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A Study of Hausdorff Measure in Fuzzy Soft Metric Spaces

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ABSTRACT

In fields such as mathematics, statistics, computer science, etc., the theory of fuzzy metric spaces is crucial. In this work, a new significant characteristics of the space of fuzzy soft metric are presented, following Kider's (2020) proposal of the fuzzy soft metric model. furthermore, to study hyperspaces in terms of this specific fuzzy soft metric notion. A fuzzy soft distance between two fuzzy soft compact sets is created. A formulation has been proposed for establishing a Hausdorff fuzzy soft metric on the collection of non-empty fuzzy soft compact subsets. Additionally, a variety of essential features of the Hausdorff fuzzy soft metric concept are described.

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1. Introduction

Since uncertainty plays a crucial role in real-world issues, Due to the sufficiency of the parameterization tool, Molodtsov [1] proposed the theory of soft sets, which offers to handle uncertainty more effectively. On the other hand, Zadeh [2] was a pioneer in the field of fuzzy theory, which was developed to solve the problem of ambiguity in control techniques. As a result, fuzzy-set theory has been effectively applied in numerous scientific disciplines. Maji [3]emphasized some of the applications of the theory of soft sets, fuzzy soft sets, and intuitionistic fuzzy soft sets.

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Other studies of fuzzy sets are also available in[4], [5],[6],[7],[8]. For instance, Kramosil and Michalek [9] presented fuzzy metric notions as a modification and continuation of the standard metric spaces by using the idea of ambiguity with the distance framework on a non - empty set. Later, George and Veeramani adjusted their original conditions to produce the Hausdorff topology that was caused by such a fuzzy metric[10], [11]. The category of areas that are strongly related to the family of metrizable topological notions is established by the modified fuzzy metric, despite the fact that it is more limited. Consequently, investigating the linguistic fuzzy metric is fascinating. Kocinac [12] investigated a few fuzzy metric space selection features. Kumar and Mihet [13] established a common fixed point theorem in M-complete fuzzy metric spaces. The subject of fuzzy metric spaces recently made significant progress thanks to Gregori et al.[14],[15], [16],[17],[18], [19]. With a view to analyze hyperspaces in provided fuzzy metric notios, Hausdorff fuzzy metric on the group of non-empty compact subsets of a specific fuzzy metric space was presented in [20]. Kider recently used continuous t-norms to study fuzzy soft metric space in [21]. In keeping with this line of research, we describe the concept of the Hausdorff fuzzy soft metric space. It is essential to look for additional characteristics of the fuzzy soft metric. As a result, we present a novel contribution to the theory of fuzzy soft metrics in a perspective that could be quite attractive given the unquestionable significance of the Hausdorff distance in other branches of computer science and mathematics in addition to general topology.

The content of the article is as follows: In Section 2 a necessary basic information and supplementary findings of the fuzzy metric space (FSM-space) are provide. Section 3 is focused to demonstrating and discussing new fundamental characteristics for FSM-space (X,$\mathrm{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})$,Δ). A fuzzy soft measure between two fuzzy soft compact sets is created, this will assist us to a formulation of the Hausdorff fuzzy soft metric notion is established in Section 4. Additionally, a variety of essential issues of the Hausdorff fuzzy soft metric concept are described.

2. Preliminaries and Basic Results

Many definitions and preliminary findings that will be utilized in this paper are provided in this section. A necessary background information as well as auxiliary findings are provide.

Throughout this paper, $X=\{x_1,\dots,x_{\mathfrak{y}}\}$, I^X and E denote the initial universe, the X-based fuzzy sets group and the collection of parameters for the universe X, respectively.

Definition 2.1[21]: A fuzzy soft set (in short FSS) is a map from the feature set E to I^X, i.e., $\mathfrak{f}_{\mathcal{B}}: E \to I^X$, where $\mathfrak{f}_{\mathcal{B}}(\bar{\mathrm{e}}) \neq 0_X$ if $\bar{\mathrm{e}} \in \mathcal{B} \subseteq \mathrm{E}$ and $\mathfrak{f}_{\mathcal{B}}(\bar{\mathrm{e}}) = 0_X$ if $\bar{\mathrm{e}} \notin \mathcal{B}$, where 0_X is empty fuzzy set on X. The family of all fuzzy soft collections over X will now be replaced $\mathfrak{F}(X,E)$.

