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On The Statistically Convergent For Quadruple Sequence Spaces of Fuzzy Numbers Described by Triple Maximal Orlicz Functions

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ABSTRACT

In this paper ,we give the statistically convergent for quadruple sequence spaces of fuzzy numbers defined by the triple maximal Orlicz function and shows explore properties such as semi-normed space, complete semi-normed space, and others.

MSC..

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1. Introduction :

Fast [1] and Schoenberg [6] were the first to establish the concept of statistical convergence. It can be found in Zygmund [9] as well . Fridy and Orhan [2] , Maddox [3] , Salat [5], Rath and Tripathay [4] , Tripathy [7, 8] and many others later examined it.

In this paper ,we offer and define the classes of quadruple sequence spaces of fuzzy numbers $(\ell_\infty)^4_{\mathbb{F}}(\mathbb{M},q)$, $(\bar{c})^4_{\mathbb{F}}(\mathbb{M},q)$, $(\bar{c}_0)^4_{\mathbb{F}}(\mathbb{M},q)$, $(\bar{c})^{4^{\mathcal{R}}}_{\mathbb{F}}(\mathbb{M},q)$, $(\bar{c}_0)^{4^{\mathcal{R}}}_{\mathbb{F}}(\mathbb{M},q)$, $(\bar{c})^{4^{\mathcal{B}}}_{\mathbb{F}}(\mathbb{M},q)$, $(\bar{c}_0)^{4^{\mathcal{B}}}_{\mathbb{F}}(\mathbb{M},q)$, $(\bar{c})^{4^{\mathcal{R}\mathcal{B}}}_{\mathbb{F}}(\mathbb{M},q)$, $(\bar{c}_0)^{4^{\mathcal{R}\mathcal{B}}}_{\mathbb{F}}(\mathbb{M},q)$ specified by the triple maximal Orlicz functions .

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2. Definitions and Preliminaries :

A map $\mathbb{G} : \mathbb{R} \to \mathbb{R}$ is called non-decreasing if $\mathfrak{W}_1 \prec \mathfrak{M}_2$ implies $\mathbb{G}(\mathfrak{M}_1) \preccurlyeq \mathbb{G}(\mathfrak{M}_2)$, $\forall\, \mathfrak{M}_1, \mathfrak{M}_2 \in \mathbb{R}$.

A map $\mathbb{G} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is said to be non-decreasing if $(\mathfrak{W}_1, \mathfrak{M}_2) \prec (\mathfrak{N}_1, \mathfrak{N}_2)$ implies $\mathbb{G}\,(\mathfrak{M}_1, \mathfrak{M}_2) \preccurlyeq \mathbb{G}\,(\mathfrak{N}_1, \mathfrak{N}_2), \forall\, (\mathfrak{W}_1, \mathfrak{M}_2), (\mathfrak{N}_1, \mathfrak{N}_2) \in \mathbb{R} \times \mathbb{R}$.

A real-valued function $\mathbb{G}(\mathfrak{N})$ of the real variable $\mathfrak{N}$ is called convex if the inequality $\mathbb{G}\left(\frac{\mathfrak{N}_1 + \mathfrak{N}_2}{2}\right) \preccurlyeq \frac{1}{2}\left(\mathbb{G}(\mathfrak{N}_1) + \mathbb{G}(\mathfrak{N}_2)\right), \forall\, \mathfrak{N}_1, \mathfrak{N}_2 \in \mathbb{R}$.

A map $\mathbb{G} : \mathbb{R} \to \mathbb{R}$ is said to be a continuous from the right at $\mathbb{a}$ if $\forall\, \varepsilon \succ 0, \exists\, \varsigma \succ 0 \ni |\mathbb{G}(\mathfrak{N}) - \mathbb{G}(\mathbb{a})| \prec \varepsilon, \forall\, \mathfrak{N} \in (\mathbb{a}\,, \mathbb{a} + \varsigma)$.

An Orlicz function is a function $\mathcal{M}: [0, \infty) \to [0, \infty)$, which is a continuous, non-decreasing , and convex with $\mathcal{M}(0) = 0, \mathcal{M}(\mathfrak{A}) \succ 0$ as $\mathfrak{A} \succ 0$ and $\mathcal{M}(\mathfrak{A}) \to \infty$ as $\mathfrak{A} \to \infty$.

A maximal Orlicz function is a function $\mathcal{H}: [0, \infty) \to [0, \infty) \ni \mathcal{H}(\mathfrak{A}) = \mathfrak{A}^2\mathcal{M}(\mathfrak{A})$ and $\mathcal{M}$ is Orlicz function , which is a continuous , non-decreasing , and convex with $\mathcal{H}(0) = 0, \mathcal{H}(\mathfrak{A}) \succ 0$ as $\mathfrak{A} \succ 0$ and $\mathcal{H}(\mathfrak{A}) \to \infty$ as $\mathfrak{A} \to \infty$.

A triple maximal Orlicz function is a function $\mathbb{M}:[0,\infty)\times[0,\infty)\times[0,\infty) \to [0,\infty)\times[0,\infty)\times[0,\infty) \ni \mathbb{M}(\mathfrak{A}\,, \mathfrak{S}\,, \mathfrak{R}) = \left(\mathbb{M}_1(\mathfrak{A}), \mathbb{M}_2(\mathfrak{S}), \mathbb{M}_3(\mathfrak{R})\right)$, where $\mathbb{M}_1 : [0, \infty) \to [0, \infty) \ni \mathbb{M}_1(\mathfrak{A}) = \mathfrak{A}^2\mathcal{M}_1(\mathfrak{A})$ and $\mathbb{M}_2 : [0, \infty) \to [0, \infty) \ni \mathbb{M}_2(\mathfrak{S}) = \mathfrak{S}^2\mathcal{M}_2(\mathfrak{S})$and $\mathbb{M}_3 : [0, \infty) \to [0, \infty) \ni \mathbb{M}_3(\mathfrak{R}) = \mathfrak{R}^2\mathcal{M}_3(\mathfrak{R})$ .These functions are non-decreasing, continuous, even, convex , that hold the following conditions :

i) $\mathbb{M}_1(0) = 0, \mathbb{M}_2(0) = 0, \mathbb{M}_3(0) = 0 \Longrightarrow \mathbb{M}(\mathfrak{A}\,, \mathfrak{S}\,, \mathfrak{R}) = \left(\mathbb{M}_1(0), \mathbb{M}_2(0), \mathbb{M}_3(0)\right) = (0,0,0)$.

ii) $\mathbb{M}_1(\mathfrak{A}) \succ 0\,, \mathbb{M}_2(\mathfrak{S}) \succ 0\,, \mathbb{M}_3(\mathfrak{R}) \succ 0 \Longrightarrow \mathbb{M}(\mathfrak{A}\,, \mathfrak{S}\,, \mathfrak{R}) = \left(\mathbb{M}_1(\mathfrak{A}), \mathbb{M}_2(\mathfrak{S}), \mathbb{M}_3(\mathfrak{R})\right) \succ (0,0,0)$, for $\mathfrak{A} \succ 0, \mathfrak{S} \succ 0, \mathfrak{R} \succ 0$,by which we say $(\mathfrak{A}\,, \mathfrak{S}\,, \mathfrak{R}) \succ (0,0,0)$ that $\mathbb{M}_1(\mathfrak{A}) \succ 0\,, \mathbb{M}_2(\mathfrak{S}) \succ 0\,, \mathbb{M}_3(\mathfrak{R}) \succ 0$.

iii) $\mathbb{M}_1(\mathfrak{A}) \to \infty, \mathbb{M}_2(\mathfrak{S}) \to \infty, \mathbb{M}_3(\mathfrak{R}) \to \infty$ as $\mathfrak{A} \to \infty, \mathfrak{S} \to \infty, \mathfrak{R} \to \infty \Longrightarrow \mathbb{M}(\mathfrak{A}\,, \mathfrak{S}\,, \mathfrak{R}) = \left(\mathbb{M}_1(\mathfrak{A}), \mathbb{M}_2(\mathfrak{S}), \mathbb{M}_3(\mathfrak{R})\right) \to (\infty, \infty, \infty)$ as $(\mathfrak{A}\,, \mathfrak{S}\,, \mathfrak{R}) \to (\infty, \infty, \infty)$ by which we say $\mathbb{M}(\mathfrak{A}\,, \mathfrak{S}\,, \mathfrak{R}) \to (\infty, \infty, \infty)$ as $\mathbb{M}_1(\mathfrak{A}) \to \infty, \mathbb{M}_2(\mathfrak{S}) \to \infty, \mathbb{M}_3(\mathfrak{R}) \to \infty$.

