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Codisk-cyclic Mixing Operators

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ABSTRACT

Let H be an infinite dimensional separable complex Hilbert space, and T be a bounded linear operator. T is called codisk mixing operator, $C\mathbb{D}$- mixing operator, if for any non-empty open subsets U, V of H, there are $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ such that $T^n(\alpha U) \cap U \neq \emptyset$ for all $n \geq N$. In this paper, we studied a necessarily and sufficiently conditions of $C\mathbb{D}$- mixing operators, and discused the direct sum of two $C\mathbb{D}$ - mixing operators.

MSC..

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1. Introduction

Let $\boldsymbol{H}$ be an infinte dimensional separable complex Hilbert space, and $\boldsymbol{T}$ be a bounded linear operator, i.e $\boldsymbol{T} \in \boldsymbol{B}(\boldsymbol{H})$. In 2002, Jamil introduced a codisk-cyclicity concept as**:** $\boldsymbol{T}$ is called codisk-cyclic operator if there is a non-zero vector $\boldsymbol{x} \in \boldsymbol{H}$ such that $\{\boldsymbol{\alpha T^n x}: \boldsymbol{n} \geq \boldsymbol{0}, \boldsymbol{\alpha} \in \mathbb{C};\ |\boldsymbol{\alpha}| \geq \boldsymbol{1}\}$ is dense in $\boldsymbol{H}$[3].
She gave criterion to investigate if an operator is codisk-cyclic or not**.** Recently, many authors studied codisk-cyclic operators[2],[4]. Jamil showed that the direct sum of two codisk **-** cyclic operators is codisk-cyclic. But, in general, the converse is not true [3]**.** This motivates us to present codiskcyclicity mixing concept. $\boldsymbol{T}$ is called a codisk cyclic operator, $\boldsymbol{C\mathbb{D}}$- mixing operator, if for any non-empty open subsets $\boldsymbol{U}, \boldsymbol{V}$ of $\boldsymbol{H}$, there are $\boldsymbol{n} \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{C}; |\boldsymbol{\alpha}| \geq \boldsymbol{1}$, such that $\boldsymbol{T^n}(\boldsymbol{\alpha U}) \cap \boldsymbol{V} \neq \emptyset$ for all $\boldsymbol{n} \geq \boldsymbol{N}$**.** After studied its characterizations, we showed that the direct sum of two operators is codidk-mixing if and if they are, also we proved that the direct sum is a codisk- cyclic operator if one of them is a codisk **-** cyclic and the other is a $\boldsymbol{C\mathbb{D}}$- mixing operator. We remark that $\mathbb{N}$ in this paper stands for the natural set with zero, and $\mathbb{B}^c$ the complement of the unit ball with zero center.

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2 The Characterization of Codisk-Cyclic Mixing Operators.

In this section, we introduce the concept of a codisk-mixing operator, and prove some characterization to $\mathbb{CD}$-mixing operators by using $\mathbb{CD}$-return sets.

Definition (2.1): Let $T \in B(H)$. T is said to be a codisk-mixing operator, abbreviate by $\mathbb{CD}$-mixing, if U, V be any non-empty open subsets of H, there exist $N \in \mathbb{N}$ and $\alpha \in \mathbb{B}^c$ such that $T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$.

Now, we discuss the characterization of $\mathbb{D}$-mixing operators by $\mathbb{CD}$-return sets which is defined as:

Definition (2.2): Let $T \in B(H)$, and U, V be any non-empty open subsets of H, $\alpha \in \mathbb{B}^c$. The set

$$N^{\mathbb{CD}}(U,V) = N_T^{\mathbb{CD}}(U,V) \coloneqq \{n \in \mathbb{N} : T^n(\alpha U) \cap V \neq \emptyset \,;\, \alpha \in \mathbb{B}^c\}$$

is called $\mathbb{CD}$-return set.

