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Strongly Pseudo Nearly Quasi-2-Absorbing Submodules(II)

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ABSTRACT

In this research, we presented the concept of Strongly Pseudo Nearly Quasi 2-Absorbing submodules . Which is a generalization of the concepts of 2-Absorbing and Quasi 2-Absorbing submodules and the ideal $[\mathcal{F}_R{:}\mathcal{H}]$ is not Strongly Pseudo Nearly Quasi 2-Absorbing submodulesand we gave an example of that . We also noted that $[\mathcal{F}_R{:}\mathcal{H}]$ is STPNS-2-Absorbing ideals under several conditions. Also in this part we introduce the characterization of the concept of STPNS-2-Absorbing ideals by special kind of submodules.

MSC..

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Introduction

In our work to start with the common notion which is 2-Absorbing ideal which introduced by Badawi in 2007[1]. Darani and Soheiline in 2011 [2] they expanded the notion of 2-Absorbing ideals to 2-Absorbing submodules. Many author's extend the notion of 2-Absorbing submodule in 2015 to (Semi-2-absorbing, Primary-2-absorbing and Almost-2-Absorbing) submodules see [3, 4, 5]. In 2018 the notion 2-Absorbing submodules they expanded to Nearly 2-Absorbing submodule by [6]. In 2019 they expanded to pseudo 2-Absorbing submodule by [7]. Also they expanded to Quasi 2-Absorbingsubmodule by [8]. In 2019 they expanded to Pseudo Quasi 2-Absorbingsubmodule by [9]. In 2023 they expanded to Nearly Quasi Primary 2-Absorbing submodules by [10].$[\mathcal{F}_R{:}\mathcal{H}] = \{r \in R: r\mathcal{H} \subseteq \mathcal{F}\}$ where $\mathcal{F}$ is a submodule of an R-module $\mathcal{H}$ [11]. An R – module $\mathcal{H}$ is said to be faithful if $Ann_R(\mathcal{H}) = (0)$. $Where\ Ann(\mathcal{H}) = \{r \in R : r\mathcal{H} = (0)\}$ [12]. An R–module $\mathcal{H}$ is said to be a multiplication , if every submodule $\mathcal{F}$ of $\mathcal{H}$ is of the form $\mathcal{F} = I\mathcal{H}$ for some ideal I of R. Equivalent to $\mathcal{F} = [\mathcal{F}_R{:}\mathcal{H}]\mathcal{H}$ [13]. an R – module $\mathcal{H}$ is a projective if for every R – epimorphism $f : \mu \to \mu'$ and every R – homomorphism $g : \mathcal{H} \to \mu'$, there exists an R – homomorphism $h : \mathcal{H} \to \mu$ such that the following diagram is commute that is $foh = g$ [12]. $Z(\mathcal{H}) = \{x \in \mathcal{H}: ann(x) essential\ ideal\ in\ R\}$. If $Z(\mathcal{H}) = 0$, then $\mathcal{H}$ is called the non-singular module [14]. An R-module $\mathcal{H}$ is said to be content module if $(\cap_{i\in I} A_i)\mathcal{H} = \cap_{i\in I} A_i\mathcal{H}$ for each family of ideals A_i in R [15]. An R–module $\mathcal{H}$ is called Z–regular if for any $x \in \mathcal{H}$ there exists $g \in \mathcal{H} = \mathrm{Hom}_R(\mathcal{H}, R)$ such that $x = g(x)x$ [16]. An R – module $\mathcal{H}$ is finitely generated if $\mathcal{H} = \langle z_1, z_2, z_3, \dots, z_n\rangle = Rz_1, Rz_2, Rz_3, \dots Rz_n$, where $z_1, z_2, z_3, \dots, z_n \in \mathcal{H}$[12]. An R–module $\mathcal{H}$ is said to be cancellation if whenever $I\mathcal{H} = J\mathcal{H}$ for any ideals I, J of R, implies that $I = J$[17].

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2. Strongly Pseudo Nearly Quasi -2-Absorbing Submodules.

In this section we introduce the definition of Strongly Pseudo Nearly Quasii-2-Absorbing submodule and we introduce several characterzations of STPNQ-2-Absorbing submodules in classes of multiplication modules and other types of modules:

Definition 2.1[18] A proper submodule $\mathcal{F}$ of an R-module $\mathcal{H}$ is said to be Strongly pseudo Nearly Quasi-2-Absorbing (for short STPNQ-2-Absorbing) submodule of $\mathcal{H}$ if whenever $abcm \in \mathcal{F}$, where $a,b,c \in R$, $m \in \mathcal{H}$ implies that either $acm \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcm \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $abm \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. And an ideal I of a ring R is called STPNQ -2- Absorbing ideal of R, if I is an STPNQ-2-Absorbing R-submodule of an R-module R.