A quadruple sequence $(\mathfrak{A}_{\ell k j i})$ is a converge in Pringsheim's sense to a number $\mathbb{L}$ if $\lim_{\ell,k,j,i\to\infty} \mathfrak{A}_{\ell k j i} = \mathbb{L}$, where $\ell, k, j,$ and i tend to ∞.

A quadruple sequence $(\mathfrak{A}_{\ell k j i})$ converges in Pringsheim's sense and it is said to be converge regularly if $\lim_{\ell\to\infty} \mathfrak{A}_{\ell k j i} = \mathbb{L}_{k j i}$ and $\lim_{k\to\infty} \mathfrak{A}_{\ell k j i} = \mathbb{J}_{\ell j i}$ and $\lim_{j\to\infty} \mathfrak{A}_{\ell k j i} = \mathbb{T}_{\ell k i}$ and $\lim_{i\to\infty} \mathfrak{A}_{\ell k j i} = \mathbb{T}_{\ell k j}$ exists .

A subset $\mathbb{E}$ of $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have natural density $p(\mathbb{E})$ if $p(\mathbb{E}) = \lim_{n} \frac{1}{n} \sum_{\ell,k,j,i=1}^{n} \mathcal{X}_{\mathbb{E}}\big((\ell, k, j, i)\big)$ exists, where $\mathcal{X}_{\mathbb{E}}(\mathfrak{s}) = \begin{cases} 1 & \text{if } (\ell, k, j, i) \in \mathbb{E} \\ 0 & \text{if } (\ell, k, j, i) \notin \mathbb{E} \end{cases}, \forall\, (\ell, k, j, i) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

A quadruple sequence $(\mathfrak{A}_{\ell k j i})$ is statistically convergent to a number $\mathbb{L}$ if $\forall\, \varepsilon \succ 0$,we have $p\big(\{(\ell, k, j, i) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} : q[\mathfrak{A}_{\ell k j i} - \mathbb{L}] \succcurlyeq \varepsilon\}\big) = 0$.

Let $\mathbb{V}$ be a vector space on a field $\mathcal{F}(\mathbb{R}$ or $\mathbb{C})$. A semi-norm is a function $k : \mathbb{V} \longrightarrow \mathbb{R}$ defined on a linear space $\mathbb{V}$, satisfies the following conditions :

1. $k(\mathbb{x})\|\mathbb{x}\| = 0 \Leftrightarrow \mathbb{x} = 0 , \forall\, \mathbb{x} \in \mathbb{V}$
2. $k(\mathbb{x} + \mathbb{y}) \le k(\mathbb{x}) + k(\mathbb{y}) , \forall\, \mathbb{x} , \mathbb{y} \in \mathbb{V}$
3. $k(\ltimes \mathbb{x}) = |\ltimes| k(\mathbb{x}) , \forall\, \mathbb{x} \in \mathbb{V} , \ltimes \in \mathcal{F}$.

A fuzzy real number $\mathbb{F}$ is a fuzzy subset of the real line $\mathbb{R}$, i.e. a mapping $\mathbb{F} : \mathbb{R} \to [0,1]$ associating each real number $\mathbb{r}$ with its grade of membership $\mathbb{F}(\mathbb{r})$, satisfies the following conditions [34] :

1. $\mathbb{F}$ is a convex if for each $\mathbb{F}(\mathbb{r}_2) \succcurlyeq \mathbb{F}(\mathbb{r}_1) \wedge \mathbb{F}(\mathbb{r}_3) = \min\{ \mathbb{F}(\mathbb{r}_1), \mathbb{F}(\mathbb{r}_3)\}$, $\forall \mathbb{r}_1 \prec \mathbb{r}_2 \prec \mathbb{r}_3$, $\forall\, \mathbb{r}_1, \mathbb{r}_2, \mathbb{r}_3 \in \mathbb{R}$.
2. $\mathbb{F}$ is normal if there is a $\mathbb{r}_0 \in \mathbb{R}$ and $\mathbb{F}(\mathbb{r}_0) = 1$.
3. $\mathbb{F}$ is upper-semi-continuous $\forall\, \mathbb{a} \in \mathbb{I} , \forall\, \varepsilon \succ 0$ and $\mathbb{F}^{-1}([0 , \mathbb{a} + \varepsilon))$ is open in the usual topology of $\mathbb{R}$
4. $\mathbb{F}$ is a non-negative fuzzy number $\forall\, \mathbb{r} \prec 0$ implies $\mathbb{F}(\mathbb{r}) = 0$.

The set of all non-negative fuzzy numbers of $\mathbb{R}(\mathbb{I})$ is denoted by $\mathbb{R}^*(\mathbb{I})$. Let $\mathbb{R}(\mathbb{I})$ denote the set of all fuzzy numbers which are upper-semi continuous , normal and have compact support, i.e. if $\mathbb{H} \in \mathbb{R}(\mathbb{I})$ then $\mathbb{H}^{\propto}$ is compact, for any $\propto \in [0,1]$, where

$$\mathbb{H}^{\propto} = \{\mathbb{r} \in \mathbb{R} : \mathbb{H}(\mathbb{r}) \succcurlyeq \propto , \text{if } \propto \in [0,1]\} .$$

$$\mathbb{H}^0 = \text{closure of } (\{\mathbb{r} \in \mathbb{R} : \mathbb{H}(\mathbb{r}) \succ 0 , \text{if } \propto = 0\}) .$$

In this paper , The quadruple sequences spaces are introduced as follows :

$$(\ell_\infty)_{\mathbb{F}}^4(\mathbb{M}, q) = \Big\{(\mathfrak{A}_{\ell k j i}) = \big((\mathfrak{A}_1)_{\ell k j i}, (\mathfrak{A}_2)_{\ell k j i}, (\mathfrak{A}_3)_{\ell k j i}\big) \in \mathbb{W}_{\mathbb{F}}^4(q) : \sup_{\ell k j i}\Big[\mathbb{M}_1\Big(q\Big(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho}\Big)\Big) \curlyvee \mathbb{M}_2\Big(q\Big(\frac{(\mathfrak{A}_2)_{\ell k j i}}{\rho}\Big)\Big) \curlyvee \mathbb{M}_3\Big(q\Big(\frac{(\mathfrak{A}_3)_{\ell k j i}}{\rho}\Big)\Big)\Big] \prec \infty , \text{for some } \rho \succ 0\Big\} .$$