Proposition (2.3): Let $T \in B(H)$ is $\mathbb{CD}$-mixing if and only if for all U, V are non-empty open subsets of H. $N^{\mathbb{D}}(U,V)$ is a co-finite set.

Proof: $\Longrightarrow$) Let $T \in B(H)$ is $\mathbb{CD}$-mixing, and U, V be any non-empty open subsets of H, then there exist $N \in \mathbb{N}$ and $\alpha \in \mathbb{B}^c$ such that

$T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$.

Note that

$$\begin{aligned} \mathbb{N} - N^{\mathbb{CD}}(U,V) &= \mathbb{N} - \{n \in \mathbb{N} : T^n(\alpha U) \cap V \neq \emptyset; \alpha \in \mathbb{B}^c\} \\ &= \mathbb{N} - \{n \in \mathbb{N} : n \geq N\} = \{0, \dots, N-1\} \end{aligned}$$

Thus $N^{\mathbb{CD}}(U,V)$ is a co-finite set.

$\Longleftarrow$) Since $N^{\mathbb{CD}}(U,V)$ is a co-finite set, then there exist $k \in \mathbb{N}$ such that

$$\mathbb{N} - \{n \in \mathbb{N};\ T^n(\alpha U) \cap V \neq \emptyset; \alpha \in \mathbb{B}^c\} = \{0,1,\dots,k\}.$$

So, $T^{k+n}(\alpha U) \cap V \neq \emptyset$, for all $n > k$.

Hence T is $\mathbb{CD}$-mixing.

Next proposition gives anther equivalent relation between the $\mathbb{CD}$-mixing operators and $\mathbb{CD}$-return sets with neighborhood of zero. But first we will need the following lemma.

Lemma (2.4) [1]: If U be a non-empty open set in H, then there is a non-empty open subset $U_1 \subset U$ and W be a neighborhood of zero such that $U_1 + W \subset U$. If W is a neighborhood of zero, then there is a neighborhood of zero W_1 such that $W_1 + W_1 \subset W$.

PROPOSITION (2.5): AN OPERATOR T IS $\mathbb{CD}$-MIXING IF AND ONLY IF, FOR ANY NONEMPTY OPEN SUBSET U IN H AND ANY NEIGHBORHOOD OF ZERO, W, THE $\mathbb{CD}$-RETURN SETS $N^{\mathbb{CD}}(U,W)$ AND $N^{\mathbb{CD}}(W,U)$ ARE CO-FINITE SET.

Proof: $\Longrightarrow$)Let T be a $\mathbb{CD}$-mixing, U, V be nonempty open sets, and W is a neighborhood of zero, thus there exist an open ball Z such that $0 \in Z \subset W$.

Since T is $\mathbb{CD}$-mixing, therefore, by Proposition (2.3), $N^{\mathbb{CD}}(U,W) \supseteq N^{\mathbb{CD}}(U,Z)$ is a co-finite set. Similarly, $N^{\mathbb{CD}}(W,V) \supseteq N^{\mathbb{CD}}(Z,V)$ is a co-finite set.

$\Longleftarrow$)Let U, V be non-empty open subsets of H. By Lemma (2.4), there are non-empty open subsets U_1, V_1 and W_0, $\widehat{W}_0$ neighborhoods of zero, such that

$$U_1 \subset U \text{ and } W_0 \subset W;\ U_1 + W_0 \subset U \dots (1)$$

and

$$V_1 \subset V \text{ and } \widehat{W}_0 \subset W; V_1 + \widehat{W}_0 \subset V \dots (2).$$

Let $W = W_0 \cap \widehat{W}_0$. By the hypothesis,

$N^{\mathbb{CD}}(U_1,W), N^{\mathbb{CD}}(W,V_1), N^{\mathbb{CD}}(V_1,W)$ and $N^{\mathbb{CD}}(W,U_1)$ are co-finite set.