Definition 2.2[21]: Let $\mathfrak{f}_{\mathcal{B}} \in \mathfrak{F}(X,E)$ and $\mathfrak{g}_{\mathcal{A}} \in \mathfrak{F}(S,U)$ Then the fuzzy product $\mathfrak{f}_{\mathcal{B}} \times \mathfrak{g}_{\mathcal{A}}$ is defined by $(\mathfrak{f}\times\mathfrak{g})_{\mathcal{B}\times\mathcal{A}}$ Where $(\mathfrak{f}\times\mathfrak{g})_{\mathcal{B}\times\mathcal{A}}\,(\mathfrak{b},\mathfrak{a}) = \mathfrak{f}_{\mathcal{B}}(\mathfrak{b}) \times \mathfrak{g}_{\mathcal{A}}(\mathfrak{a}) \in I^{X\times S}$ $\forall\ (\mathfrak{b},\mathfrak{a}) \in \mathcal{B}\times\mathcal{A}$ .Also $\forall\ (u,v) \in X\times S$ where $[\,\mathfrak{f}_{\mathcal{B}}(\mathfrak{b}) \times \mathfrak{g}_{\mathcal{A}}(\mathfrak{a})](u,v) = \mathfrak{f}_{\mathcal{B}}(\mathfrak{b})(u) \wedge \mathfrak{g}_{\mathcal{A}}(\mathfrak{a})(v)$. According to this definition a mapping $\mathfrak{f}_{\mathcal{B}} \times \mathfrak{g}_{\mathcal{A}}$ is a FSS over $X\times S$ and its parameter universe is $E\times U$.

Definition 2.3[21]: Let $\mathfrak{F}(X,E)$ be the family of all FSS over X, $\mathfrak{f}_{\mathcal{B}}$, $\mathfrak{g}_{\mathcal{A}} \in \mathfrak{F}(X,E)$ and put $\mathrm{P}_{\mathcal{B}\times\mathcal{A}} = (\mathfrak{f}\times\mathfrak{g})_{\mathcal{B}\times\mathcal{A}} = \mathfrak{f}_{\mathcal{B}} \times \mathfrak{g}_{\mathcal{A}}$. Let Δ be a continuous t-norm then $\mathrm{P}_{\mathcal{B}\times\mathcal{A}}$ is a fuzzy soft metric on $X\times X$ if $\forall\, \mathfrak{b} \in \mathcal{B}$, $\mathfrak{a} \in \mathcal{A}$.

(S1) $\mathrm{P}_{\mathcal{B}\times\mathcal{A}}\,(\mathfrak{b},\mathfrak{a})[u,v] > 0 \quad \forall\ u,v\ \in X$

(S2) $\mathrm{P}_{\mathcal{B}\times\mathcal{A}}\,(\mathfrak{b},\mathfrak{a})[u,v] = 1$ if and only if $u = v$

(S3) $\mathrm{P}_{\mathcal{B}\times\mathcal{A}}\,(\mathfrak{b},\mathfrak{a})[u,v] = \mathrm{P}_{\mathcal{B}\times\mathcal{A}}\,(\mathfrak{b},\mathfrak{a})[v,u] \quad \forall\, u,v \in X$

(S4) $\mathrm{P}_{\mathcal{B}\times\mathcal{A}}\,(\mathfrak{b},\mathfrak{a})[u,w] \geq \mathrm{P}_{\mathcal{B}\times\mathcal{A}}\,(\mathfrak{b},\mathfrak{a})[u,v]\ \Delta\ \mathrm{P}_{\mathcal{B}\times\mathcal{A}}\,(\mathfrak{b},\mathfrak{a})[v,w] \quad \forall\, u,v,w \in X$

(S5) $\mathrm{P}_{\mathcal{B}\times\mathcal{A}}\,(\mathfrak{b},\mathfrak{a})[u,v]$ is a continuous function $\forall\ u,v\ \in X$.

Then the triple $(X, \mathrm{P}_{\mathcal{B}\times\mathcal{A}}\,(\mathfrak{b},\mathfrak{a}), \Delta)$ is called a fuzzy soft metric space (briefly, FSM-space) .

The notion for a FSM-space that was considered in [21] is given below.

Proposition2.4[21]: Let (X, d) be metric space and $\alpha \Delta \beta = \alpha\beta \ \forall \ 0 \le \alpha, \beta \le 1$ .Put $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a})[u, v] = [\bar{e} \times \mathcal{P}[d(u, v)]]^{-1}$ then $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$ is a FSM-space $\forall\, \mathfrak{b} \in \mathcal{B}, \mathfrak{a} \in \mathcal{A}$.

Definition 2.5[21]: Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$ be a FSM-space, put $SB(u, r) = \{v \in X: \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a})[v, u] > 1 - r$. Then $SB(u, r)$ is called fuzzy soft open ball centered at $u \in X$ with radius $r \in l$.