The following proposition gives characterization of STPNQ-2-Absorbing submodules in classes of multiplication modules.

Before proving the following proposition we need the following lemma.

Lemma 2.2 [18, prop(2.7)] 2.7 Let $\mathcal{H}$ be module and $\mathcal{F} \subset \mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$ if and only if for every submodule A of $\mathcal{H}$ and for every ideals I_1, I_2, I_3 of R such that $I_1 I_2 I_3 A \subseteq \mathcal{F}$ implies that either $I_1 I_2 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1 I_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2 I_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Proposition 2.3 Let $\mathcal{H}$ be a multiplication R-module and $\mathcal{F} \subset \mathcal{H}$. Then $\mathbb{N}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$ if and only if whenever $H_1 H_2 H_3 A \subseteq \mathcal{F}$ for some submodules H_1, H_2, H_3, A of $\mathcal{H}$, implies that either $H_1 H_2 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $H_1 H_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $H_2 H_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})$.

Proof (⇒) Let $H_1 H_2 H_3 A \subseteq \mathcal{F}$for some submodules H_1, H_2, H_3, A of $\mathcal{H}$. Since $\mathcal{H}$ is a multiplication, then $H_1 = J_1\mathcal{H}$, $H_2 = J_2\mathcal{H}$, $H_3 = J_3\mathcal{H}$ and $A = J_4\mathcal{H}$ for some ideals J_1, J_2, J_3 and $J_3$of R. That is $H_1 H_2 H_3 A = J_1 J_2 J_3 (J_4\mathcal{H}\ ) \subseteq \mathbb{N}$. But $\mathbb{N}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$, hence from lemma 2.2 we get either $J_1 J_2 (J_4\mathcal{H}) \subseteq \mathbb{N} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $J_1 J_3 (J_4\mathcal{H}) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $J_2 J_3 (J_4\mathcal{H}) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Next, following either $H_1 H_2 A \subseteq \mathcal{F} + \big(J(\mathcal{H}) \cap soc(\mathcal{H})\big)$ or $H_1 H_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$or $H_2 H_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

(⇐) Let $J_1 J_2 J_3 A \subseteq \mathcal{F}$ for J_1, J_2, J_3 are ideals of R and A is a submodule of $\mathcal{H}$. Put $H_1 = J_1\mathcal{H}$, $H_2 = J_2\mathcal{H}$ and $H_3 = J_3\mathcal{H}$. That is $H_1 H_2 H_3 A \subseteq \mathbb{N}$. Now, by hypotheses either $H_1 H_2 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $H_1 H_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $H_2 H_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, thus $J_1 J_2 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $J_1 J_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $J_2 J_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore by lemma 2.2 $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

Before proving the following proposition we need the following lemma.

Lemma 2.4 [18, coro.(2.9)] Let $\mathcal{H}$ be module and $\mathcal{F} \subset \mathcal{H}$. Then $\mathcal{F}$ is STPNQ -2-Absorbing submodule of $\mathcal{H}$ if and only if for every ideals I_1, I_2, I_3 of R and $x \in \mathcal{H}$ such that $I_1 I_2 I_3 x \subseteq \mathcal{F}$implies that either $I_1 I_2 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1 I_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2 I_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Proposition 2.5 Let $\mathcal{H}$ be a multiplication R-module and $\mathcal{F} \subset \mathcal{H}$. Then $\mathcal{F}$is STPNQ-2-Absorbing submodule of $\mathcal{H}$ if and only if whenever $Ꝁ_1 Ꝁ_2 Ꝁ_3 x \subseteq \mathcal{F}$ for some submodules $Ꝁ_1, Ꝁ_2$, $Ꝁ_3$ of $\mathcal{H}, x \in \mathcal{H}$, then either $Ꝁ_1 Ꝁ_2 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $Ꝁ_1 Ꝁ_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $Ꝁ_2 Ꝁ_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Proof (⇒) Let $Ꝁ_1 Ꝁ_2 Ꝁ_3 x \subseteq \mathcal{F}$ for some submodules $Ꝁ_1, Ꝁ_2$, $Ꝁ_3$ of $\mathcal{H}$ and $x \in \mathcal{H}$. Since $\mathcal{H}$ is a multiplication, then $Ꝁ_1 = J_1\mathcal{H}$, $Ꝁ_2 = J_2\mathcal{H}$ and $Ꝁ_3 = J_3\mathcal{H}$for some ideals J_1, $J_2$and J_3 of R. That is $Ꝁ_1 Ꝁ_2 Ꝁ_3 x = J_1 J_2 J_3 x \subseteq \mathcal{F}$. But $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$, hence from lemma 2.4 we get either $J_1 J_3 x \subseteq \mathbb{N} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $J_2 J_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $J_1 J_2 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Next, following either $Ꝁ_1 Ꝁ_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $Ꝁ_2 Ꝁ_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $Ꝁ_1 Ꝁ_2 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