$$(\bar{c})_{\mathbb{F}}^4(\mathbb{M}, q) = \Big\{(\mathfrak{A}_{\ell k j i}) = \big((\mathfrak{A}_1)_{\ell k j i}, (\mathfrak{A}_2)_{\ell k j i}, (\mathfrak{A}_3)_{\ell k j i}\big) \in \mathbb{W}_{\mathbb{F}}^4(q) : \operatorname*{stat-lim}_{\ell, k, j, i \to \infty}\Big[\mathbb{M}_1\Big(q\Big(\frac{(\mathfrak{A}_1)_{\ell k j i} - \mathbb{L}_1}{\rho}\Big)\Big) \curlyvee \mathbb{M}_2\Big(q\Big(\frac{(\mathfrak{A}_2)_{\ell k j i} - \mathbb{L}_2}{\rho}\Big)\Big) \curlyvee \mathbb{M}_3\Big(q\Big(\frac{(\mathfrak{A}_3)_{\ell k j i} - \mathbb{L}_3}{\rho}\Big)\Big)\Big] = 0 , \text{for some } \rho \succ 0 , \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3 \in \mathbb{R} \text{ or } \mathbb{C}\Big\} .$$

$$(\bar{c}_0)_{\mathbb{F}}^4(\mathbb{M}, q) = \Big\{(\mathfrak{A}_{\ell k j i}) = \big((\mathfrak{A}_1)_{\ell k j i}, (\mathfrak{A}_2)_{\ell k j i}, (\mathfrak{A}_3)_{\ell k j i}\big) \in \mathbb{W}_{\mathbb{F}}^4(q) : \operatorname*{stat-lim}_{\ell, k, j, i \to \infty}\Big[\mathbb{M}_1\Big(q\Big(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho}\Big)\Big) \curlyvee \mathbb{M}_2\Big(q\Big(\frac{(\mathfrak{A}_2)_{\ell k j i}}{\rho}\Big)\Big) \curlyvee \mathbb{M}_3\Big(q\Big(\frac{(\mathfrak{A}_3)_{\ell k j i}}{\rho}\Big)\Big)\Big] = 0 , \text{for some } \rho \succ 0\Big\} .$$

A quadruple sequence $(\mathfrak{A}_{\ell k j i}) \in (\bar{c})_{\mathbb{F}}^{4^{\mathcal{R}}}(\mathbb{M}, q)$ if $(\mathfrak{A}_{\ell k j i}) \in (\bar{c})_{\mathbb{F}}^4(\mathbb{M}, q)$ and there are the following statistical limits exist

$$\operatorname*{stat-lim}_{\ell \to \infty}\Big[\mathbb{M}_1\Big(q\Big(\frac{(\mathfrak{A}_1)_{\ell k j i} - (\mathbb{L}_1)_{k j i}}{\rho}\Big)\Big) \curlyvee \mathbb{M}_2\Big(q\Big(\frac{(\mathfrak{A}_2)_{\ell h g} - (\mathbb{L}_2)_{k j i}}{\rho}\Big)\Big) \curlyvee \mathbb{M}_3\Big(q\Big(\frac{(\mathfrak{A}_3)_{\ell h g} - (\mathbb{L}_3)_{k j i}}{\rho}\Big)\Big)\Big] = 0,$$

for $k, j, i = 1,2, \ldots$.

$$\operatorname*{stat-lim}_{k \to \infty}\Big[\mathbb{M}_1\Big(q\Big(\frac{(\mathfrak{A}_1)_{\ell k j i} - (\mathbb{J}_1)_{\ell j i}}{\rho}\Big)\Big) \curlyvee \mathbb{M}_2\Big(q\Big(\frac{(\mathfrak{A}_2)_{\ell k j i} - (\mathbb{J}_2)_{\ell j i}}{\rho}\Big)\Big) \curlyvee \mathbb{M}_3\Big(q\Big(\frac{(\mathfrak{A}_3)_{\ell k j i} - (\mathbb{J}_3)_{\ell j i}}{\rho}\Big)\Big)\Big] = 0,$$

for $\ell, j, i = 1,2, \ldots$.

$$\operatorname{stat}-\lim_{j\to\infty}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}-(\mathbb{T}_1)_{\ell k i}}{\rho}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}-(\mathbb{T}_2)_{\ell k i}}{\rho}\right)\right)\curlyvee\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}-(\mathbb{T}_3)_{\ell k i}}{\rho}\right)\right)\right]=0,$$

for $\ell, k, i = 1,2, \ldots.$

$$\operatorname{stat}-\lim_{i\to\infty}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}-(\mathbb{S}_1)_{\ell k j}}{\rho}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}-(\mathbb{S}_2)_{\ell k j}}{\rho}\right)\right)\curlyvee\right.$$
$$\left.\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}-(\mathbb{S}_3)_{\ell k j}}{\rho}\right)\right)\right]=0, \text{ for } \ell, k, j = 1,2, \ldots.$$

A quadruple sequence $(\mathfrak{A}_{\ell k j i}) \in (\bar{c}_0)_{\mathbb{F}}^{4^{\mathcal{R}}}(\mathbb{M}, q)$, if $(\mathfrak{A}_{\ell k j i}) \in (\bar{c}_0)_{\mathbb{F}}^{4}(\mathbb{M}, q)$ and there are the following statistical limits exist

$$\operatorname{stat}-\lim_{\ell\to\infty}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k g}}{\rho}\right)\right)\curlyvee\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k g}}{\rho}\right)\right)\right]=0, \text{ for } k, j, i = 1,2, \ldots.$$

$$\operatorname{stat}-\lim_{k\to\infty}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}}{\rho}\right)\right)\curlyvee\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}}{\rho}\right)\right)\right]=0, \text{ for } \ell, j, i = 1,2, \ldots.$$

$$\operatorname{stat}-\lim_{j\to\infty}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}}{\rho}\right)\right)\curlyvee\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}}{\rho}\right)\right)\right]=0, \text{ for } \ell, k, i = 1,2, \ldots.$$

$$\operatorname{stat}-\lim_{i\to\infty}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}}{\rho}\right)\right)\curlyvee\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}}{\rho}\right)\right)\right]=0, \text{ for } \ell, k, j = 1,2, \ldots .$$

These spaces $(\bar{c})_{\mathbb{F}}^{4}(\mathbb{M}, q)$, $(\bar{c}_0)_{\mathbb{F}}^{4}(\mathbb{M}, q)$, $(\bar{c})_{\mathbb{F}}^{4^{\mathcal{R}}}(\mathbb{M}, q)$, $(\bar{c}_0)_{\mathbb{F}}^{4^{\mathcal{R}}}(\mathbb{M}, q)$, $(\bar{c})_{\mathbb{F}}^{4^{\mathcal{B}}}(\mathbb{M}, q)$, $(\bar{c}_0)_{\mathbb{F}}^{4^{\mathcal{B}}}(\mathbb{M}, q)$, $(\bar{c})_{\mathbb{F}}^{4^{\mathcal{R}\mathcal{B}}}(\mathbb{M}, q)$, $(\bar{c}_0)_{\mathbb{F}}^{4^{\mathcal{R}\mathcal{B}}}(\mathbb{M}, q)$ denoted the spaces of statistically convergent of fuzzy numbers in the Pringsheim sense, statistically null of fuzzy numbers in the Pringsheim sense, bounded statistically convergent of fuzzy numbers in the Pringsheim sense, bounded statistically null of fuzzy numbers in the Pringsheim sense, regularly statistically convergent of fuzzy numbers, regularly statistically null of fuzzy numbers, bounded regularly statistically convergent of fuzzy numbers, bounded regularly statistically null of fuzzy numbers.

