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Strongly Pseudo Nearly Quasi-2-Absorbing Submodules(I)

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ABSTRACT

In this research, we presented the notion of Strongly Pseudo Nearly Quasi 2-Absorbing submodules . Which is a generalization of the notions of 2-Absorbing and Quasi 2-Absorbing submodules and stronger than the notions (Nearly 2-Absorbing, pseudo 2-Absorbing , Nearly Quasi 2-Absorbing and pseudo Quasi 2-Absorbing) submodules. Give us the properties, characterizations and examples of this new concept. We studied the relationship between the notion of Strongly Pseudo Nearly Quasi 2-Absorbingsubmodules and notion (Nearly 2-Absorbing, pseudo 2-Absorbing , Nearly Quasi 2-Absorbing and pseudo Quasi 2-Absorbing) submodules with special types of modules.

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Introduction

The famous notion in our work to start with is 2-Absorbingideal which introduced by Badawi in 2007[1], where a proper ideal I of a ring R is called 2-Absorbing if whenever $abc \in I$, for $a, b, c \in R$, then $ab \in I$ or $bc \in I$ or $ac \in I$. Darani and Soheiline in 2011 [2] extend the notion of 2-Absorbingideals to 2-Absorbing submodules, let $\mathcal{H}$ be an $\mathcal{R}$-module and $\mathcal{F} \subset \mathcal{H}$ is called 2-absorbing if $\forall\, uvh \in \mathcal{F}$, for $u, v \in \mathcal{R}, h \in \mathcal{H}$, then either $uh \in \mathcal{F}$ or $vh \in \mathcal{F}$ or $uv \in [\mathcal{F}_{\mathcal{R}}^{:}\mathcal{H}]$. Many author's extend the notion of 2-Absorbing submodule in 2015 to (Semi-2-absorbing, Primary-2-absorbing and Almost-2-Absorbing) submodules see [3, 4, 5]. In 2018 the notion 2-Absorbing submodules extend to Nearly 2-Absorbingsubmodule by [6], let $\mathcal{H}$ be an R-module and $\mathcal{F} \subset \mathcal{H}$ is called Nearly-2-Absorbing submodule if $\forall\, uvm \in \mathcal{F}$, for $u, v \in R$, $h \in \mathcal{H}$, implies that either $uh \in \mathcal{F} + J(\mathcal{H})$ or $vh \in \mathcal{F} + J(\mathcal{H})$ or $uv \in [\mathcal{F} + J(\mathcal{H})_R : \mathcal{H}]$. In 2019 extend to pseudo 2-Absorbingsubmodule by [7], let $\mathcal{H}$ be an R-module and $\mathcal{F} \subset \mathcal{H}$ is called Pseudo-2-Absorbing submodule if $\forall\, uvm \in \mathcal{F}$, for $u, v \in R$, $h \in \mathcal{H}$, implies that either $uh \in \mathcal{F} + Soc(\mathcal{H})$ or $vh \in \mathcal{F} + Soc(\mathcal{H})$ or $uv \in [\mathcal{F} + Soc(\mathcal{H})_R : \mathcal{H}]$. Also extend to Quasi 2-Absorbingsubmodule by [8], where a proper submodule $\mathcal{F}$ of an R-module $\mathcal{H}$ is called quasi-2-Absorbing submodule if $\forall\, abcm \in \mathcal{F}$, for $a, b, c \in R, m \in \mathcal{H}$ then $abm \in \mathcal{F}$ or $acm \in \mathcal{F}$ or $bcm \in \mathcal{F}$. In 2018 extend to Nearly Quasi 2-Absorbingsubmodules by [9], let $\mathcal{H}$ an R-module $\mathcal{F} \subset \mathcal{H}$ is called Nearly quasi-2-Absorbing submodule if $\forall$ abcm $\in \mathcal{F}$, for a, b, c $\in$ R, m $\in \mathcal{H}$ then abm $\in \mathcal{F} + \mathrm{J}(\mathcal{H})$ or acm $\in \mathcal{F} + \mathrm{J}(\mathcal{H})$ or bcm $\in \mathcal{F} + \mathrm{J}(\mathcal{H})$. In 2019 extend to Pseudo Quasi 2-Absorbingsubmodule by [10], a proper submodule $\mathcal{F}$ of an R-module $\mathcal{H}$ is called Pseudo quasi-2-Absorbingsubmodule if $\forall\, abcm \in \mathcal{F}$, for $a, b, c \in R, m \in \mathcal{H}$ then $abm \in \mathcal{F} +$