(⇐) Let $J_1 J_2 J_3 x \subseteq \mathcal{F}$ for J_1, J_2, J_3 are ideals of R and $x \in \mathcal{H}$. Put $Ꝁ_1 = J_1\mathcal{H}$, $Ꝁ_2 = J_2\mathcal{H}$ and $Ꝁ_3 = J_3\mathcal{H}$. That is $Ꝁ_1 Ꝁ_2 Ꝁ_3 x \subseteq \mathcal{F}$. Now, by hypotheses either $Ꝁ_1 Ꝁ_2 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $Ꝁ_1 Ꝁ_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $Ꝁ_2 Ꝁ_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, thus $J_1 J_2 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $J_1 J_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $J_2 J_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore by lemma 2.4 $\mathcal{F}$is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

Remark 2.6 If $\mathcal{F}$ is an STPNQ_2_Absorbing submodule of an R_module $\mathcal{H}$, then $[\mathcal{F}:_R\mathcal{H}]$ need not to be an STPNQ_2_Absorbing ideal of R, the following example explain that:

Consider the Z-module Z_{72}, the submodule $\mathcal{F} = \langle\overline{36}\rangle$ is an STPNQ_2_Absorbing submodule of the Z_module Z_{72}, since $3.3.4.\overline{1} \in \langle\overline{36}\rangle$, implies that $3.4.\overline{1} \in \langle\overline{36}\rangle + \left(J(Z_{72}) \cap soc(Z_{72})\right) = \langle\overline{36}\rangle + (\langle\overline{6}\rangle \cap \langle\overline{12}\rangle) = \langle\overline{6}\rangle$, but $[\langle\overline{36}\rangle:_R Z_{72}] = 36Z$ is not to be an STPNQ_2_Absorbing ideal of Z, because $3.3.4.1 \in 36Z$, but $3.4.1 \notin 36Z + \left(soc(Z) \cap J(Z)\right) = 36Z$ and $3.3.1 \notin 36Z + \left(soc(Z) \cap J(Z)\right) = 36Z$.

Under certain conditions, the above observation is fulfilled.

Before proving the following proposition we need the following lemmas.

Lemma 2.7 [19 , corollary. (2 .1. 14) (i)] Let $\mathcal{H}$ be a faithful multiplication R – module , then $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$.

Lemma 2.8 [20] Let $\mathcal{H}$ be a faithful multiplication R – module , then $J(\mathcal{H}) = J(R)\mathcal{H}$.

Lemma 2.9 [18, proposition.(2.5)] A proper submodule $\mathcal{F}$ of $\mathcal{H}$ is STPNQ-2-A submodule of $\mathcal{H}$ if and only if $abcL \subseteq \mathcal{F}$, for $a, b, c \in R$ and L is a submodule of $\mathcal{H}$, implies that either $acL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $abL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Proposition 2.10 Let $\mathcal{F}$ be a proper submodule of a faithful multiplication R-module $\mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$ if and only if $[\mathcal{F}:_R\mathcal{H}]$ is STPNQ-2-Absorbing ideal of R.

Proof (⇒) Assume that $I_1I_2I_3I_4 \in [\mathcal{F}:_R\mathcal{H}]$ for some ideals I_1, I_2, I_3 and I_4 of R, then $I_1I_2I_3I_4\mathcal{H} \subseteq \mathcal{F}$. But $\mathcal{H}$ is a multiplication, then $I_1I_2I_3I_4\mathcal{H} = Ҡ_1Ҡ_2Ҡ_3Ҡ_4 \subseteq \mathcal{F}$, by taking $I_1\mathcal{H} = Ҡ_1$, $I_2\mathcal{H} = Ҡ_2$, $I_3\mathcal{H} = Ҡ_3$ and $I_4\mathcal{H} = Ҡ_4$. But $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$, then by proposition 2.3 either $Ҡ_1Ҡ_3Ҡ_4 \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $Ҡ_2Ҡ_3Ҡ_4 \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $Ҡ_1Ҡ_2Ҡ_4 \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Since $\mathcal{H}$ is multiplication, then $\mathcal{F} = [\mathcal{F}:_R\mathcal{H}]\mathcal{H}$, and since $\mathcal{H}$ is faithful multiplication, then by lemmas 2.7, 2.8 $soc(\mathcal{H}) = soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $I_1I_3I_4\mathcal{H} \subseteq [\mathcal{F}:_R\mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $I_2I_3I_4\mathcal{H} \subseteq [\mathcal{F}:_R\mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $I_1I_2I_4\mathcal{H} \subseteq [\mathcal{F}:_R\mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$. Hence either $I_1I_3I_4 \subseteq [\mathcal{F}:_R\mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $I_2I_3I_4 \subseteq [\mathcal{F}:_R\mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $I_1I_2I_4 \subseteq [\mathcal{F}:_R\mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$. Therefore $[\mathcal{F}:_R\mathcal{H}]$ is STPNQ-2-Absorbing ideal of R.