3. Main results

Theorem 3.1:

The quadruple sequence spaces $\mathbb{Z}(\mathbb{M}, q)$,where $\mathbb{Z} = (\ell_\infty)^4_{\mathbb{F}}, (\bar{c})^{4^{\mathcal{B}}}_{\mathbb{F}}, (\bar{c_0})^{4^{\mathcal{B}}}_{\mathbb{F}}, (\bar{c})^{4^{\mathcal{R}^{\mathcal{B}}}}_{\mathbb{F}}, (\bar{c_0})^{4^{\mathcal{R}^{\mathcal{B}}}}_{\mathbb{F}}$ are semi-normed spaces semi-normed by :

$$\mathfrak{f}\left[\left((\mathfrak{A}_1)_{\ell kji}, (\mathfrak{A}_2)_{\ell kji}, (\mathfrak{A}_3)_{\ell kji}\right)\right] = \inf\left\{\rho \succ 0 : \sup_{\ell kji}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell kji}}{\rho}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell kji}}{\rho}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell kji}}{\rho}\right)\right)\right] \preccurlyeq 1\right\}.$$

Proof:

Since q is a semi-norm , then $\mathfrak{f}(\mathbb{A}) \succcurlyeq 0$, $\forall\, \mathbb{A}$, $\mathfrak{f}(\theta^4) = 0$ and $\mathfrak{f}(\mathfrak{y}\mathbb{A}) = |\mathfrak{y}|\mathfrak{f}(\mathbb{A}), \forall$ scalar $\mathfrak{y}$.

Assume $(\mathfrak{A})_{\ell kji} = \left((\mathfrak{A}_1)_{\ell kji}, (\mathfrak{A}_2)_{\ell kji}, (\mathfrak{A}_3)_{\ell kji}\right)$ and $(\mathfrak{H})_{\ell kji} = \left((\mathfrak{H}_1)_{\ell kji}, (\mathfrak{H}_2)_{\ell kji}, (\mathfrak{H}_3)_{\ell kji}\right)$ and $(\mathfrak{S})_{\ell kji} = \left((\mathfrak{S}_1)_{\ell kji}, (\mathfrak{S}_2)_{\ell kji}, (\mathfrak{S}_3)_{\ell kji}\right) \in (\bar{c})^{4^{\mathcal{B}}}_{\mathbb{F}}(\mathbb{M}, q)$. There is a $\rho_1, \rho_2 \succ 0 \ni$

$$\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell kji}}{\rho}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell kji}}{\rho}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell kji}}{\rho}\right)\right)\right] \preccurlyeq 1 \text{ and } \left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{H}_1)_{\ell kji}}{\rho}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{H}_2)_{\ell kji}}{\rho}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{H}_3)_{\ell kji}}{\rho}\right)\right)\right] \preccurlyeq 1 \text{ and } \left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{S}_1)_{\ell kji}}{\rho}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{S}_2)_{\ell kji}}{\rho}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{S}_3)_{\ell kji}}{\rho}\right)\right)\right] \preccurlyeq 1 .$$

Let $\rho = \rho_1 + \rho_2 + \rho_3$. Then there's ,

$$\sup_{\ell kji}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell kji}+(\mathfrak{H}_1)_{\ell kji}+(\mathfrak{S}_1)_{\ell kji}}{\rho_1+\rho_2+\rho_3}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell kji}+(\mathfrak{H}_2)_{\ell kji}+(\mathfrak{H}_3)_{\ell kji}}{\rho_1+\rho_2+\rho_3}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell kji}+(\mathfrak{H}_3)_{\ell kji}+(\mathfrak{S}_3)_{\ell kji}}{\rho_1+\rho_2+\rho_3}\right)\right)\right] = \sup_{\ell kji}\left[\left(\frac{\rho_1}{\rho_1+\rho_2+\rho_3}\right)\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell kji}+(\mathfrak{H}_1)_{\ell kji}+(\mathfrak{S}_1)_{\ell kji}}{\rho_1}\right)\right) \curlyvee \left(\frac{\rho_2}{\rho_1+\rho_2+\rho_3}\right)\mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell kji}+(\mathfrak{H}_2)_{\ell kji}+(\mathfrak{H}_3)_{\ell kji}}{\rho_2}\right)\right) \curlyvee \left(\frac{\rho_3}{\rho_1+\rho_2+\rho_3}\right)\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell kji}+(\mathfrak{H}_3)_{\ell kji}+(\mathfrak{S}_3)_{\ell kji}}{\rho_3}\right)\right)\right] =$$

$$\sup_{\ell kji}\left(\frac{\rho_1}{\rho_1+\rho_2+\rho_3}\right)\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell kji}}{\rho_1}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell kji}}{\rho_1}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell kji}}{\rho_1}\right)\right)\right] +$$

$$\sup_{\ell kji}\left(\frac{\rho_2}{\rho_1+\rho_2+\rho_3}\right)\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{H}_1)_{\ell kji}}{\rho_2}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{H}_2)_{\ell kji}}{\rho_2}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{H}_3)_{\ell kji}}{\rho_2}\right)\right)\right] +$$

$$\sup_{\ell kji}\left(\frac{\rho_3}{\rho_1+\rho_2+\rho_3}\right)\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{S}_1)_{\ell kji}}{\rho_3}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{S}_2)_{\ell kji}}{\rho_3}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{S}_3)_{\ell kji}}{\rho_3}\right)\right)\right] \preccurlyeq$$

$$\left(\frac{\rho_1}{\rho_1+\rho_2+\rho_3}\right)\sup_{\ell kji}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell kji}}{\rho_1}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell kji}}{\rho_1}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell kji}}{\rho_1}\right)\right)\right] +$$

$\left(\frac{\rho_2}{\rho_1+\rho_2+\rho_3}\right)\sup_{\ell k j i}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{H}_1)_{\ell k j i}}{\rho_2}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{H}_2)_{\ell k j i}}{\rho_2}\right)\right)\curlyvee\mathbb{M}_3\left(q\left(\frac{(\mathfrak{H}_3)_{\ell k j i}}{\rho_2}\right)\right)\right]+$

$\left(\frac{\rho_3}{\rho_1+\rho_2+\rho_3}\right)\sup_{\ell k j i}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{S}_1)_{\ell k j i}}{\rho_3}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{S}_2)_{\ell k j i}}{\rho_3}\right)\right)\curlyvee\mathbb{M}_3\left(q\left(\frac{(\mathfrak{S}_3)_{\ell k j i}}{\rho_3}\right)\right)\right]\preccurlyeq 1.$

Since $\rho_1,\rho_2,\rho_3\succ 0$, we have , $\mathbb{f}\big[\big((\mathfrak{A}_1)_{\ell k j i},(\mathfrak{A}_2)_{\ell k j i},(\mathfrak{A}_3)_{\ell k j i}\big)+$

$\big((\mathfrak{H}_1)_{\ell k j i},(\mathfrak{H}_2)_{\ell k j i},(\mathfrak{H}_3)_{\ell k j i}\big)+\big((\mathfrak{S}_1)_{\ell k j i},(\mathfrak{S}_2)_{\ell k j i},(\mathfrak{S}_3)_{\ell k j i}\big)\big]=\inf\Big\{\rho_1+\rho_2+\rho_3\succ 0:$

$\sup_{\ell k j i}\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}+(\mathfrak{H}_1)_{\ell k j i}+(\mathfrak{S}_1)_{\ell k j i}}{\rho_1+\rho_2+\rho_3}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}+(\mathfrak{H}_2)_{\ell k j i}+(\mathfrak{S}_2)_{\ell k j i}}{\rho_1+\rho_2+\rho_3}\right)\right)\curlyvee\right.$

$\left.\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}+(\mathfrak{H}_3)_{\ell k j i}+(\mathfrak{S}_3)_{\ell k j i}}{\rho_1+\rho_2+\rho_3}\right)\right)\right]\preccurlyeq 1\Big\}\preccurlyeq\inf\Big\{\rho_1\succ 0:\sup_{\ell k j i}\Big[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho_1}\right)\right)\curlyvee$