Hence there exist $N_i \in \mathbb{N}$, and $\alpha_i \in \mathbb{B}^c$; $i = 1,2,3,4.$ such that

$$\begin{gathered} sT^{n_1}(\alpha_1 U_1) \cap W \neq \emptyset, \text{ for all } n_1 \geq N_1 \\ T^{n_2}(\alpha_2 W) \cap V_1 \neq \emptyset, \text{ for all } n_2 \geq N_2 \\ T^{n_3}(\alpha_3 V_1) \cap W \neq \emptyset, \text{ for all } n_3 \geq N_3 \\ T^{n_4}(\alpha_4 W) \cap U_1 \neq \emptyset, \text{ for all } n_4 \geq N_4. \end{gathered}$$

Put $N = \max\{N_1, N_2, N_3, N_4\}$ for all $n \geq N$, and $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, we get

$$T^n(\alpha U_1) \cap W \neq \emptyset \quad \dots (3)$$

$$T^n(\alpha W) \cap V_1 \neq \emptyset \quad \dots (4)$$

$$T^n(\alpha V_1) \cap W \neq \emptyset$$

$$T^n(\alpha W) \cap U_1 \neq \emptyset$$

From (3) there exist $u \in U$ such that $T^n(\alpha u) \in W$. While from (4) there exist $w \in W$ such that $T^n(\alpha w) \in V_1$. Thus from (1), $u + w \in U$ and from (2) we get $T^n\alpha(u + w) \subset V$. Then $T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$. Which implies T is $\mathbb{CD}$-mixing.

3. The Sufficient Conditions Of C𝔻-Mixing Operator

In this section we investigate a sufficient condition of an operator to be C𝔻-mixing.

Proposition (3.1) [C𝔻-mixing Comparism Principle]:

Let $T \in B(H)$, $S \in B(K)$. If there $X \in B(H, K)$ such that $SX = XT$, then S is C𝔻-mixing if T is C𝔻-mixing

$$\begin{tikzcd} H \arrow[r, "T"] \arrow[d, "Y"'] \arrow[rd, bend left] & H \arrow[d, "Y"] \\ K \arrow[r, "S"'] & K \end{tikzcd}$$

Proof:

Let U, V be non-empty open sets of K, by continuity of X, $X^{-1}U, X^{-1}V$ are non-empty open sets of H. Since T is C𝔻-mixing, then there exist $N \in \mathbb{N}$ and $\alpha \in \mathbb{B}^c$, such that for all $n \geq N$, $T^n(X^{-1}(\alpha U)) \cap X^{-1}V \neq \emptyset$. So, there are $u \in X^{-1}(U)$ such that $T^n(\alpha u) \in X^{-1}(V)$, hence $X(T^n(\alpha u)) \subset V$. Therefore,

$$S^n(X(\alpha u)) = X(T^n(\alpha u)) \subset V \quad \dots (8).$$

Since $u \in X^{-1}(U)$, so

$$\alpha X u \in \alpha U \quad \dots (9).$$

From (8) and (9) we get $S^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$.Then S is C𝔻-mixing.

Theorem (3.2):[C𝔻-mixing Criterion]

Let $T \in B(H)$. If there exists $\alpha \in \mathbb{B}^c$ and $N \in \mathbb{N}$, For which there are a dense subsets Y, X in H and a sequence of mappings, $S_n: Y \to H$, for all $n \geq N$ such that:

1) $\alpha T^n x \to 0$ for all $x \in X$
2) a) $\frac{1}{\alpha} S_n y \to 0$ for all $y \in Y$
   b) $T^n S_n y \to y$ for all $y \in Y$

Then T is C𝔻-mixing.

Proof: Let U, V be non-empty open sets of H, let $x \in X \cap U$, $y \in Y \cap V$.

