Codisk-cyclic Mixing Operators

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1. Introduction

Let $H$ be an infinite dimensional separable complex Hilbert space, and $T$ be a bounded linear operator, i.e $T \in B(H)$. In 2002, Jamil introduced a codisk-cyclicity concept as: $T$ is called codisk-cyclic operator if there is a non-zero vector $x \in H$ such that $\{\alpha T^n x: n \geq 0, \alpha \in \mathbb{C}; |\alpha| \geq 1\}$ is dense in $H$[3].

She gave criterion to investigate if an operator is codisk-cyclic or not. Recently, many authors studied codisk-cyclic operators[2],[4]. Jamil showed that the direct sum of two codisk - cyclic operators is codisk-cyclic. But, in general, the converse is not true [3]. This motivates us to present codiskccyclicly mixing concept. $T$ is called a codisk cyclic operator, $CD$ - mixing operator, if for any non-empty open subsets $U, V$ of $H$, there are $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$; such that $T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$. After studied its characterizations, we showed that the direct sum of two operators is codiskk-mixing if and if they are, also we proved that the direct sum is a codisk- cyclic operator if one of them is a codisk - cyclic and the other is a $CD$ - mixing operator. We remark that $\mathbb{N}$ in this paper stands for the natural set with zero, and $\mathbb{B}^c$ the complement of the unit ball with zero center.
2 The Characterization of Codisk-Cyclic Mixing Operators.

In this section, we introduce the concept of a codisk-mixing operator, and prove some characterization to CD-mixing operators by using CD-return sets.

Definition (2.1): Let $T \in B(H)$, $T$ is said to be a codisk-mixing operator, abbreviate by CD-mixing, if $U, V$ be any non-empty open subsets of $H$, there exist $N \in \mathbb{N}$ and $\alpha \in \mathbb{B}^c$ such that $T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$.

Now, we discuss the characterization of D-mixing operators by CD-return sets which is defined as:

Definition (2.2): Let $T \in B(H)$, and $U, V$ be any non-empty open subsets of $H$, $\alpha \in \mathbb{B}^c$. The set $N_{CD}(U, V) = N_{T}^{CD}(U, V) := \{n \in \mathbb{N} : T^n(\alpha U) \cap V \neq \emptyset ; \alpha \in \mathbb{B}^c\}$

is called CD-return set.

Proposition (2.3): Let $T \in B(H)$ is CD-mixing if and only if for all $U, V$ are non-empty open subsets of $H$. $N^B(U, V)$ is a co-finite set.

Proof: (⇒) Let $T \in B(H)$ is CD-mixing, and $U, V$ be any non-empty open subsets of $H$, then there exist $N \in \mathbb{N}$ and $\alpha \in \mathbb{B}^c$ such that $T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$.

Note that

$$N - N_{CD}(U, V) = N - \{n \in \mathbb{N} : T^n(\alpha U) \cap V \neq \emptyset ; \alpha \in \mathbb{B}^c\}$$

Thus $N_{CD}(U, V)$ is a co-finite set.

(⇐) Since $N_{CD}(U, V)$ is a co-finite set, then there exist $k \in \mathbb{N}$ such that

$$N - \{n \in \mathbb{N} : T^n(\alpha U) \cap V \neq \emptyset ; \alpha \in \mathbb{B}^c\} = \{0, 1, \ldots, k\}$$

So, $T^{k+n}(\alpha U) \cap V \neq \emptyset$, for all $n > k$. Hence $T$ is CD-mixing.

Next proposition gives another equivalent relation between the CD-mixing operators and CD-return sets with neighborhood of zero. But first we will need the following lemma.

Lemma (2.4) [1]: If $U$ be a non-empty open set in $H$, then there is a non-empty open subset $U_1 \subset U$ and $W$ be a neighborhood of zero such that $U_1 + W \subset U$. If $W$ is a neighborhood of zero, then there is a neighborhood of zero $W_1$ such that $W_1 + W_1 \subset W$.