$\mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}}{\rho_1}\right)\right)\curlyvee\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}}{\rho_1}\right)\right)\Big]\preccurlyeq 1\Big\}+\inf\Big\{\rho_2\succ 0:\sup_{\ell k j i}\Big[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{H}_1)_{\ell k j i}}{\rho_2}\right)\right)\curlyvee$

$\mathbb{M}_2\left(q\left(\frac{(\mathfrak{H}_2)_{\ell k j i}}{\rho_2}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{H}_3)_{\ell k j i}}{\rho_2}\right)\right)\Big]\preccurlyeq 1\Big\}+\inf\Big\{\rho_3\succ 0:\sup_{\ell k j i}\Big[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{S}_1)_{\ell k j i}}{\rho_2}\right)\right)\curlyvee$

$\mathbb{M}_2\left(q\left(\frac{(\mathfrak{S}_2)_{\ell k j i}}{\rho_2}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{S}_3)_{\ell k j i}}{\rho_2}\right)\right)\Big]\preccurlyeq 1\Big\}=\mathbb{f}\big[\big((\mathfrak{A}_1)_{\ell k j i},(\mathfrak{A}_2)_{\ell k j i},(\mathfrak{A}_3)_{\ell k j i}\big)\big]+$

$\mathbb{f}\big[\big((\mathfrak{H}_1)_{\ell k j i},(\mathfrak{H}_2)_{\ell k j i},(\mathfrak{H}_3)_{\ell k j i}\big)\big]++\mathbb{f}\big[\big((\mathfrak{S}_1)_{\ell k j i},(\mathfrak{S}_2)_{\ell k j i},(\mathfrak{S}_3)_{\ell k j i}\big)\big]$.

Theorem 3.2:

Let $(\mathbb{X},q)$ to be a complete semi-normed space .Then the quadruple sequences spaces $\mathbb{Z}(\mathbb{M},q)$,where $\mathbb{Z}=(\ell_\infty)^4_{\mathbb{F}},(\bar{c})^{4^{\mathcal{B}}}_{\mathbb{F}},(\bar{c_0})^{4^{\mathcal{B}}}_{\mathbb{F}},(\bar{c})^{4^{\mathcal{R}^{\mathcal{B}}}}_{\mathbb{F}},(\bar{c_0})^{4^{\mathcal{R}^{\mathcal{B}}}}_{\mathbb{F}}$ are complete semi-normed spaces semi-normed by :

$\mathbb{f}\big[\big((\mathfrak{A}_1)_{\ell k j i},(\mathfrak{A}_2)_{\ell k j i},(\mathfrak{A}_3)_{\ell k j i}\big)\big]=\inf\Big\{\rho\succ 0:\sup_{\ell k j i}\Big[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho}\right)\right)\curlyvee\mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}}{\rho}\right)\right)\curlyvee$

$\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}}{\rho}\right)\right)\Big]\preccurlyeq 1\Big\}$.

Proof:

Suppose $\left(\mathfrak{A}^{gfed}_{\ell k j i}\right)$ be a Cauchy quadruple sequence in $(\bar{c})^{4^{\mathcal{B}}}_{\mathbb{F}}(\mathbb{M},q)$. We must to demonstrate the following:

$i)$ $(\mathfrak{A}_1)^{gfed}_{\ell k j i}\longrightarrow(\mathfrak{A}_1)_{\ell k j i}$ and $(\mathfrak{A}_2)^{gfed}_{\ell k j i}\longrightarrow(\mathfrak{A}_2)_{\ell k j i}$ and $(\mathfrak{A}_2)^{gfed}_{\ell k j i}\longrightarrow(\mathfrak{A}_3)_{\ell k j i}$ as $g,f,e,d\longrightarrow\infty$, $\forall\,(\ell,k,j,i)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

$ii)$ $(\mathfrak{A}_1)_{gfed} \longrightarrow (\mathfrak{A}_1)$ and $(\mathfrak{A}_2)_{gfed} \longrightarrow (\mathfrak{A}_2)$ and $(\mathfrak{A}_3)_{gfed} \longrightarrow (\mathfrak{A}_3)$ as $g,f,e,d \longrightarrow \infty$, where stat-lim $(\mathfrak{A}_1)^{gfed}_{\ell kji} = (\mathfrak{A}_1)_{gfed}$ and stat-lim $(\mathfrak{A}_2)^{gfed}_{\ell kji} = (\mathfrak{A}_2)_{gfed}$ and stat-lim $(\mathfrak{A}_3)^{gfed}_{\ell kji} = (\mathfrak{A}_3)_{gfed}$, $\forall\, g,f,e,d \in \mathbb{N}$.

$iii)$ $(\mathfrak{A}_1)_{\ell kji} \xrightarrow{\text{stat}} (\mathfrak{A}_1)$ and $(\mathfrak{A}_2)_{\ell kji} \xrightarrow{\text{stat}} (\mathfrak{A}_2)$ and $(\mathfrak{A}_3)_{\ell kji} \xrightarrow{\text{stat}} (\mathfrak{A}_3)$.

Assume $\varepsilon \succ 0$, for a fixed $x_0 \succ 0$, choose $r \succ 0 \ni \left[\mathbb{M}_1\left(\frac{rx_0}{3}\right) \curlyvee \mathbb{M}_2\left(\frac{rx_0}{3}\right) \curlyvee \mathbb{M}_3\left(\frac{rx_0}{3}\right)\right] \succcurlyeq 1$ and $\exists\, n_0 \in \mathbb{N}$, $\mathbb{f}\left[\left(\mathfrak{A}^{gfed}_{\ell kji} - \mathfrak{A}^{nmcb}_{\ell kji}\right)\right] \prec \frac{\varepsilon}{rx_0}$, $\forall\, g,f,e,d,n,m,c,b \succcurlyeq n_0$. By the definition of $\mathbb{f}$, we have ,

$$\left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)^{gfed}_{\ell kji} - (\mathfrak{A}_1)^{nmcb}_{\ell hg}}{\mathbb{f}\left[\left((\mathfrak{A}_1)^{gfed}_{\ell kji} - (\mathfrak{A}_1)^{nmcb}_{\ell hg}\right)\right]}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)^{gfed}_{\ell kji} - (\mathfrak{A}_2)^{nmcb}_{\ell kji}}{\mathbb{f}\left[\left((\mathfrak{A}_2)^{gfed}_{\ell kji} - (\mathfrak{A}_2)^{nmcb}_{\ell hg}\right)\right]}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)^{gfed}_{\ell kji} - (\mathfrak{A}_3)^{nmcb}_{\ell kji}}{\mathbb{f}\left[\left((\mathfrak{A}_3)^{gfed}_{\ell kji} - (\mathfrak{A}_3)^{nmcb}_{\ell hg}\right)\right]}\right)\right)\right] \preccurlyeq 1 \preccurlyeq \left[\mathbb{M}_1\left(\frac{rx_0}{3}\right) \curlyvee \mathbb{M}_2\left(\frac{rx_0}{3}\right) \curlyvee \mathbb{M}_3\left(\frac{rx_0}{3}\right)\right],$$

$\forall\, g,f,e,d,n,m,c,b \succcurlyeq n_0 \Longrightarrow q\left((\mathfrak{A}_1)^{gfed}_{\ell kji} - (\mathfrak{A}_1)^{nmcb}_{\ell hg}, (\mathfrak{A}_2)^{gfed}_{\ell kji} - (\mathfrak{A}_2)^{nmcb}_{\ell kji}, (\mathfrak{A}_3)^{gfed}_{\ell kji} - (\mathfrak{A}_3)^{nmcb}_{\ell kji}\right) \prec \frac{rx_0}{3} \cdot \frac{\varepsilon}{rx_0} = \frac{\varepsilon}{3}$, $\forall\, g,f,e,d,n,m,c,b \succcurlyeq n_0$.