(⟸) Let $rstA \subseteq \mathcal{F}$ for $r, s, t \in R$ and A is a submodule of $\mathcal{H}$, since $\mathcal{H}$ is a multiplication, then $A = I\mathcal{H}$ for some ideal I of R, that is $rstI\mathcal{H} \subseteq \mathcal{F}$, implies that $rstI \subseteq [\mathcal{F}:_R\mathcal{H}]$, but $[\mathcal{F}:_R\mathcal{H}]$ is STPNQ-2-Absorbing ideal of R, then by lemma 2.9 either $rsI \subseteq [\mathcal{F}:_R\mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $rtI \subseteq [\mathcal{F}:_R\mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $stI \subseteq [\mathcal{F}:_R\mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$. Thus either $rsI\mathcal{H} \subseteq [\mathcal{F}:_R\mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $rtI\mathcal{H} \subseteq [\mathcal{F}:_R\mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $stI\mathcal{H} \subseteq [\mathcal{F}:_R\mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$. Since $\mathcal{H}$ is a faithful multiplication, then $[\mathcal{F}:_R\mathcal{H}]\mathcal{H} = \mathcal{F}$ and by lemmas 2.7, 2.8 either $rsA \subseteq (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rtA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $stA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus by lemma 2.9 $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

Before proving the following proposition we need the following lemmas.

Lemma 2.11 [20 , proposition. (3 . 24)] Let $\mathcal{H}$ be a projective R – module, then $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$.

Lemma 2.12 [12 , Theorem. (9 . 2.1) (a)] For any projective R – module $\mathcal{H}$, we have $J(\mathcal{H}) = J(R)\mathcal{H}$.

Lemma 2.13 [18, corollary. (2.11)] Let $\mathcal{H}$ be module and $\mathcal{F} \subset \mathcal{H}$. Then $\mathcal{F}$ is STPNQ -2-Absorbing submodule of $\mathcal{H}$ if and only if for any $r, s \in R$ and any ideal I of R and every submodule A of $\mathcal{H}$ with $rsIA \subseteq \mathcal{F}$ implies that either $rsA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$or $rIA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sIA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Proposition 2.14 Let $\mathcal{F}$ be a proper submodule of a multiplication projective R-module$\mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$ if and only if $[\mathcal{F}:_R\mathcal{H}]$ is STPNQ-2-Absorbing ideal of R.

Proof (⇒)Assume that $I_1I_2I_3b \subseteq [\mathcal{F}:_R\mathcal{H}]$ for some ideals I_1, I_2, I_3 of R and $b \in R$, then $I_1I_2I_3(b\mathcal{H}) \subseteq \mathcal{F}$. But $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$, then by lemma 2.4 either $I_1I_3b\mathcal{H} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2I_3b\mathcal{H} \subseteq \mathcal{F} +$

$(J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1 I_2 b\mathcal{H} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Since $\mathcal{H}$ is multiplication, then$\mathcal{F} = [\mathcal{F}:_R \mathcal{H}]\mathcal{H}$, and since $\mathcal{H}$ is projective R-module $\mathcal{H}$, then by lemmas 2.11, 2.12 $soc(\mathcal{H}) = soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $I_1 I_3 b\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $I_2 I_3 b\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $I_1 I_2 b\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$.Henceeither$I_1 I_3 b \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$or $I_3 b \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$or $I_1 I_2 b \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$. Therefore by lemma 2.4 $[\mathcal{F}:_R \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R.

(⟸) Let $rsIA \subseteq \mathcal{F}$ for $r, s \in R$ and some submodule A of $\mathcal{H}$ and for some ideal I of R since $\mathcal{H}$ is a multiplication, then $A = J\mathcal{H}$ for some ideal J of R, that is $rsIJ\mathcal{H} \subseteq \mathcal{F}$, implies that $rsIJ \subseteq [\mathcal{F}:_R \mathcal{H}]$, but $[\mathcal{F}:_R \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R, then by lemma 2.13 either $rsJ \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $rIJ \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $sIJ \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$. Thus either $rsJ\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $rIJ\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $sIJ\mathcal{H} \subseteq [Ҡ:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$. Hence by lemmas 2.11, 2.12 either $rsA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rIA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sIA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus by lemma 2.4 $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

Before proving the following proposition we need the following lemmas.