Definition 2.6[21]: Suppose that $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$ is a FSM-space and $C \subseteq X$. Then

1- C is fuzzy soft open if $\forall\, c$ in C, $\exists\, r \in l$ with $SB(c, r) \subset C$

2- C is fuzzy soft closed if C^c is fuzzy soft open, that is $C^c = X/C$.

Definition 2.7[21]: In a FSM-space $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a})), \Delta)$, a subset C of X is fuzzy soft bounded if $\exists\ 0 < (1 - r) < 1$ with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a})[u, v] > (1 - r) \ \forall\ u, v$ in C.

The fuzzy soft convergent and fuzzy soft Cauchy sequence concepts, as well as the FSC of a set, are presented in the following group of definitions.

Definition 2.8[21]: A sequence (u_η) in the FSM-space $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$ is referred to as

(1) Fuzzy soft convergent if $\varepsilon > 0$ exists such that $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a})[u_\eta, u]$> 1-$\varepsilon$,$\forall\, \varepsilon > 0$. This is known as the fuzzy soft limit of (u_η), and it is written $\lim_{\eta\to\infty} u_\eta = u$ or maybe just written $u_\eta \to^{s} u$. Is referred to as the (u) fuzzy soft limit.

(2) Fuzzy soft Cauchy if $\forall < 1 - \varepsilon < 0$, $\exists$ an integer number $N \in \mathbb{N}$ such that $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a})[u_\eta, u_m]$> 1-$\varepsilon$, $\forall\, \eta, m \ge N$.

Definition 2.9[21]: Let C be a subset of a FSM-space $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$. Then $FSC\,[C]$ is called the fuzzy soft closure of C which is the smallest fuzzy soft closed set contains.

Definition 2.10[21]: Let (u_η) be a sequence in a FSM-space $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$ fuzzy soft converges to u in X then every subsequence (u_{η_l}) of (u_η) fuzzy soft converges to u.

3. Fuzzy Soft Metric Space's Relation to Other Fuzzy Soft Concepts

This section proves several novel characteristics of a FSM-space $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$. Also provided are some connections between FSM-space and other fuzzy ideas.

Definition 3.1: A FSM-space $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$ is known to be FS-complete if (u_η) is fuzzy soft Cauchy sequence then $u_\eta \to u$ in X.

Definition 3.2: Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$ be a FSM-space. If every sequence in A has a fuzzy soft convergent subsequence with a fuzzy soft limit on an element of E, then the subset E of X is known as fuzzy soft compact.

Definition 3.3: Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$ be a FSM-space. A subset E of X is called fuzzy soft totally bounded if for each $0 < \alpha < 1$, $\exists$ a finite set $\{v_1, v_2, v_3, \dots, v_\eta\} \subset X$ with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a})[u, v_\eta] > 1 - \alpha$ for some v_η in $\{v_1, v_2, v_3, \dots, v_\eta\}$ whenever u in X. The set $\{v_1, v_2, v_3, \dots, v_\eta\}$ is called fuzzy soft α-net.

The following theorem discusses the space's fuzzy soft completeness property in a FSM-space.

Theorem 3.4: Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b}, \mathfrak{a}), \Delta)$ be a FSM-space with the property of the fuzzy soft completeness. A subset E of X is a fuzzy soft compact if it is fuzzy soft closed and fuzzy soft totally bounded.

Proof :

Let E be a fuzzy soft to totally bounded . Let (v_η) be a sequence in E whose range may be assumed to be infinite . choose a finite fuzzy soft (1/2) -net in E then one of the balls of radius (1/2) with center in a fuzzy soft (1/2) -net contains infinitely many elements of the range of the sequence .we shall denote a subsequence formed by these elements by $({v_\eta}^{(1)})$ choose a finite fuzzy soft (1/4)-net in E, then one of the balls of radius (1/4) with center in the finite fuzzy soft (1/4)-net contains infinitely many elements of the range of $(v_{\eta}{}^{(1)})$. We shall denote the subsequence formed as $(v_{\eta}{}^{(2)})$ proceeding in this way, we obtain a sequence of sequences, each one is a subsequence of the preceding one, so that at the rth stage, the terms $(v_{\eta}{}^{(r)})$ lie in the ball of radius $(1/2^r)$ with center in a fuzzy soft $(1/2^r)$-net. Now $(v_{\eta}{}^{(\eta)})$ is a subsequence of (v_{η}). Suppose that $0 < \alpha < 1$ be given choose N so large that $(1-(1/2^{\eta}))\,\Delta\,(1-(1/2^{\eta+1}))\,\Delta \ldots \Delta\,(1-(1/2^{m-1})) > 1-\alpha$.