By (2(a)) we get $x + \frac{1}{\alpha} S_n y \to x \in U$ for all $n \geq N$. thus,

$$\alpha T^n \left(x + \frac{1}{\alpha} S_n y\right) = \alpha T^n x + T^n S_n y \to y \in V \text{ for all } n \geq N.$$

Therefore, $T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$. Hence T is C𝔻-mixing.

4. The Sufficient Condition of C𝔻-Mixing Operator

The goal of this section is studying the direct sum of two C𝔻-mixing operators.

Proposition (4.1): Let $T, S \in B(H)$. Then $S \oplus T$ is C𝔻-mixing in $H \oplus H$ if and only if S and T are C𝔻-mixing operators.

Proof: ⟹)Let U_1, U_2, V_1, V_2 be any non-empty open subsets of H, since $S \oplus T$ is C𝔻-mixing, then there exist $N \in \mathbb{N}$ and $\alpha \in \mathbb{B}^c$ such that for all $n \geq N$,

$$(S^n(\alpha U_1) \cap V_1) \oplus (T^n(\alpha U_2) \cap V_2) = (S \oplus T)^n(\alpha(U_1 \oplus U_2)) \cap (V_1 \oplus V_2) \neq \emptyset .$$

Thus, for all $n \geq N$, $S^n(\alpha U_1) \cap V_1 \neq \emptyset$ and $T^n(\alpha U_2) \cap V_2) \neq \emptyset$. So S and T are C𝔻-mixing.

⟸)Let S and T be C𝔻-mixing operators, then for any U_1, U_2, V_1, V_2 of non-empty open subsets of H, there exist $N_1, N_2 \in \mathbb{N}$ and $\alpha_i \in \mathbb{B}^c: i = 1,2$, such that for all $n_1 \geq N_1$ and $n_2 \geq N_2$ we get

$$S^{n_1}(\alpha_1 U_1) \cap V_1 \neq \emptyset \text{ and } T^{n_2}(\alpha_2 U_2) \cap V_2 \neq \emptyset$$

Let $N = \max\{N_1, N_2\}$ and $\alpha = \max\{ \alpha_1, \alpha_2\}$. Hence, for all $n \geq N$

$$(S^n(\alpha U_1) \cap V_1) \oplus (T^n(\alpha U_2) \cap V_2) = (S \oplus T)^n(\alpha(U_1 \oplus U_2)) \cap (V_1 \oplus V_2) \neq \emptyset$$

So $S \oplus T$ is C𝔻-mixing.

Recall that a bounded linear operator T is called codisk-cyclic operator if there is a non-zero vector $x \in H$ such that $\{\alpha T^n x: n \geq 0, \alpha \in \mathbb{B}^c\}$ is dense in H.[3]. It is well – know that there is a direct sum of two codisk cyclic operators which is not codisk -cyclic operator [3]. The following proposition discuss this case. But first we need the following lemma

