compact .

Proof : by theorem (1.13) and proposition (1.14) we get $A \cup B$ is compact .

Theorem(1.15) : Let $(X, \mathcal{M}, *)$ be a fuzzy metric space and $A \subseteq X$, then A is compact iff every sequence has a converge subsequence to a point in X. (i.e. A is compact iff it is sequentially compact)

Proof : Let A is compact set , let $\{x_n\}$ be a sequence in A
let $\{B(x, r, t) : x \in A\}$ is open cover of A such that $A \subseteq \bigcup_{x \in A} B(x, r, t)$

Since $\{x_n\} \in A \Rightarrow \{x_n\} \in \bigcup_{x \in A} B(x, r, t) \Rightarrow \{x_n\} \in B(x, r, t)$ for some x , $x \in A$

$\forall\, r_1 \in (0,1)$, $\frac{t}{3} > 0$ such that $M\left(x_n, x, \frac{t}{3}\right) > 1 - r_1$.

Since A is compact there exist finite subcover $\{B(x_i, r, t) : x_i \in A\}$

Such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, r, t)$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\} \Rightarrow \{x_{n_k}\} \in \bigcup_{i=1}^{n} B(x_i, r, t) \Rightarrow \{x_{n_k}\} \in B(x_i, r, t)$ for some $x_i \in A$

$\Rightarrow \forall\, r_1 \in (0,1)$, $\frac{t}{3} > 0$ such that $M\left(x_{n_k}, x_i, \frac{t}{3}\right) > 1 - r_1$

Now : $M(x_{n_k}, x, t) \geq M\left(x_{n_k}, x_i, \frac{t}{3}\right) * M\left(x_n, x_i, \frac{t}{3}\right) * M\left(x_n, x, \frac{t}{3}\right)$

$$\geq (1 - r_1) * (1 - r_1) * (1 - r_1) > 1 - r$$

Hence $x_{n_k} \to x$.

Conversely : suppose A is not compact and let $\{x_{n_k}\}$ subsequence of $\{x_n\}$ such that $x_{n_k} \to x$.

Let $\{B(x,r,t): x \in A\}$ is open cover of A such that $A \subseteq \bigcup_{x\in A} B(x,r,t) \quad \Rightarrow M(x_n,x,t) > 1-r$,there exist finite subcover $\{B(x_i,r,t): x_i \in A\}$

but $A \nsubseteq \bigcup_{x_i\in A} B(x_i,r,t)$

$\Rightarrow M\left(x_{n_k},x_i,t\right) \leq 1-r$

$\Rightarrow M\left(x_{n_k},x,t\right) \leq 1-r$ which is contradiction .

Hence A is compact .

Theorem(1.16)[6] : In fuzzy metric space every closed subset of compact set is compact .

Theorem(1.17): Let $(X,M,*)$ be a fuzzy metric space , $A \subseteq X$ is a compact such that for any $x \notin A$ then there exist two open set U,V in X such that $A \subseteq U$, $x \in V \ni U \cap V = \emptyset$.

Proof: Let $x \notin A$, $a \in A \quad \Rightarrow \quad x \neq a$

Since $(X,M,*)$ is Hausdorff

Then there exist $U = B(a,r,t)$, $V = B(x,r,t)$ is open set $\ni$

$a \in U, x \in V$ and $U \cap V = \emptyset$

Since $\{U_a : a \in A\}$ is open cover of A and A is compact

Then there exist finite subcover $\left\{U_{a_i}, i = 1,2,\dots.n\right\}$ such that

$A \subseteq \bigcup_{i=1}^{n} U_{a_i}$

Take $V = \bigcap_{i=1}^{n} V_{x_i}$, $x_i \notin A$

U,V is open set , $A \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.

Theorem(1.18): Let $\{x_n\}$ be a Cauchy sequence in a compact A in fuzzy metric space ,show that there exist $x \in A$ such that $x_n \longrightarrow x$.

Proof:

Let $\{x_n\}$ be a cauchy sequence in A

Suppose $\{x_n\}$ has not converge to x

Let $1-r = M(x_n,x,t)$, $0 < r < 1$

Since $0 < 1-r < 1$, there exist $0 < r_1 < 1$ such that

$(1-r_1) * (1-r_1) > 1-r$

Since $\{x_n\}$ be a Cauchy sequence in A

For all $0 < r_1 < 1$, $t > 0$ there exist $k \in N$

$M\left(x_n,x_m,\frac{t}{2}\right) > 1-r_1 \quad , m,n \geq k$

Since A compact

Then A is closed $\Longrightarrow A = \bar{A}$

Let $\{x_m\}$ any sequence in A

Since $A = \bar{A}$ $\Longrightarrow$ there exist $x \in A$ and for all $0 < r_1 < 1$, $t > 0$

Such that $$M\left(x_m, x, \frac{t}{2}\right) > 1 - r_1$$

Now: $$M(x_n, x, t) \geq M\left(x_n, x_m, \frac{t}{2}\right) * M\left(x_m, x, \frac{t}{2}\right)$$

$$\geq (1 - r_1) * (1 - r_1) > 1 - r \text{ (contradiction)}$$

Hence $x_n \to x$.

Theorem(1.19) : Let f be a function from $(X, M_1, *)$ to $(Y, M_2, *)$ and $f: X \to Y$ be continuous function . If A be a compact subset of X ,then $f(A)$ is compact subset of Y.

Proof : Let $\{y_n\}$ be a sequence in $f(A)$,then for each n there exist $\{x_n\}$ in A such that $f(x_n) = y_n$

. Since A is compact , there exists $\{x_{n_k}\}$ a subsequence of $\{x_n\}$
and $x_0 \in A$ such that $x_{n_k} \to x_0$ in A
Since f is continuous at x_0
if for all $r \in (0,1)$ and $t > 0$ there exist $r_1 \in (0,1)$ and $s > 0$ such that for all $x \in X$

$$M(x, x_0, s) > 1 - r_1 \quad \text{implies} \quad M(f(x), f(x_0), t) > 1 - r$$

Now ; $x_{n_k} \to x_0$ in $A \Longrightarrow$ there exist $n \in N$such that for all $k \geq n$

$$M(x_{n_k}, x_0, s) > 1 - r_1 \Longrightarrow M(f(x_{n_k}), f(x_0), t) > 1 - r$$

i.e. $M(y_{n_k}, f(x_0), t) > 1 - r$ for all $k \geq n_0$.
$\Rightarrow f\ (\ A\ )$ is a compact subset of Y.

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بعض النتائج الجديدة للمجموعات المتراصة في الفضاء المتري الضبابي

أ.م.د. نوري فرحان المياحي ، صارم حازم هادي
(قسم الرياضيات ، كلية علوم الحاسوب والرياضيات ، جامعة القادسية)

الملخص

في هذا البحث سنقدم تعريف المجموعة المتراصة والمجموعة المتراصة التتابعية و سنبرهن بعض الحقائق المتعلقة بالمجموعات المتراصة وكذلك سنبرهن علاقة التكافؤ بين المجموعة المتراصة والمجموعة المتراصة التتابعية في الفضاء المتري الضبابي .