Lemma 2.15 [18, corollary. (2.10)] Let $\mathcal{H}$ be module and $\mathcal{F} \subset \mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$ if and only if for each $r \in R$ and any ideals I, J of R and every submodule A of $\mathcal{H}$ with $rIJA \subseteq \mathcal{F}$ implies that either $rIA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rJA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $IJA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Lemma 2.16[14 , corollary. (1 . 26)] Let $\mathcal{H}$ be a non-singular R – module , then $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$.

Lemma 2.17[20 , proposition (1 . 11)] If $\mathcal{H}$ is content module, then $J(\mathcal{H}) = J(R)\mathcal{H}$.

Proposition 2.18 Let $\mathcal{F}$ be a proper submodule of a content multiplication non-singular R-module $\mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$ if and only if $[\mathcal{F}:_R \mathcal{H}]$ is STPNQ-2-bsorbing ideal of R.

Proof (⇒)Let $bIJa \in [\mathcal{F}:_R \mathcal{H}]$ for J, I are ideals of R and $a, b \in R$, so $bIJ(a\mathcal{H}) \subseteq \mathcal{F}$. But $\mathcal{F}$is STPNQ-2-Absorbing submodule of $\mathcal{H}$, then by lemma 2.15 either $bI(a\mathcal{H}) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bJ(a\mathcal{H}) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $IJ(a\mathcal{H}) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Since $\mathcal{H}$is multiplication, then $\mathcal{F} = [Ҡ:_R \mathcal{H}]\mathcal{H}$, and since $\mathcal{H}$is non-singular multiplication content R-module , then by lemmas 2.16, 2.17 $soc(\mathcal{H}) = soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $bIa\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $IJa\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$, it follows that either $bIa \subseteq [\mathcal{F}:_R \mathcal{H}] + [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $bJa \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $IJa \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$. Hence by lemma 2.15 $[\mathcal{F}:_R \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R.

(⟸) Let $rIJL \subseteq \mathcal{F}$ for some submodule L of $\mathcal{H}$, $r \in \mathcal{H}$and I, J are ideals of R. Since $\mathcal{H}$is a multiplication, then $L = \mathcal{A}\mathcal{H}$ for some ideal $\mathcal{A}$ of R, that is $rIJ\mathcal{A}\mathcal{H} \subseteq \mathcal{F}$, implies that $rIJ\mathcal{A} \subseteq [\mathcal{F}:_R \mathcal{H}]$, but $[\mathcal{F}:_R \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R, then by lemma 2.9 either $rI\mathcal{A} \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $rJ\mathcal{A} \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $IJ\mathcal{A} \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(\mathrm{R}) \cap soc(\mathrm{R}))$. Thus either $rI\mathcal{A}\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $rJ\mathcal{A}\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $IJ\mathcal{A}\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$. Hence by lemmas 2.16, 2.17 either $rIL \subseteq Ҡ + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rJL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $IJL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus by lemma 2.15 $\mathcal{F}$is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

Before proving the following proposition we need the following lemma.

Lemma 2.19[20 , proposition (3 . 25)]Let $\mathcal{H}$ be a Z – regular R – module then $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$.

Proposition 2.20 Let $\mathcal{F}$ be a proper submodule of a content multiplication Z-regular R-module $\mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$ if and only if $[\mathcal{F}:_R \mathcal{H}]$ is STPNQ-2-bsorbing ideal of R.

Proof By proof of proposition 2.18 and use lemma 2.19.

3. Characterization of STPNQ-2-Absorbing ideals by special kind of submodules

In this section we introduce the characterization of the concept of STPNQ-2-Absorbing ideals by special kind of submodules.

The following proposition gives characterization of the concept of STPNQ-2-Absorbing ideals

But before that we need the following lemmas.

Lemma 3.1 [21 ,coro. of theo. (9)] Let $\mathcal{H}$ be a finitely generated multiplication R – module and I , J ideals of R . Then $I\mathcal{H} \subseteq J\mathcal{H}$ if and only if $I \subseteq J + ann_R(\mathcal{H})$.

Lemma 3.2 [18, proposition. (2.7)] Let $\mathcal{H}$ be module and $\mathcal{F} \subset \mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ ifandonlyif for every submodule A of $\mathcal{H}$ and for every ideals I_1, I_2, I_3 of R such that $I_1I_2I_3A \subseteq \mathcal{F}$ implies that either $I_1I_2A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1I_3A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2I_3A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Proposition 3.3 Let $\mathcal{H}$ be a finitely generated multiplication projective R-module, and B is an ideal of R with $ann_R(\mathcal{H}) \subseteq B$. Then B is STPNQ-2-Absorbing ideal of R if and only if $B\mathcal{H}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