Then for $m > \eta > N$, we have $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[v_{\eta}{}^{(\eta)}, v_{m}{}^{(m)}\right] \geq$
$\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[v_{\eta}{}^{(\eta)}, v_{\eta+1}{}^{(\eta+1)}\right]\ \Delta\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[v_{\eta+1}{}^{(\eta+1)}, v_{\eta+2}{}^{(\eta+2)}\right]\Delta \ldots \Delta\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[v_{m}{}^{(m-1)}, v_{m}{}^{(m)}\right]$

$$\geq (1-1/2^{\eta}))\,\Delta\,(1-1/2^{\eta+1}))\,\Delta \ldots \Delta\,(1-1/2^{m-1})) > 1-\alpha.$$

So that the sequence $(v_{\eta}{}^{(\eta)})$ is a fuzzy soft Cauchy in E. Since E is fuzzy soft closed and hence E is FS- complete as $(v_{\eta}{}^{(\eta)})$ fuzzy soft converge to v in E. Therefore, E is fuzzy soft compact .

The behavior of a fuzzy soft compact set in a FSM-space is described by the following theorem.

Theorem 3.5: A fuzzy soft compact subset E of a FSM -space $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ is fuzzy soft closed and fuzzy soft totally bounded .

Proof :

Let E be a fuzzy soft compact. In E $\forall\ u \in \mathrm{FSC}[E]$, $\exists$ a sequence with $u_{\eta} \to u$. So $u \in E$ by assumption, hence E is fuzzy soft closed since $u \in FSC[E']$ was arbitrary. Let $0 < \alpha < 1$ be given, assume that does not exists a fuzzy soft α-net for E then there is an infinite sequence (u_{η}) in E with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[u_l, u_{k}\right] \leq 1-\alpha\ \ \forall\ l \neq k$. Nevertheless, this sequence must contain the fuzzy soft convergent subsequence (u_{N_l}) this sequence is a fuzzy soft Cauchy thus that we may select two positive numbers N_l and N_{k} with $N_l \neq N_{k}$ that is $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[u_{N_l}, u_{N_{k}}\right] > 1-\alpha$. But $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[u_{N_l}, u_{N_{k}}\right] \leq 1-\alpha$ hence, we have a contradiction. Thus $\exists$ a fuzzy soft α-net.

4.Basic Characteristics of Fuzzy Soft Hausdorff Measure

The generation of Hausdorff fuzzy soft measure based on FSM-space $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ and its fundamental characteristics are the focus of this section. This notion is the cornerstone to all outcomes. First, we designate the space created by every fuzzy soft compact subset of X as $FSH(X)$. The following auxiliray definition is essential for this structure.

Definition 4.1: Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ be a FSM-space . For each u in X and C in FSH, $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[u, C] = sup\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[u, c]: c \in C\}$ where $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[u, C]$ is called a fuzzy soft distance from the point u to the set C.

Remark 4.2: The set of real numbers $\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}))[u, c]: c \in C\}$ is subset of I it has a supremum point and now we show that it contains its supremum point define $K: C \to I$ by $K(c) = \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[u, c]\ \forall\, c \in C$ then K is fuzzy continuous since $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})$ continuous. Let $\delta = sup\,\{K(c): c \in C\}$ to show that there is $c' \in c$ with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}))[u, c'] = \delta$ we can find sequence (c_l) in C with $(c_l) - c' < \frac{1}{l}$. Now by fuzzy soft compactness of C we have (c_l) has a subsequence (c_{l_i}) with $c_{l_i} \to c'$ in C. Finally, we use fuzzy continuity of K to get $K(c') = \delta$.

Definition 4.3: Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ be a FSM-space. For C, W in $FSH\,(X)$, define $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C, W]$

$= inf\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c, W]: c \text{ in } C\}$. Then $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C, W]$ is called a fuzzy soft distance from C to W.

The following example shows that in general, in a FSM-space $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ if $C, W \in FSH(X)$ then $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C, W]$ and $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[W, C]$ are different (i.e., they are not equal).

Example 4.4: Suppose that $(X = R)$ and $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}): R \to I$ defined by$\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[u, v] = 1/|u - v|$, if $u \neq v$ and $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[u, v] = 1$, if $u = v$. Also, we have $u\,\Delta v = uv, \forall\ 0 < u, v \leq 1$. Then $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[u, v] \neq \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[v, u]$.

Proof:

Let $C = [2,4]$, $W = [5,6]$ be two compact intervals in X. Let us now establish fuzzy soft distance from C to W.