Proposition (2.5): An operator $T$ is CD-mixing if and only if, for any nonempty open subset $U$ in $H$ and any neighborhood of zero, $W$, the CD-return sets $N_{CD}(U, W)$ and $N_{CD}(W, U)$ are co-finite set.

Proof: (⇒) Let $T$ be a CD-mixing, $U, V$ be nonempty open sets, and $W$ is a neighborhood of zero, thus there exist an open ball $Z$ such that $0 \in Z \subset W$.

Since $T$ is CD-mixing, therefore, by Proposition (2.3), $N_{CD}(U, W) \supseteq N_{CD}(U, Z)$ is a co-finite set. Similarly, $N_{CD}(W, V) \supseteq N_{CD}(Z, V)$ is a co-finite set.

(⇐) Let $U, V$ be non-empty open subsets of $H$. By Lemma (2.4), there are non-empty open subsets $U_1, V_1$ and $W_0, \tilde{W}_0$ neighborhoods of zero, such that

$$U_1 \subset U \text{ and } W_0 \subset W; \ U_1 + W_0 \subset U \ldots (1)$$

and

$$V_1 \subset V \text{ and } \tilde{W}_0 \subset V; \ V_1 + \tilde{W}_0 \subset V \ldots (2)$$

Let $W = W_0 \cap \tilde{W}_0$. By the hypothesis, $N_{CD}(U, W), N_{CD}(W, V_1), N_{CD}(V_1, W)$ and $N_{CD}(W, U_1)$ are co-finite set.

Hence there exist $N_i \in \mathbb{N}$, and $\alpha_i \in \mathbb{B}^c$; $i = 1, 2, 3, 4$, such that

$$sT^{n_i}(\alpha_i U_1) \cap W \neq \emptyset, \text{ for all } n_i \geq N_1$$

$$T^{n_i}(\alpha_i W) \cap V_i \neq \emptyset, \text{ for all } n_i \geq N_2$$

$$T^{n_i}(\alpha_i V_i) \cap W \neq \emptyset, \text{ for all } n_i \geq N_3$$

$$T^{n_i}(\alpha_i U_1) \cap W \neq \emptyset, \text{ for all } n_i \geq N_4$$

Put $N = \max\{N_1, N_2, N_3, N_4\}$ for all $n \geq N$, and $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, we get

$$T^n(\alpha U_1) \cap W \neq \emptyset \ldots (3)$$

$$T^n(\alpha W) \cap V_1 \neq \emptyset \ldots (4)$$

From (3) there exist $u \in U$ such that $T^n(\alpha u) \in W$. While from (4) there exist $w \in W$ such that $T^n(\alpha w) \in V_1$. Thus from (1), $u + w \in U$ and from (2) we get $T^n(\alpha (u + w)) \subset V$. Then $T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$. Which implies $T$ is CD-mixing.
3. The Sufficient Conditions Of CD-Mixing Operator

In this section we investigate a sufficient condition of an operator to be CD-mixing.

Proposition (3.1) [CD-mixing Comparison Principle]:
Let $T \in B(H), S \in B(K)$. If there $X \in B(H, K)$ such that $SX = XT$, then $S$ is CD-mixing if $T$ is CD-mixing.

Proof:
Let $U, V$ be non-empty open sets of $K$, by continuity of $X, X^{-1}U, X^{-1}V$ are non-empty open sets of $H$. Since $T$ is CD-mixing, then there exist $N \in \mathbb{N}$ and $\alpha \in \mathbb{B}^c$, such that for all $n \geq N$, $T^n(X^{-1}(\alpha U)) \cap X^{-1}V \neq \emptyset$. So, there are $u \in X^{-1}(U)$ such that $T^n(\alpha u) \in X^{-1}(V)$, hence $X(T^n(\alpha u)) \subset V$. Therefore,

$$S^n(X(\alpha u)) = X(T^n(\alpha u)) \subset V \quad \ldots (8).$$

Since $u \in X^{-1}(U)$, so

$$\alpha Xu \in \alpha U \quad \ldots (9).$$

From (8) and (9) we get $S^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$. Then $S$ is CD-mixing.