Proof (⇒) Let $sI_1I_2A \subseteq B\mathcal{H}$, for $s \in R$, I_1, I_2 are ideals of R and A is a submodule of $\mathcal{H}$. Since $\mathcal{H}$ is a multiplication, then $A = I_3\mathcal{H}$, for some ideal I_3 of R, that is $sI_1I_2A = sI_1I_2I_3\mathcal{H} \subseteq B\mathcal{H}$. But $\mathcal{H}$ is a finitely generated multiplication R-module then by lemma 3.1 $sI_1I_2I_3 \subseteq B + ann_R(\mathcal{H})$, but $ann_R(\mathcal{H}) \subseteq B$, implies that $B + ann_R(\mathcal{H}) = B$, thus $sI_1I_2I_3 \subseteq B$. Now, by assumption B is STPNQ-2-Absorbing ideal of R then by lemma 2.15 either $sI_1I_3 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $sI_2I_3 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $I_1I_2I_3 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$, it follows that either $sI_1I_3\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $sI_2I_3\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $I_1I_2I_3\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$. Since $\mathcal{H}$ is a projective then by lemmas 2.11, 2.12 $(J(\mathcal{H}) \cap soc(\mathcal{H})) = (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$, it follows that either $sI_1A \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sI_2A \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1I_2A \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Hence by corollary lemma 2.15 $B\mathcal{H}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

(⇐) Let $I_1I_2I_3I_4 \subseteq B$, for I_1, I_2, I_3 and I_4 are ideals in R, implies that $I_1I_2I_3(I_4\mathcal{H}) \subseteq B\mathcal{H}$. But $B\mathcal{H}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$, then by lemma 3.2 either $I_1I_2(I_4\mathcal{H}) \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1I_3(I_4\mathcal{H}) \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2I_3(I_4\mathcal{H}) \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. But $\mathcal{H}$ is a projective then $(J(\mathcal{H}) \cap soc(\mathcal{H})) = (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$. Thus either $I_1I_2I_4\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $I_1I_3I_4\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $I_2I_3I_4\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$, it follows that either $I_1I_2I_4 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $I_1I_3I_4 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $I_2I_3I_4 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$. Hence by lemma 3.2 B is STPNQ-2-Absorbing ideal of R.

Proposition 3.4 Let $\mathcal{H}$ be a faithful finitely generated multiplication R-module and $\mathcal{A}$ is ideal of R with $ann_R(\mathcal{H}) \subseteq \mathcal{A}$. Then $\mathcal{A}$ is STPNQ-2-Absorbing ideal of R if and only if $\mathcal{A}\mathcal{H}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

Proof (⟹) Let $rIJx \subseteq \mathcal{A}\mathcal{H}$ for any $r \in R$, $x \in \mathcal{H}$ and I, J are ideals of R, it follows that $rIJ\langle x\rangle \subseteq \mathcal{A}\mathcal{H}$. Since $\mathcal{H}$ is a multiplication, then $\langle x\rangle = I_1\mathcal{H}$ for some ideal I_1 of R, that is $rIJI_1\mathcal{H} \subseteq \mathcal{A}\mathcal{H}$. Thus by lemma 3.1 we get $rIJI_1 \subseteq \mathcal{A} + ann(\mathcal{H})$, but $\mathcal{H}$ is faithful, it follows $ann(\mathcal{H}) = (0)$, that is $rIJI_1 \subseteq \mathcal{A}$. Since $\mathcal{A}$ is STPNQ-2-Absorbing ideal of R, then by lemma 2.4 either $rII_1 \subseteq \mathcal{A} + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $rJI_1 \subseteq \mathcal{A} + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $IJI_1 \subseteq \mathcal{A} + (J(\mathrm{R}) \cap soc(\mathrm{R}))$, hence either $rII_1\mathcal{H} \subseteq \mathcal{A}\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $rJI_1\mathcal{H} \subseteq \mathcal{A}\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $IJI_1\mathcal{H} \subseteq \mathcal{A}\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$, hence by lemmas 2.7, 2.8 either $rI\langle x\rangle \subseteq \mathcal{A}\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rJ\langle x\rangle \subseteq \mathcal{A}\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $IJ\langle x\rangle \subseteq \mathcal{A}\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus by lemma 2.15 $\mathcal{A}\mathcal{H}$ is an STPNQ-2-Absorbing submodule of $\mathcal{H}$.

(⟸) Let $rstI \subseteq \mathcal{A}$ for $r, s, t \in R$ and I ideal of R, hence $rst(I\mathcal{H}) \subseteq \mathcal{A}\mathcal{H}$, but $\mathcal{A}\mathcal{H}$ is an STPNQ-2-Absorbing submodule of $\mathcal{H}$, then either $rs(I\mathcal{H}) \subseteq \mathcal{A}\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rt(I\mathcal{H}) \subseteq \mathcal{A}\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $st(I\mathcal{H}) \subseteq \mathcal{A}\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus by lemmas 2.7,2.8 either $rsI\mathcal{H} \subseteq \mathcal{A}\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $rtI\mathcal{H} \subseteq \mathcal{A}\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $stI\mathcal{H} \subseteq \mathcal{A}\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$, hence either $rsI \subseteq \mathcal{A} + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $rtI \subseteq \mathcal{A} + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $stI \subseteq \mathcal{A} + (J(\mathrm{R}) \cap soc(\mathrm{R}))$. Therefore $\mathcal{A}$ is EXNPQ-2-Absorbing ideal of R.