$\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[2, W] = sup\,\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[2,5],\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[2,6]\} = sup\,\{1/3\,, 1/4\} = 1/3.$

And $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[4,W] = sup\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[4,5],\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[4,6]\} = sup\{1, 1/2\} = 1$.

Then $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,W] = inf\{1/3, 1\} = 1/3$.

Now, we establish fuzzy soft distance from W to C.

$\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[5,C] = sup\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[5,2],\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[5,4]\} = sup\{1/3, 1\} = 1$.

And $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[6,C] = sup\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[6,2],\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[6,4]\} = sup\{1/4, 1/2\} = 1/2$. Then $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[W,C] = inf\{1/3, 1/4\} = 1/4$.

So $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,W] = 1/3$ and $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[W,C] = 1/4$. Therefore $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,W] \neq \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[W,C]$.

Remark 4.5: By confirming that C and W are fuzzy soft compact, we could demonstrate that $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,W]$ is properly defined. As a consequence $\exists\ c'$ in C and w' in W which indicates that $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,W] = \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c',w']$.

Definition 4.6: Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ be a FSM-space, a Hausdorff fuzzy soft distance between C, D in $FSH(X)$ is defined $H_{fs}(C,D) = \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,D]\ \Delta\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[D,C]$.

Lemma 4.7 : Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ be a FSM-space. $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,D] = 1$ if and only if $C \subseteq D$.

Proof :

$\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,D] = 1$ if and only if $inf\ \{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,D]: c \text{ in } C\} = 1$ if and only if $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,D] = 1\ \forall\ c \in C$ if and only if $sup\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,d]: d \text{ in } D\} = 1\ \forall\ c \in C$ if and only if $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,d] = 1\ \forall c \in C$ and for some $d \in D$ if and only if $c = d\ \ \forall\ c \in C$ and for some $d \in D$ if and only if $C \subseteq D$.

Lemma 4.8 : Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ be a FSM-space. If $C, D, W \in FSH(X)$ then $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,W] \geq \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,D]\ \Delta\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[D,W]$.

Proof :

We show firstly that $\forall\ c \text{ in } C\ \ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,W] \geq \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,D]\ \Delta\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[d_c,W]$, where $d_c \in D$ satisfies $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,D] = \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,d_c]$. Note that $d_c \in D$ satisfying $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,D] = \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,d_c]$ exists by Remark 4.2.

Now, $\forall\ w \text{ in } W$ there have been $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,W] \geq \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,w] \geq$

$\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,d_c]\ \Delta\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[d_c,w]$. Therefore, by continuity of Δ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,W] \geq \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,d_c]\ \Delta\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[d_c,W]$.

So $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,W] \geq \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,D]\ \Delta\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[d_c,W]$.

Hence, $inf\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,W]: c \text{ in } C\} \geq inf\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,D]: c \text{ in } C\}\ \Delta\ \ inf\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[d_c,W]: c \text{ in } C\}$. Since $\{d_c: c \in C\} \subseteq D$,

$inf\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[d_c,W]: c \text{ in } C\} \geq inf\ \{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[d,W]: d \text{ in } D\}$.

Therefore, $inf\{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,W]: c \text{ in } C\} \geq inf\ \{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c,D]: c \text{ in } C\}\ \Delta\ inf\ \{\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[d,W] : d \text{ in } D\}$ implies that $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,W] \geq \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[C,D]\ \Delta\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[D,W]$.

The new structure of fuzzy soft Hausdorff $(FSH(X), H_{fs}, \Delta)$, as shown by the following theorem, must be FSM-space.

Theorem 4.9: If $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ is a FSM-space then $(FSH(X), H_{fs}, \Delta)$ is a FSM-space .

Proof:

(S1) Since $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,D] > 0$ and $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,C] > 0$ $\forall$ C,D in $FSH(X)$ hence $H_{fs}(C,D) > 0$.

(S2) $H_{fs}(C,D) = 1$ if and only if $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,D]$ Δ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,C] = 1$ if and only if $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,D] = 1$ and $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,C] = 1$ if and only if $C \subseteq D$ and $D \subseteq C$ if and only $= D$.

(S3) $H_{fs}(C,D) =$ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,D]$ Δ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,C] =$ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,C]$ Δ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,D] = H_{fs}(D,C)$ $\forall$ C,D in $FSH(X)$

(S4) $\forall$ C,D,W in $FSH(X)$

$\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,W] \geq$ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,D]\Delta$ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,W]$.