Theorem (3.2): [CD-mixing Criterion]
Let $T \in B(H)$. If there exists $\alpha \in \mathbb{B}^c$ and $N \in \mathbb{N}$, For which there are a dense subsets $Y, X$ in $H$ and a sequence of mappings, $S_n: Y \to H$, for all $n \geq N$ such that:

1) $\alpha T^n x = 0$ for all $x \in X$
2) a) $\frac{1}{\alpha} S_n y \to 0$ for all $y \in Y$
   b) $T^n S_n y \to y$ for all $y \in Y$

Then $T$ is CD-mixing.

Proof:
Let $U, V$ be non-empty open sets of $H$, let $x \in X \cap U, \ y \in Y \cap V$.

By (2(a)) we get $x + \frac{1}{\alpha} S_n y \to x \in U$ for all $n \geq N$, thus,

$$\alpha T^n \left( x + \frac{1}{\alpha} S_n y \right) = \alpha T^n x + T^n S_n y \to y \in V \quad \text{for all } n \geq N.$$ 

Therefore, $T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$. Hence $T$ is CD-mixing.

4. The Sufficient Condition of CD-Mixing Operator

The goal of this section is studying the direct sum of two CD-mixing operators.

Proposition (4.1): Let $T, S \in B(H)$. Then $S \oplus T$ is CD-mixing in $H \oplus H$ if and only if $S$ and $T$ are CD-mixing operators.

Proof: ($\Rightarrow$) Let $U_1, U_2, V_1, V_2$ be any non-empty open subsets of $H$, since $S \oplus T$ is CD-mixing, then there exist $N \in \mathbb{N}$ and $\alpha \in \mathbb{B}^c$ such that for all $n \geq N$,

$$(S^n(\alpha U_1) \cap V_1) \oplus (T^n(\alpha U_2) \cap V_2) = (S \oplus T)^n(\alpha(U_1 \oplus U_2)) \cap (V_1 \oplus V_2) \neq \emptyset.$$ 

Thus, for all $n \geq N$, $S^n(\alpha U_1) \cap V_1 \neq \emptyset$ and $T^n(\alpha U_2) \cap V_2 \neq \emptyset$. So $S$ and $T$ are CD-mixing.

($\Leftarrow$) Let $S$ and $T$ be CD-mixing operators, then for any $U_1, U_2, V_1, V_2$ of non-empty open subsets of $H$, there exist $N_1, N_2 \in \mathbb{N}$ and $\alpha_i \in \mathbb{B}^c$: $i = 1, 2$, such that for all $n_1 \geq N_1$ and $n_2 \geq N_2$ we get

$$S^{n_1}(\alpha_1 U_1) \cap V_1 \neq \emptyset \text{ and } T^{n_2}(\alpha_2 U_2) \cap V_2 \neq \emptyset.$$ 

Let $N = \max\{N_1, N_2\}$ and $\alpha = \max\{\alpha_1, \alpha_2\}$. Hence, for all $n \geq N$

$$(S^n(\alpha U_1) \cap V_1) \oplus (T^n(\alpha U_2) \cap V_2) = (S \oplus T)^n(\alpha(U_1 \oplus U_2)) \cap (V_1 \oplus V_2) \neq \emptyset.$$ 

So $S \oplus T$ is CD-mixing.

Recall that a bounded linear operator $T$ is called codisk-cyclic operator if there is a non-zero vector $x \in H$ such that $\{\alpha T^n x: n \geq 0, \alpha \in \mathbb{B}^c\}$ is dense in $H$. [3]. It is well – know that there is a direct sum of two codisk cyclic operators which is not codisk-cyclic operator [3]. The following proposition discuss this case. But first we need the following lemma.
Lemma (4.2): Let $T \in \mathcal{CD}(H)$, then for any pair $U, V$ of non-empty open subsets of $H$, $\alpha \in \mathbb{B}^c$, $T^n(\alpha U) \cap V \neq \emptyset$ is infinite.