Therefore $\left((\mathfrak{A}_1)^{gfed}_{\ell kji}\right), \left((\mathfrak{A}_2)^{gfed}_{\ell kji}\right), \left((\mathfrak{A}_3)^{gfed}_{\ell kji}\right)$ are Cauchy quadruple sequences in $\mathbb{X}$, $\forall\, (\ell,k,j,i) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

Since $\mathbb{X}$ be a complete then there is quadruple sequences $(\mathfrak{A}_1)_{\ell kji}, (\mathfrak{A}_2)_{\ell kji}, (\mathfrak{A}_3)_{\ell kji} \in \mathbb{X} \ni (\mathfrak{A}_1)^{gfed}_{\ell kji} \longrightarrow (\mathfrak{A}_1)_{\ell kji}$ and $(\mathfrak{A}_2)^{gfed}_{\ell kji} \longrightarrow (\mathfrak{A}_2)_{\ell kji}$ and $(\mathfrak{A}_3)^{gfed}_{\ell kji} \longrightarrow (\mathfrak{A}_3)_{\ell kji}$ as $g,f,e,d \longrightarrow \infty$, $\forall\, (\ell,k,j,i) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

$ii)$ We have get stat-lim $(\mathfrak{A}_1)^{gfed}_{\ell kji} = (\mathfrak{A}_1)_{gfed}$ and stat-lim $(\mathfrak{A}_2)^{gfed}_{\ell kji} = (\mathfrak{A}_2)_{gfed}$ and stat-lim $(\mathfrak{A}_3)^{gfed}_{\ell kji} = (\mathfrak{A}_3)_{gfed}$, $\forall\, g,f,e,d \in \mathbb{N}$. then there exists subset $\mathbb{E}_{gfed} \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N} \ni$

$$\mathbb{r}\left(\mathbb{E}_{gfed}\right) = 1, \forall\, g,f,e,d \in \mathbb{N} \text{ and } \forall\, \varepsilon \succ 0 , \left[\mathbb{M}_1\left(q\left(\frac{(\mathfrak{A}_1)^{gfed}_{\ell kji} - (\mathfrak{A}_1)_{gfed}}{\rho}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)^{gfed}_{\ell kji} - (\mathfrak{A}_2)_{gfed}}{\rho}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)^{gfed}_{\ell kji} - (\mathfrak{A}_3)_{gfed}}{\rho}\right)\right)\right] \preccurlyeq \left[\mathbb{M}_1\left(\frac{\varepsilon}{3\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\varepsilon}{3\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\varepsilon}{3\rho}\right)\right],$$

$\forall\, (\ell,k,j,i) \in \mathbb{E}_{gfed}$, $\forall\, g,f,e,d \in \mathbb{N}$ and some $\rho \succ 0 \Longrightarrow q\left(\left((\mathfrak{A}_1)^{gfed}_{\ell kji} - (\mathfrak{A}_1)_{gfed}\right), \left((\mathfrak{A}_2)^{gfed}_{\ell kji} - (\mathfrak{A}_2)_{gfed}\right), \left((\mathfrak{A}_3)^{gfed}_{\ell kji} - (\mathfrak{A}_3)_{gfed}\right)\right) \prec \frac{\varepsilon}{3}$, $\forall\, (\ell,k,j,i) \in \mathbb{E}_{gfed}$, $\forall\, g,f,e,d \in \mathbb{N}$ and by the continuity of $\mathbb{M}$.

Let $g,f,e,d,n,m,c,b \succcurlyeq n_0$ and $(\ell,k,j,i) \in \mathbb{E}_{gfed} \cap \mathbb{E}_{nmcb}$. Then there's ,

$q\left(\left((\mathfrak{A}_1)_{gfed} - (\mathfrak{A}_1)_{nmcb}\right), \left((\mathfrak{A}_2)_{gfed} - (\mathfrak{A}_2)_{nmcb}\right), \left((\mathfrak{A}_3)_{gfed} - (\mathfrak{A}_3)_{nmcb}\right)\right) \preccurlyeq q\left(\left((\mathfrak{A}_1)_{gfed} - (\mathfrak{A}_1)^{gfed}_{\ell kji}\right), \left((\mathfrak{A}_2)_{gfed} - (\mathfrak{A}_2)^{gfed}_{\ell kji}\right), \left((\mathfrak{A}_3)_{gfed} - (\mathfrak{A}_3)^{gfed}_{\ell kji}\right)\right) + q\left(\left((\mathfrak{A}_1)^{gfed}_{\ell kji} - (\mathfrak{A}_1)^{nmcb}_{\ell kji}\right), \left((\mathfrak{A}_2)^{gfed}_{\ell kji} - (\mathfrak{A}_2)^{nmcb}_{\ell kji}\right), \left((\mathfrak{A}_3)^{gfed}_{\ell kji} - (\mathfrak{A}_3)^{nmcb}_{\ell kji}\right)\right) + q\left(\left((\mathfrak{A}_1)^{nmcb}_{\ell kji} - (\mathfrak{A}_1)_{nmcb}\right), \left((\mathfrak{A}_2)^{nmcb}_{\ell kji} - (\mathfrak{A}_2)_{nmcb}\right), \left((\mathfrak{A}_3)^{nmcb}_{\ell kji} - (\mathfrak{A}_3)_{nmcb}\right)\right) \prec \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Consequently $(\mathfrak{A}_1)_{gfed}$, $(\mathfrak{A}_2)_{gfed}$, $(\mathfrak{A}_3)_{gfed}$ are Cauchy quadruple sequences in $\mathbb{X}$, so is complete .

Thus ,

$(\mathfrak{A}_1)_{gfed}$, $(\mathfrak{A}_2)_{gfed}$, $(\mathfrak{A}_3)_{gfed}$ are converges in $\mathbb{X}$ and $(\mathfrak{A}_1)_{gfed} \longrightarrow (\mathfrak{A}_1)$ and $(\mathfrak{A}_2)_{gfed} \longrightarrow (\mathfrak{A}_2)$ and $(\mathfrak{A}_3)_{gfed} \longrightarrow (\mathfrak{A}_3)$.

iii) For $\varepsilon_1 \succ 0$, let's assume $g, f, e, d \succcurlyeq n_0$ and $\rho \succ 0$ be so determined that $\left[\mathbb{M}_1\left(\frac{\varepsilon}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\varepsilon}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{\varepsilon}{\rho}\right)\right] \prec \varepsilon_1$ and the following be a satisfy .

From the (ii), we have get a subset $\mathbb{E} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \ni q\left(\left((\mathfrak{A}_1)^{gfed}_{\ell kji} - (\mathfrak{A}_1)_{gfed}\right), \left((\mathfrak{A}_2)^{gfed}_{\ell kji} - (\mathfrak{A}_2)_{gfed}\right), \left((\mathfrak{A}_3)^{gfed}_{\ell kji} - (\mathfrak{A}_3)_{gfed}\right)\right) \prec \frac{\varepsilon}{3}$.