$\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[W,C] \geq$ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[W,D]\Delta$ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,C]$ by Lemma 4.8 . Now,

$H_{fs}(C,W) = \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,W]\Delta$ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[W,C] \geq$
$[\,\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,D]$ Δ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,W]]$ $\Delta[\,\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[W,D]\Delta$ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,C]] =$
$[\,\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,D]\Delta$ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,C]]\Delta[\,\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,W]\Delta$ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[W,D]] = H_{fs}(C,D)\Delta H_{fs}(D,W)$.

(S5) Suppose that $(C_{\mathfrak{y}})$, $(D_{\mathfrak{y}})$ are two sequences in $H_{fs}(X)$ with $C_{\mathfrak{y}} \to C$ and $D_{\mathfrak{y}} \to D$,

$\lim_{\mathfrak{y}\to\infty} H_{fs}(C_{\mathfrak{y}}, D_{\mathfrak{y}}) = \lim_{\mathfrak{y}\to\infty} \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C_{\mathfrak{y}}, D_{\mathfrak{y}}]$ Δ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D_{\mathfrak{y}}, C_{\mathfrak{y}}] = \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,D]\Delta\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[D,C] = H_{fs}(C,D)$. It means that $H_{fs}(C_{\mathfrak{y}}, D_{\mathfrak{y}}) \to H_{fs}(C,D)$. So H_{fs} is continuous. Therefore, $(FSH(X), H_{fs}\,,\Delta)$ is a FSM-space.

Definition 4.10: Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a}),\Delta)$ be a FSM-space and let $C \in FSH(X)$ then we define $C\Delta\,\mu = \{x \in X{:}\, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[c,x] > 1-\mu$ for some c in C, $0 < \mu < 1\}$.

The remaining paragraphs of this section are devoted to demonstrating various FM-space theorems and properties using the fuzzy soft Hausdorff construct.

Theorem 4.11: Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a}), min\,)$ be a FSM-space and Let C, W belongs to $FSH(X)$. Then for given $0 < \mu < 1$, $FSH(C,W) > 1-\mu$ if and only if $C \subset W\Delta\,\mu$ and $\subset C\,\Delta\,\mu$.

Proof :

To show $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,W] > 1-\mu$ if and only if $C \subset W\Delta\,\mu$. Consider that for C,W in $FSH(X)$, $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,W] > 1-\mu$ then $\inf\{\,\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[c,W]{:}\, c$ in $C\} > 1-\mu$ implies $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[c,W] > 1-\mu$ $\forall\, c \in C \subseteq X$. So $\forall\, c \in C$ we have $\in W\Delta\,\mu$. Hence $C \subset W\Delta\,\mu$. Conversely suppose that $C \subset W\Delta\,\mu$. Let $c \in C$ so $\exists\, w \in W$ with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[c,w] > 1-\mu$ $\forall\, c \in C$. So $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[c,W] > 1-\mu$ this is genuine $\forall\, c$ in C thus $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,W] > 1-\mu$. Correspondingly, we may conclude that $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[W,C] > 1-\mu$ if and only if $W \subset C\Delta\,\mu$. Now, $H_{fs}(C,W) > 1-\mu$ if and only if $[\,\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,W]$ Δ $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[W,C]] > 1-\mu$ if and only if $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[C,W] > 1-\mu$ and $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[W,C] > 1-\mu$ if and only if $C \subset W\Delta\,\mu$ and $\subset C\,\Delta\,\mu$.

Remark 4.12: Let $(FSH(X), H_{fs}, \Delta)$ be the space of fuzzy soft Hausdorff and $(c_{\mathfrak{y}})$ be a fuzzy soft Cauchy sequence in $FSH(X)$ then for any $0 < \mu < 1, \exists$ $N \in \mathbb{N}$ with $H_{fs}(C_{\mathfrak{y}}, C_{\mathfrak{m}}) > 1-\mu$ or $C_{\mathfrak{y}} \subseteq C_{\mathfrak{m}}\,\Delta\,\mu$ and $C_{\mathfrak{m}} \subseteq C_{\mathfrak{y}}\,\Delta\,\mu$ $\forall\, \mathfrak{y}, \mathfrak{m} \geq N$.

Theorem 4.13: Let $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a}),\Delta)$ be a FSM-space and Let $(c_{\mathfrak{y}})$ be a fuzzy soft Cauchy sequence in $(FSH(X), H_{fs}, \Delta)$. let $(\mathfrak{y}_l)$ represent an infinite sequence of positive numbers, $0 < \mathfrak{y}_1 < \mathfrak{y}_2 < \ldots < \mathfrak{y}_l < \cdots$. Suppose that $(c_{\mathfrak{y}_i}) \in C_{n_i}$ is fuzzy soft Cauchy in X. Then there is a fuzzy soft Cauchy $(\bar{c}_{\mathfrak{y}}) \in C_{\mathfrak{y}}$ with $\bar{c}_{\mathfrak{y}_i} =$ $c_{\mathfrak{y}_i}$ $\forall\, i \in \mathbb{N}$.