Proof: Let $U, V$ be non-empty open subsets of $H$. Since $T \in \mathcal{CD}(H)$, there exist $n_1 \in \mathbb{N}$, $\alpha_1 \in \mathcal{D}$; $i = 1, 2$, such that $T^{n_i}(\alpha_1 U) \cap V \neq \emptyset$. So, $U \cap T^{-n_1}(\frac{1}{\alpha_1} V) \neq \emptyset$. Let $W = T^{-n_1}(\frac{1}{\alpha_1} V)$, since $T$ is continuous, then $W$ is open. But $T \in \mathcal{CD}(H)$, so there exist $n_2 \in \mathbb{N}, \alpha_2 \in \mathbb{B}^c$, such that $T^{n_2}(\alpha_2 U) \cap W \neq \emptyset$.

So, $T^{n_2}(\alpha_2 U) \cap T^{-n_1}(\frac{1}{\alpha_1} V) \neq \emptyset$, hence $T^{n_2+n_1}(\alpha_1 \alpha_2 U) \cap V \neq \emptyset$, and so on. Therefore, there are infinite natural number $n$ such that $T^n(\alpha U) \cap V \neq \emptyset$.

Proposition (4.3): Let $T, S \in B(H)$. If $T$ is a $\mathcal{CD}$-mixing operator and $S$ is a codisk-cyclic operator, then $S \oplus T$ is a codisk-cyclic operator.

Proof: Let $U_1, U_2, V_1, V_2$ be any non-empty open subsets of $H$, since $T$ is $\mathcal{CD}$-mixing and $S$ is codisk-cyclic, then there exist $N, k \in \mathbb{N}$ and $\alpha_1, \alpha_2 \in \mathbb{B}^c$

such that $T^n(\alpha_1 U_1) \cap V_2 \neq \emptyset$ for all $n \geq N$ and $S^k(\alpha_2 U_2) \cap V_1 \neq \emptyset$.

Now, if $k < N$, then by Lemma (4.2), there exist $m \geq N$, such that $S^m(\alpha_1 U_1) \cap V_1 \neq \emptyset$. Put $\alpha = \max \{\alpha_1, \alpha_2\}$,

$$S \oplus T)^m(\alpha(U_1 \oplus U_2) \cap (V_1 \oplus V_2) \neq \emptyset$$

Hence $S \oplus T$ is a codisk-cyclic operator. The result is done when $k \geq N$.

Here a natural question appears, can we generalize the proposition (4.3) to a finite number of direct summand operators?

Corollary (3.4): Let $T_i$ be a codisk-cyclic operator, and $(T_i)_{i=1}^n$ be a sequence of $\mathcal{CD}$-mixing operator, then for all $n \in \mathbb{N}$, $\bigoplus_{i=1}^n T_i$ is a codisk-cyclic operator.

Proof: By induction. If $n = 2$, then by proposition (2.3), $T_1 \oplus T_2$ is a disk-cyclic operator. Suppose it is true when $n = k$. Now that $n = k + 1$ thus $\bigoplus_{i=1}^n T_i = \bigoplus_{i=1}^{k} T_i \oplus T_{k+1}$. So by Proposition (4.3), $\bigoplus_{i=1}^{k} T_i \in \mathcal{CD}(H)$.

4. Conclusion

Let $H$ be an infinite dimensional separable complex Hilbert space, and $T$ be a bounded linear operator. $T$ is called codisk-mixing operator, $\mathcal{CD}$-mixing operator, if for any non-empty open subsets $U, V$ of $H$, there are $n \in \mathbb{N}$ and $\alpha \in \mathbb{B}^c$ such that $T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$.

In this paper, we studied a characterization of $\mathcal{CD}$-mixing operators, and discussed when a direct sum of two codisk cyclic operators which is codisk-cyclic operator. We showed that if one of them is $\mathcal{CD}$-mixing operator and the other is codisk cyclic operator, then the direct sum of them is codisk-cyclic operator.

References