By means of (i), we arrive that ,

$$q\left(\left((\mathfrak{A}_1)_{\ell kji} - (\mathfrak{A}_1)^{gfed}_{\ell kji}\right), \left((\mathfrak{A}_2)_{\ell kji} - (\mathfrak{A}_2)^{gfed}_{\ell kji}\right), \left((\mathfrak{A}_3)_{\ell kji} - (\mathfrak{A}_3)^{gfed}_{\ell kji}\right)\right) \prec \frac{\varepsilon}{3}, \forall\, g, f, e, d \succcurlyeq n_0 .$$

By means of (ii), we obtain

$$q\left(\left((\mathfrak{A}_1)^{gfed} - (\mathfrak{A}_1)\right), \left((\mathfrak{A}_2)^{gfed} - (\mathfrak{A}_2)\right), \left((\mathfrak{A}_3)^{gfed} - (\mathfrak{A}_3)\right)\right) \prec \frac{\varepsilon}{3}, \forall\, g, f, e, d \succcurlyeq n_0.$$

Consequently $\forall\, g, f, e, d \succcurlyeq n_0$ and for some $\rho \succ 0$ and $\forall\, (\ell, k, j, i) \in \mathbb{E}$ with $\mathbb{r}(\mathbb{E}) = 1$, we get

$$q\left(\left((\mathfrak{A}_1)_{\ell kji} - (\mathfrak{A}_1)\right), \left((\mathfrak{A}_2)_{\ell kji} - (\mathfrak{A}_2)\right), \left((\mathfrak{A}_3)_{\ell kji} - (\mathfrak{A}_3)\right)\right) \preccurlyeq$$

$$q\left(\left((\mathfrak{A}_1)_{\ell kji} - (\mathfrak{A}_1)^{gfed}_{\ell kji}\right), \left((\mathfrak{A}_2)_{\ell kji} - (\mathfrak{A}_2)^{gfed}_{\ell kji}\right), \left((\mathfrak{A}_3)_{\ell kji} - (\mathfrak{A}_3)^{gfed}_{\ell kji}\right)\right) + q\left(\left((\mathfrak{A}_1)^{gfed}_{\ell kji} - (\mathfrak{A}_1)^{gfed}\right), \left((\mathfrak{A}_2)^{gfed}_{\ell kji} - (\mathfrak{A}_2)^{gfed}\right), \left((\mathfrak{A}_3)^{gfed}_{\ell kji} - (\mathfrak{A}_3)^{gfed}\right)\right) + q\left(\left((\mathfrak{A}_1)^{gfed} - (\mathfrak{A}_1)\right), \left((\mathfrak{A}_2)^{gfed} - (\mathfrak{A}_2)\right), \left((\mathfrak{A}_3)^{gfed} - (\mathfrak{A}_3)\right)\right) \prec \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

$\Rightarrow \left[\mathbb{M}_1\left(q\left(\frac{\left((\mathfrak{A}_1)_{\ell k j i}-(\mathfrak{A}_1)\right)}{\rho}\right)\right) \curlyvee \mathbb{M}_2\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}-(\mathfrak{A}_2)}{\rho}\right)\right) \curlyvee \mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}-(\mathfrak{A}_3)}{\rho}\right)\right)\right] \preccurlyeq \left[\mathbb{M}_1\left(\frac{\varepsilon}{\rho}\right) \curlyvee\right.$ $\left.\mathbb{M}_2\left(\frac{\varepsilon}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{\varepsilon}{\rho}\right)\right] = \varepsilon_1$, for some $\rho \succ 0$ and $\forall\ (\ell, k, j, i) \in \mathbb{E}$ with $\mathbb{r}(\mathbb{E}) = 1$. Stat-lim $\mathfrak{A}_{\ell k j i} = \mathfrak{A}$. Consequently $(\mathfrak{A}_{\ell k j i}) \in (\bar{c})_{\mathbb{F}}^{4^{\mathcal{B}}}(\mathbb{M}, q)$.

Thus,

$(\bar{c})_{\mathbb{F}}^{4^{\mathcal{B}}}(\mathbb{M}, q)$ be a complete semi-normed space.

Theorem 3.3:

Let $\mathbb{M}, \mathbb{M}_1$ be two triple maximal Orlicz function , then $\mathbb{Z}(\mathbb{M}_1, q) \subseteq \mathbb{Z}(\mathbb{M} \circ \mathbb{M}_1, q)$, where $\mathbb{Z} = (\ell_\infty)_{\mathbb{F}}^4, (\bar{c})_{\mathbb{F}}^4, (\bar{c_0})_{\mathbb{F}}^4, (\bar{c})_{\mathbb{F}}^{4^{\mathcal{R}}}, (\bar{c_0})_{\mathbb{F}}^{4^{\mathcal{R}}}, (\bar{c})_{\mathbb{F}}^{4^{\mathcal{B}}}, (\bar{c_0})_{\mathbb{F}}^{4^{\mathcal{B}}}, (\bar{c})_{\mathbb{F}}^{4^{\mathcal{R}^{\mathcal{B}}}}, (\bar{c_0})_{\mathbb{F}}^{4^{\mathcal{R}^{\mathcal{B}}}}$ and $\mathbb{M} = (\mathbb{M}_2, \mathbb{M}_3, \mathbb{M}_4)$, $\mathbb{M}_1 = (\mathbb{M}_5, \mathbb{M}_6, \mathbb{M}_7)$.

Proof:

We prove that for the case $\mathbb{Z} = (\bar{c_0})_{\mathbb{F}}^4$. Let $\varepsilon \succ 0$ be a given. Since $\mathbb{M} = (\mathbb{M}_2, \mathbb{M}_3, \mathbb{M}_4)$ be a continuous , so there is $\mathfrak{z} \succ 0 \ni [\mathbb{M}_2(\mathfrak{z}) \curlyvee \mathbb{M}_3(\mathfrak{z}) \curlyvee \mathbb{M}_4(\mathfrak{z})] = \varepsilon$. Let $(\mathfrak{A}_{\ell k j i}) \in (\bar{c_0})_{\mathbb{F}}^4(\mathbb{M}_1, q)$. Then there's a subset $\mathbb{K} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with $\mathbb{r}(\mathbb{K}) = 1 \ni \left[\mathbb{M}_5\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho}\right)\right) \curlyvee \mathbb{M}_6\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}}{\rho}\right)\right) \curlyvee\right.$ $\left.\mathbb{M}_7\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}}{\rho}\right)\right)\right] \prec \mathfrak{z}, \forall\ (\ell, k, j, i) \in \mathbb{K}$.

$[\mathbb{M} \circ \mathbb{M}_1]\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho}\right) \curlyvee q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}}{\rho}\right) \curlyvee q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}}{\rho}\right)\right) = [\mathbb{M}_2, \mathbb{M}_3, \mathbb{M}_4]\left[\mathbb{M}_5\left(q\left(\frac{(\mathfrak{A}_1)_{\ell k j i}}{\rho}\right)\right) \curlyvee\right.$ $\left.\mathbb{M}_6\left(q\left(\frac{(\mathfrak{A}_2)_{\ell k j i}}{\rho}\right)\right) \curlyvee \mathbb{M}_7\left(q\left(\frac{(\mathfrak{A}_3)_{\ell k j i}}{\rho}\right)\right)\right] \prec [\mathbb{M}_2, \mathbb{M}_3, \mathbb{M}_4](\mathfrak{z}) \prec [\mathbb{M}_2(\mathfrak{z}) \curlyvee \mathbb{M}_3(\mathfrak{z}) \curlyvee \mathbb{M}_4(\mathfrak{z})] \prec \varepsilon$.

Consequently $(\mathfrak{A}_{\ell k j i}) \in (\bar{c_0})_{\mathbb{F}}^4(\mathbb{M} \circ \mathbb{M}_1, q)$. Therefore $(\bar{c_0})_{\mathbb{F}}^4(\mathbb{M}_1, q) \subseteq (\bar{c_0})_{\mathbb{F}}^4(\mathbb{M} \circ \mathbb{M}_1, q)$.