Proof :

The construction of the sequence $(\,\bar{c}_{\mathfrak{y}}\,)$ in $C_{\mathfrak{y}}$ is as follows: for $\mathfrak{y} \in \{1,2,\ldots,\mathfrak{y}_1\}$ choose $\bar{c}_{\mathfrak{y}} \in \{c \in C_{\mathfrak{y}}{:}\, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[c, c_{\mathfrak{y}_i}] = \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[c_{\mathfrak{y}_i}, C_{\mathfrak{y}}]\}$ given that $C_{\mathfrak{y}}$ is fuzzy soft compact, $\bar{c}_{\mathfrak{y}}$ must exist. Similarly for $i \in \{2,3,\ldots\}$ and $\forall$ $\mathfrak{y} \in$ $\{\mathfrak{y}_i + 1\,, \mathfrak{y}_i + 2, \ldots, \mathfrak{y}_i + 1\}$ choose $\bar{c}_{\mathfrak{y}} \in \{c \in C_{\mathfrak{y}}{:}\, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[c, c_{\mathfrak{y}_i}] = \mathcal{P}_{\mathcal{B}\times\mathcal{A}}$ $(\mathfrak{b},\mathfrak{a})[c_{\mathfrak{y}_i}, C_{\mathfrak{y}}]\}$. Clearly $\bar{c}_{\mathfrak{y}_i} = c_{\mathfrak{y}_i}$ using our design. Given that $(c_{\mathfrak{y}_i}) \in C_{\mathfrak{y}_i}$ is a fuzzy soft Cauchy sequence in X. Let $0 < \rho < 1$ be given, then there will be $\eta_i, \eta_l \geq N_1$ with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{\eta_i}, c_{\eta_l}] \geq 1-\rho$. Additionally, because (C_η) is a fuzzy soft Cauchy sequence in $FSH(X)$, There really is N_2 with $H_{fs}(C_i, C_l) > 1-\rho\ \forall\, i, l \geq N_2$. Put currently $N = min\{N_1, N_2\}$ and for $j, l \geq N$ there are $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[\bar{c}_j, \bar{c}_l] \geq \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[\bar{c}_j, c_{\eta_i}]\,\Delta\, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{\eta_i}, c_{\eta_l}]\,\Delta\,\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{\eta_l}, \bar{c}_l]$ where $j \in \{\eta_{i-1}+1, \eta_{i-1}+2, \ldots, \eta_i\}$ and $l \in \{\eta_{l-1}+1, \eta_{l-1}+2, \ldots, \eta_l\}$. But, $H_{fs}(c_j, c_{\eta_i}) > 1-\rho$ so there is $\bar{c}_j \in C_j \cap [(c_{\eta_i})\,\Delta\rho]$ hence $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[\bar{c}_j, c_{l_i}] > 1-\rho$. Similar, we can demonstrate $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{\eta_l}, \bar{c}_\eta] > 1-\rho$. Therefore, $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[\bar{c}_j, \bar{c}_l] \geq 1-\rho\,\Delta 1-\rho\,\Delta 1-\rho$. So that we can find $0<\mu<1$ with $1-\rho\,\Delta 1-\rho\,\Delta 1-\rho > 1-\mu$ hence $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[\bar{c}_j, \bar{c}_l] > 1-\mu\ \ \forall\, j, l \geq N$. Therefore, $(\bar{c}_\eta)$ in C_η is a fuzzy soft Cauchy sequence.

Proposition 4.14: $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ be a FSM-space with the property of the fuzzy soft completeness. If (C_η) is a fuzzy soft Cauchy sequence in $(FSH(X), H_{fs}, \Delta)$ that satisfy the condition $C_\eta \to^s C$ in $FSH(X)$ with the following characteristic $C = \{\, x \in X : \exists\ a\ fuzzy\ soft\ Cauchy\ \ (c_\eta)\ in\ C_\eta\ with\ c_\eta \to^s x\ \ \}$ then C is a non-empty set.