Lemma (4.2): Let $T \in C\mathbb{D}(H)$, then for any pair U, V of non-empty open subsets of $H, \alpha \in \mathbb{B}^c$, $T^n(\alpha U) \cap V \neq \emptyset$ is infinite.
Proof: Let U, V be non-empty open subsets of H. Since $T \in C\mathbb{D}(H)$, there exist $n_1 \in \mathbb{N}$, $\alpha_i \in \mathbb{D}; i = 1,2$, such that $T^{n_1}(\alpha_1 U) \cap V \neq \emptyset$. So, $U \cap T^{-n_1}\left(\frac{1}{\alpha_1}V\right) \neq \emptyset$. Let $W = T^{-n_1}\left(\frac{1}{\alpha_1}V\right)$, since T is continues, then W is open. But $T \in C\mathbb{D}(H)$, so there exist $n_2 \in \mathbb{N}, \alpha_2 \in \mathbb{B}^c$, such that $T^{n_2}(\alpha_2 U) \cap W \neq \emptyset$.
So, $T^{n_2}(\alpha_2 U) \cap T^{-n_1}\left(\frac{1}{\alpha_1}V\right) \neq \emptyset$, hance $T^{n_2+n_1}(\alpha_1\alpha_2 U) \cap V \neq \emptyset$,. and so on. Therefore, there are infinite natural number n such that $T^n(\alpha U) \cap V \neq \emptyset$.
Proposition (4.3): Let $T, S \in B(H)$. If T is a $C\mathbb{D}$-mixing operator and S is a codisk-cyclic operator, then $S \oplus T$ is a codisk-cyclic operator.
Proof: Let U_1, U_2, V_1, V_2 be any non-empty open subsets of H, since T is $C\mathbb{D}$-mixing and S is codisk-cyclic, then there exist $N, k \in \mathbb{N}$ and $\alpha_1, \alpha_2 \in \mathbb{B}^c$
such that $T^n(\alpha_2 U_2) \cap V_2 \neq \emptyset$ for all $n \geq N$ and $S^k(\alpha_1 U_1) \cap V_1 \neq \emptyset$.
Now, if $k < N$, then by Lemma (4.2), there exist $m \geq N$, such that $S^m(\alpha_1 U_1) \cap V_1 \neq \emptyset$. Put $\alpha = \max\{\alpha_1, \alpha_2\}$,

$$(S^m(\alpha U_1) \cap V_1 \oplus T^m(\alpha U_2) \cap V_2) = (S \oplus T)^m(\alpha(U_1 \oplus U_2) \cap (V_1 \oplus V_2).$$

Therefore $(S \oplus T)^m(\alpha(U_1 \oplus U_2) \cap (V_1 \oplus V_2) \neq \emptyset$. Hence $S \oplus T$ is a codisk-cyclic operator.
The result is done when $k \geq N$.
Here a natural question appears, can we generalize the proposition (4.3) to a finite number of direct summand operators?
Corollary (3.4): Let T_1 be a codisk-cyclic operator, and $(T_i)_{i=2}^n$ be a sequence of $C\mathbb{D}$-mixing operator, then for all $n \in \mathbb{N}$, $\bigoplus_{i=1}^n T_i$ is a codisk-cyclic operator.
Proof: By induction. If $n = 2$, then by proposition (2.3), $T_1 \oplus T_2$ is a disk-cyclic operator. Suppose it is true when $n = k$. Now that $n = k + 1$ thus
$\bigoplus_{i=1}^n T_i = \bigoplus_{i=1}^k T_i \oplus T_{k+1}$. So by Proposition (4.3), $\bigoplus_{i=1}^k T_i \in C\mathbb{D}(H)$.

4. Conclusion

Let H be an infinite dimensional separable complex Hilbert space, and T be a bounded linear operator. T is called codisk-mixing operator, $C\mathbb{D}$- mixing operator, if for any non-empty open subsets U, V of H, there are $n \in \mathbb{N}$ and $\alpha \in \mathbb{B}^c$ such that $T^n(\alpha U) \cap V \neq \emptyset$ for all $n \geq N$.
In this paper, we studied a characterization of $C\mathbb{D}$- mixing operators, and discussed when a direct sum of two codisk cyclic operators which is codisk -cyclic operator. We showed that if one of them is $C\mathbb{D}$- mixing operator and the other is codisk- cyclic operator, then the direct sum of them is codisk-cyclic operator.

References

[1] K.Grosse-Erdmann and A. Manguillot ," Linear chaos", Springer VerLag London Limited, 2011.
[2] Y. Wang and H. Zeng, Disk-cyclic and codisk-cyclic weighted pseudo-shifts, Bull. Belg. Math. Soc. Simon Stevin 25(2): 209-224 (june 2018).
[3] Z. Jamil, Cyclic phenomena of operators on Hilbert space, Ph.D. Thesis, University of Baghdad, 2002.
[4] Z. Jamil, On hereditarily codisk-cyclic operators, Baghdad Science Journal, 2022, 19(2), pp. 309–312.