Proposition 3.5 Let $\mathcal{H}$ be a finitely generated multiplication non-singular content R-module, and B is an ideal of R with $ann_R(\mathcal{H}) \subseteq B$. Then B is STPNQ-2-Absorbing ideal of R if and only if $B\mathcal{H}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

Proof (⇒) Let $sI_1I_2A \subseteq B\mathcal{H}$, for $s \in R$, I_1, I_2 are ideals of R and A is a submodule of $\mathcal{H}$. Since $\mathcal{H}$ is a multiplication, then $A = I_3\mathcal{H}$, for some ideal I_3 of R, that is $sI_1I_2A = sI_1I_2I_3\mathcal{H} \subseteq B\mathcal{H}$. But $\mathcal{H}$ is a finitely generated multiplication R-module then by lemma 3.1 $sI_1I_2I_3 \subseteq B + ann_R(\mathcal{H})$, but $ann_R(\mathcal{H}) \subseteq B$, implies that $B + ann_R(\mathcal{H}) = B$, thus $sI_1I_2I_3 \subseteq$

B. Now, by assumption B is STPNQ-2-Absorbing ideal of R then by lemma 2.15 either $sI_1I_3 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $sI_2I_3 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $I_1I_2I_3 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$, it follows that either $sI_1I_3\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $sI_2I_3\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $I_1I_2I_3\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$. Since $\mathcal{H}$ is a projective then by lemmas 2.16, 2.17 $(J(\mathcal{H}) \cap soc(\mathcal{H})) = (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$, it follows that either $sI_1A \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sI_2A \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1I_2A \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Hence by corollary lemma 2.15 $B\mathcal{H}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

($\Leftarrow$) Let $I_1I_2I_3I_4 \subseteq B$, for I_1, I_2, I_3 and I_4 are ideals in R, implies that $I_1I_2I_3(I_4\mathcal{H}) \subseteq B\mathcal{H}$. But $B\mathcal{H}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$, then by lemma 3.2 either $I_1I_2(I_4\mathcal{H}) \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1I_3(I_4\mathcal{H}) \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2I_3(I_4\mathcal{H}) \subseteq B\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. But $\mathcal{H}$ is a non-singular content R-module then $(J(\mathcal{H}) \cap soc(\mathcal{H})) = (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$. Thus either $I_1I_2I_4\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $I_1I_3I_4\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$ or $I_2I_3I_4\mathcal{H} \subseteq B\mathcal{H} + (J(\mathrm{R})\mathcal{H} \cap soc(\mathrm{R})\mathcal{H})$, it follows that either $I_1I_2I_4 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $I_1I_3I_4 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$ or $I_2I_3I_4 \subseteq B + (J(\mathrm{R}) \cap soc(\mathrm{R}))$. Hence by lemma 3.2 B is STPNQ-2-Absorbing ideal of R.

Proposition 3.6 Let $\mathcal{H}$ be a finitely generated multiplication Z-regular content R-module, and B is an ideal of R with $ann_R(\mathcal{H}) \subseteq B$. Then B is STPNQ-2-Absorbing ideal of R if and only if $B\mathcal{H}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

Proof Similar to the proof of proposition 3.5 by using lemma 2.19.

Proposition 3.7 Let $\mathcal{H}$ be a faithful finitely generated multiplication R-module and $\mathcal{F} \subset \mathcal{H}$, then the following statements are equivalent:

1. $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R.

3. $\mathcal{F} = B\mathcal{H}$ for some STPNQ-2-Absorbing ideal B of R.

Proof (1) $\Leftrightarrow$ (2) It follows by Proposition 2.10.

(2) $\Longrightarrow$ (3) Since $[\mathcal{F}:_\mathrm{R} \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R and $\mathcal{H}$ is a faithful, that is $(0) = \mathrm{ann}_\mathrm{R}(\mathcal{H}) = [0:_\mathrm{R} \mathcal{H}] \subseteq [\mathcal{F}:_\mathrm{R} \mathcal{H}]$ and $\mathcal{H}$ is a multiplication, so $\mathcal{F} = [\mathrm{L}:_\mathrm{R} \mathcal{H}]\mathcal{H}$, implies that $\mathcal{F} = \mathrm{J}\mathcal{H}$ for some STPNQ-2-Absorbing ideal $\mathrm{J} = [\mathcal{F}:_\mathrm{R} \mathcal{H}]$ of R.