Proof :

Suppose that (N_η) be a fuzzy soft Cauchy sequence in $FSH(X)$ with $H_{fs}(C_i, C_l) > 1-(1/2^\eta)\ \ \forall\, i, l \geq N_\eta$ where $N_\eta \in \mathbb{N}$ with $N_1 < N_2 < \ldots < N_m < \cdots$. Choose c_{N_1} in C_{N_1} then since $H_{fs}(C_{N_1}, C_{N_2}) > 1-(1/2)$ so that we can discover c_{N_2} in C_{N_2} with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{N_1}, c_{N_2}] > 1-(1/2)$. Suppose we have specified a finite sequence c_{N_i} in C_{N_i}, $i = 1,2\ldots,\eta$ with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{N_{i-1}}, c_{N_i}] > 1-(1/2^{i-1})$. So because $H_{fs}\left(C_{N_\eta}, C_{N_{\eta-1}}\right) > 1-(1/2^\eta)$ and c_{N_η} in C_{N_η} we can discover $c_{N_{\eta+1}}$ in $C_{N_{\eta+1}}$ with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[c_{N_\eta}, c_{N_{\eta-1}}\right] > 1-(1/2^\eta)$. As an illustration, $c_{N_{\eta-1}}$ be a point in $C_{N_{\eta+1}}$ the nearest to c_{N_η}. A sequence (c_{N_i}) in C_{N_i} can be discovered via induction with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{N_i}, c_{N_{i-1}}] > 1-(1/2^i)$. Next, we demonstrate that (c_{N_i}) is a fuzzy soft Cauchy in X, let $0<\alpha<1$ and choose N_α with $1-(1/2)\,\Delta\,1-(1/2^2)\,\Delta \ldots \Delta\ 1-(1/2^{N_i-N_l}) > 1-\alpha$. Then we have

$\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{N_i}, c_{N_l}] \ \geq\ \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{N_i}, c_{N_{i+1}}]\,\Delta\,\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{N_{i+1}}, c_{N_{i+2}}]\,\Delta \ldots \Delta\,[c_{N_{l-1}}, c_{N_l}]$ then

$\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})[c_{N_{l-1}}, c_{N_l}] \geq 1-(1/2)\,\Delta\,1-(1/2^2)\,\Delta \ldots \Delta\,1-\left(1/2^{N_i-N_l}\right) > 1-\alpha$. Thus, by Theorem 4.13 $\bar{v}_{N_j} = c_{N_j}$ exists for a fuzzy soft convergent subsequence $(\bar{v}_j)$ in C_j and in C , $lim\bar{v}_j$ exists. Therefore, C is a non-empty set.

Proposition 4.15: $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$ be a FSM-space with the property of the fuzzy soft completeness. If (C_η) is a fuzzy soft Cauchy sequence in $(FSH(X), H_{fs}, \Delta)$ that satisfy the condition $C_\eta \to^s C$ in $FSH(X)$ with the following characteristic $C = \{\, x \in X : \exists\ a\ fuzzy\ soft\ Cauchy\ \ (c_\eta)\ in\ C_\eta\ with\ c_\eta \to^s x\ \ \}$ then C is fuzzy soft complete.

Proof :

Let (c_j) in C_j with $c_j \to^s c$, we demonstrate that c in C. $\forall\, j, \exists$ a sequence $c_j^{(\eta)}$ in C_η with $c_j^{(\eta)} \to^s c_j$ as $\eta \to \infty$.There is an increasing sequence (N_j) with $N_j \in \mathbb{N}$ such that $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[c_{N_j}, c\right] > 1-\ (1/j)$.Further,$\exists$ a sequence (l_j) with $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[c_{N_j}^{(l_j)}, c_{N_j}\right] > 1-(1/j)$.Therefore, $\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[c_{N_j}^{(l_j)}, c\right] \geq \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[c_{N_j}^{(l_j)}, c_{N_j}\right]\Delta\mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a})\left[c_{N_j}, c\right]$

$$\geq 1-(1/j)\,\Delta\,1-(1/j)$$

Put $c_{N_j}^{(l_j)} = u_{l_j}$ we note that u_{l_j} in C_{l_j} and $u_{l_j} \to^s c$. u_{l_j} can now be expanded to a convergent sequence (v_j) in C_j hence c in C thus C is fuzzy soft closed. As X is fuzzy soft Complete, C is also fuzzy soft Complete.

5.Conclusion

Between two point sets, the Hausdorff distance is a measurement that is defined. Following the presentation and discussion of a few novel fundamental properties for the FSM-space $(X, \mathcal{P}_{\mathcal{B}\times\mathcal{A}}(\mathfrak{b},\mathfrak{a}), \Delta)$. The Hausdorff metric in the sense of fuzzy soft metric notion are formulated because of the undoubted importance of the Hausdorff measure in general topology as well as other fields of mathematics and computer science. In addition, a number of significant problems with the Hausdorff fuzzy soft metric approach are described.

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