Theorem 3.4:

Suppose $\mathbb{M}_1, \mathbb{M}_2$ be two triple maximal Orlicz function, then $\mathbb{Z}(\mathbb{M}_1, q) \cap \mathbb{Z}(\mathbb{M}_2, q) \subseteq \mathbb{Z}(\mathbb{M}_1 + \mathbb{M}_2, q)$,where $\mathbb{Z} = (\ell_\infty)_{\mathbb{F}}^4, (\bar{c})_{\mathbb{F}}^4, (\bar{c_0})_{\mathbb{F}}^4, (\bar{c})_{\mathbb{F}}^{4^{\mathcal{R}}}, (\bar{c_0})_{\mathbb{F}}^{4^{\mathcal{R}}}, (\bar{c})_{\mathbb{F}}^{4^{\mathcal{B}}}, (\bar{c_0})_{\mathbb{F}}^{4^{\mathcal{B}}}, (\bar{c})_{\mathbb{F}}^{4^{\mathcal{R}^{\mathcal{B}}}}, (\bar{c_0})_{\mathbb{F}}^{4^{\mathcal{R}^{\mathcal{B}}}}$, where $\mathbb{M}_1 = (\mathbb{M}_3, \mathbb{M}_4, , \mathbb{M}_5)$, $\mathbb{M}_2 = (\mathbb{M}_6, \mathbb{M}_7, \mathbb{M}_8)$.

Proof:

We prove that the case $\mathbb{Z} = (\bar{c})^4_{\mathbb{F}}$. Assume $(\mathfrak{A}_{\ell \mathscr{k} j i}) \in (\bar{c})^4_{\mathbb{F}}(\mathbb{M}_1, q) \cap (\bar{c})^4_{\mathbb{F}}(\mathbb{M}_2, q)$. Let $\varepsilon \succ 0$ be a given .Then there's a subset $\mathbb{K}$ and $\mathbb{D}$ of $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with

$$\mathbb{r}(\mathbb{K}) = \mathbb{r}(\mathbb{D}) = 1 \ni \left[\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_1)_{\ell \mathscr{k} j i} - \mathbb{L}_1}{\rho_1}\right)\right) \curlyvee \mathbb{M}_4\left(q\left(\frac{(\mathfrak{A}_2)_{\ell \mathscr{k} j i} - \mathbb{L}_2}{\rho_1}\right)\right) \curlyvee \mathbb{M}_5\left(q\left(\frac{(\mathfrak{A}_3)_{\ell \mathscr{k} j i} - \mathbb{L}_3}{\rho_1}\right)\right)\right] \prec \frac{\varepsilon}{2}$$

, $\forall (\ell, \mathscr{k}, j, i) \in \mathbb{K}$, for some $\rho_1 \succ 0$, and $\left[\mathbb{M}_6\left(q\left(\frac{(\mathfrak{A}_1)_{\ell \mathscr{k} j i} - \mathbb{L}_1}{\rho_2}\right)\right) \curlyvee \mathbb{M}_7\left(q\left(\frac{(\mathfrak{A}_2)_{\ell \mathscr{k} j i} - \mathbb{L}_2}{\rho_2}\right)\right) \curlyvee \mathbb{M}_8\left(q\left(\frac{(\mathfrak{A}_3)_{\ell \mathscr{k} j i} - \mathbb{L}_3}{\rho_2}\right)\right)\right] \prec \frac{\varepsilon}{2}$, $\forall (\ell, \mathscr{k}, j, i) \in \mathbb{D}$, for some $\rho_2 \succ 0$,

Suppose $\rho = \max\{\rho_1, \rho_2\}$, then $\forall (\ell, \mathscr{k}, j, i) \in \mathbb{K} \cap \mathbb{D}$, we get ,

$$[\mathbb{M}_1 + \mathbb{M}_2]\left(q\left(\frac{(\mathfrak{A}_1)_{\ell \mathscr{k} j i} - \mathbb{L}_1}{\rho}\right) \curlyvee q\left(\frac{(\mathfrak{A}_2)_{\ell \mathscr{k} j i} - \mathbb{L}_2}{\rho}\right) \curlyvee q\left(\frac{(\mathfrak{A}_3)_{\ell \mathscr{k} j i} - \mathbb{L}_3}{\rho}\right)\right) = [(\mathbb{M}_3, \mathbb{M}_4, \mathbb{M}_5) + (\mathbb{M}_6, \mathbb{M}_7, \mathbb{M}_8)]\left(q\left(\frac{(\mathfrak{A}_1)_{\ell \mathscr{k} j i} - \mathbb{L}_1}{\rho}\right) \curlyvee q\left(\frac{(\mathfrak{A}_2)_{\ell \mathscr{k} j i} - \mathbb{L}_2}{\rho}\right) \curlyvee q\left(\frac{(\mathfrak{A}_3)_{\ell \mathscr{k} j i} - \mathbb{L}_3}{\rho}\right)\right) = \left[\mathbb{M}_3\left(q\left(\frac{(\mathfrak{A}_1)_{\ell \mathscr{k} j i} - \mathbb{L}_1}{\rho_1}\right)\right) \curlyvee \mathbb{M}_4\left(q\left(\frac{(\mathfrak{A}_2)_{\ell \mathscr{k} j i} - \mathbb{L}_2}{\rho_1}\right)\right) \curlyvee \mathbb{M}_5\left(q\left(\frac{(\mathfrak{A}_3)_{\ell \mathscr{k} j i} - \mathbb{L}_3}{\rho_1}\right)\right)\right] + \left[\mathbb{M}_6\left(q\left(\frac{(\mathfrak{A}_1)_{\ell \mathscr{k} j i} - \mathbb{L}_1}{\rho_2}\right)\right) \curlyvee \mathbb{M}_7\left(q\left(\frac{(\mathfrak{A}_2)_{\ell \mathscr{k} j i} - \mathbb{L}_2}{\rho_2}\right)\right) \curlyvee \mathbb{M}_8\left(q\left(\frac{(\mathfrak{A}_3)_{\ell \mathscr{k} j i} - \mathbb{L}_3}{\rho_2}\right)\right)\right] \prec \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \prec \varepsilon$$

. Therefore $(\mathfrak{A}_{\ell \mathscr{k} j i}) \in (\bar{c})^4_{\mathbb{F}}(\mathbb{M}_1 + \mathbb{M}_2, q)$.

4. References

[1] H. Fast "**Sur la convergence statistique**", Colloq. Math., 2(1951), 241-244.

[2] J.A. Fridy & C. Orhan,"**Statistical limit superior and limit inferior**", Proc. Amer. Math. Soc., 125:12(1997), 3625-3631.

[3] I.J. Maddox "**A Tauberion condition for statistical convergence**", Math. Proc. Camb . Phil. Soc., (1989), 272-280.

[4] D. Rath & B.C. Tripathy "**Matrix maps on sequence spaces associated with sets of integers**" ,Indian J. Pure Appl. Math., 27:2(1996), 197-206.

[5] T. Salat," **On statistically convergent sequences of real numbers**", Math. Slovaca, 30:2(1980), 139-150.

[6] I.J. Schoenberg, "**The integrability of certain functions and related summability methods**",Amer. Math. Monthly, 66 (1959), 361-375.

[7] B.C. Tripathy," **On statistical convergence**", Proc. Estonian Acad. Sci., Phy. Math., 47:4(1998), 299-303.

[8] B.C. Tripathy, "**Statistically convergent double sequences**", Tamkang J. Math.,

34:3(2003), 231-237.

[9] A. Zygumd,"**Trigonometric Series**", vol II, Cambridge, 1993.