(3) $\Longrightarrow$ (2) Clearly.

But before that we need the following lemma.

Lemma 3.8 [17 , proposition. (3 . 1)] If $\mathcal{H}$ is a multiplication R–module, then $\mathcal{H}$ is cancellation if and only if $\mathcal{H}$ is a faithful finitely generated.

Proposition 3.9 Let $\mathcal{H}$ be a finitely generated multiplication projective R-module and $\mathcal{F} \subset \mathcal{H}$ with $ann_R(\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]$**.** Then the following statements are equivalent:

1. $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R.

3. $\mathcal{F} = B\mathcal{H}$ for some STPNQ-2-Absorbing ideal B of R.

Proof (1) $\Leftrightarrow$ (2) It follows by proposition 2.14.

(2) $\Longrightarrow$ (3) Suppose that $[\mathcal{F}:_\mathrm{R} \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R. Since $\mathcal{H}$ is a multiplication, then $\mathcal{F} = [\mathcal{F}:_\mathrm{R} \mathcal{H}]\mathcal{H} = \mathrm{B}\mathcal{H}$, where $\mathrm{B} = [\mathcal{F}:_\mathrm{R} \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R with $\mathrm{ann}_\mathrm{R}(\mathcal{H}) \subseteq [\mathcal{F}:_\mathrm{R} \mathcal{H}] = \mathrm{B}$, implies that $\mathrm{ann}_\mathrm{R}(\mathcal{H}) \subseteq \mathrm{B}$.

(3) $\Longrightarrow$ (2) Assume that $\mathcal{F} = \mathrm{B}\mathcal{H}$ .....(1) for some STPNQ-2Absorbing ideal B of R with $\mathrm{ann_R}(\mathcal{H}) \subseteq \mathrm{B}$. But $\mathcal{H}$ is a multiplication, then $\mathcal{F} = [\mathcal{F}:_{\mathrm{R}} \mathcal{H}]\mathcal{H}$.....(2), from (1) and (2) we have $[\mathcal{F}:_{\mathrm{R}} \mathcal{H}]\mathcal{H} = \mathrm{B}\mathcal{H}$. Since $\mathcal{H}$ is a finitely generated, then by lemma 3.8 $\mathcal{H}$ is weak cancellation, it follows that $[\mathcal{F}:_{\mathrm{R}} \mathcal{H}] + \mathrm{ann_R}(\mathcal{H}) = \mathrm{B} + \mathrm{ann_R}(\mathcal{H})$, but $\mathrm{ann_R}(\mathcal{H}) \subseteq \mathrm{B}$, and $\mathrm{ann_R}(\mathcal{H}) \subseteq [\mathcal{F}:_{\mathrm{R}} \mathcal{H}]$ implies that $\mathrm{ann_R}(\mathcal{H}) + \mathrm{B} = \mathrm{B}$ and $[\mathcal{F}:_{\mathrm{R}} \mathcal{H}] + \mathrm{ann_R}(\mathcal{H}) = [\mathcal{F}:_{\mathrm{R}} \mathcal{H}]$. Thus $\mathrm{B} = [\mathcal{F}:_{\mathrm{R}} \mathcal{H}]$, but B is STPNQ-2-Absorbing ideal of R, hence $[\mathcal{F}:_{\mathrm{R}} \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R.

(3) $\Leftrightarrow$ (1) It follows by proposition 3.3.

Proposition 3.10 Let $\mathcal{H}$ be a finitely generated multiplication non-singular contentR-module and $\mathcal{F} \subset \mathcal{H}$ with $ann_R(\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]$. Then the following statements are equivalent:

1. $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R.

3. $\mathcal{F} = B\mathcal{H}$ for some STPNQ-2-Absorbing ideal B of R.

Proof (1) $\Leftrightarrow$ (2) It follows by proposition 2.18.

(2) $\Leftrightarrow$ (3) Follows in the same way as the proof of Proposition 3.9.

Proposition 3.11 Let $\mathcal{H}$ be a finitely generated multiplication Z-regular content R-module and $\mathcal{F} \subset \mathcal{H}$ with $ann_R(\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]$. Then the following statements are equivalent:

1. $\mathcal{F}$ is STPNQ-2-Absorbing submodule of $\mathcal{H}$.

2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNQ-2-Absorbing ideal of R.

3. $\mathcal{F} = B\mathcal{H}$ for some STPNQ-2-Absorbing ideal B of R.

Proof (1) $\Leftrightarrow$ (2) It follows by proposition 2.20.

(2) $\Leftrightarrow$ (3) Follows in the same way as the proof of Proposition 3.9.

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