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$soc(\mathcal{H})$ or $acm \in \mathcal{F} + soc(\mathcal{H})$ or $bcm \in \mathcal{F} + soc(\mathcal{H})$. Also in 2019 extend to Nearly Semi 2-Absorbingsubmodules in [11] and in 2021 extend 2-Absorbingsubmodules to Nearly Primary 2-Absorbingsubmodules in [12].The notion Pseudo Primary 2-Absorbingsubmodules are introduced in 2019 [13]. In 2023 extend to Nearly Quasi Primary 2-Absorbingsubmodules by [14]. $J(\mathcal{H})$ is the junction of all maximal submodule of $\mathcal{H}$ [15]. $soc(\mathcal{H})$ is the junction of all total submodule of $\mathcal{H}$ and a nonzero submodule $\mathbb{Q}$ of $\mathcal{H}$ is a total in $\mathcal{H}$ if $\mathbb{Q} \cap \mathbb{C} \neq (0)$ for any nonzero submodule $\mathbb{C}$ of $\mathcal{H}$ [16]. An R-module H is called regular module if every submodule of $\mathcal{H}$ is a pure[17]. An R – module $\mathcal{H}$ is an injective if for every R – monomorphism $f : \mu \to \mu'$ and every R – homomorphism $g : \mathcal{H} \to \mu'$, there exists an R – homomorphism $h : \mathcal{H} \to \mu$ such that the following diagram is commute that is $foh = g$ [15]. An R – module $\mathcal{H}$ is said to be a multiplication , if every submodule $\mathcal{F}$ of $\mathcal{H}$ is of the form $\mathcal{F} = I\mathcal{H}$ for some ideal I of R . Equivalent to $\mathcal{F} = [\mathcal{F} :_R \mathcal{H}]\mathcal{H}$ [18].

2. Strongly Pseudo Nearly Quasi -2-Absorbing Submodules.

In this section we introduce the definition of Strongly Pseudo Nearly Quasi-2-Absorbing submodule and give examples characterizations some basic property of this concept.

Definition 2.1 Let $\mathcal{H}$ be an R-module and $\mathcal{F} \subset \mathcal{H}$ is said to be Strongly pseudo Nearly Quasi-2-Absorbing (for short STPNQ-2-A) submodule of $\mathcal{H}$ if whenever $abcm \in \mathcal{F}$, where $a,b,c \in R$, $m \in \mathcal{H}$ implies that either $acm \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcm \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $abm \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. And an ideal I of a ring R is called STPNQ-2-A ideal of R, if I is an STPNQ-2-A R-submodule of an R-module R.

Remarks and Examples 2.2

1. Let $\mathcal{H} = Z_{36}$, $R = Z$ and the submodule $\mathcal{F} = \langle\bar{4}\rangle$ is STPNQ-2-A submodule of $\mathcal{H}$, since $soc(Z_{36}) = \langle\bar{6}\rangle$ and $J(Z_{36}) = \langle\bar{6}\rangle$. That is for all $a,b,c \in Z$ and $m \in Z_{36}$ such that $abcm \in \langle\bar{4}\rangle$, implies that either $acm \in \langle\bar{4}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\bar{4}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{2}\rangle$ or $bcm \in \langle\bar{4}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\bar{4}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{2}\rangle$ or $bm \in \langle\bar{4}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\bar{4}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{2}\rangle$. That is $2.2.1.\bar{1} \in \langle\bar{2}\rangle$, implies that $2.1.\bar{1} = \bar{2} \in \langle\bar{2}\rangle$ and $2.2.\bar{1} = \bar{4} \in \langle\bar{2}\rangle$.

2. Any 2-Absorbing submodule of an R-module$\mathcal{H}$ is STPNQ-2-A submodule, but not contrary.

Proof clear.

For the converse consider the following example:

Let $\mathcal{H} = Z_{36}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{12}\rangle$ is STPNQ-2-A submodule of $\mathcal{H}$, since $J(Z_{36}) = \langle\bar{6}\rangle$ and $soc(Z_{36}) = \langle\bar{6}\rangle$. That is for all $a,b,c \in Z$ and $m \in Z_{36}$ such that $abcm \in \langle\overline{12}\rangle$, implies that either $acm \in \langle\overline{12}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$ or$bcm \in \langle\overline{12}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$or $abm \in \langle\overline{12}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$, that is $2.3.2.\bar{1} \in \langle\overline{12}\rangle$ and $3.2.\bar{1} \in \langle\overline{12}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$. But $\mathcal{F}$ is not 2_Absorbing submodule of $\mathcal{H}$, since $2.3.\bar{2} \in \langle\overline{12}\rangle$, but $2.\bar{2} \notin \langle\overline{12}\rangle$ and $3.\bar{2} \notin \langle\overline{12}\rangle$ and $2.3 \notin [\langle\overline{12}\rangle :_R Z_{36}] = 12Z$.

3. Any STPNQ-2-A submodule of a cyclic R-module $\mathcal{H}$ is Nearly-2-Absorbing submodule, but not contrary.

Proof Let $abh \in \mathcal{F}$ for $a, b \in R$, $h \in \mathcal{H}$, then there exists an element $c \in R$ such that $h = cw$, hence $abh = abcw \in \mathcal{F}$. Since $\mathcal{H}$ is STPNQ-2-A , then either $abw \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcw \in \mathcal{F} + (J(W) \cap soc(W))$or $acw \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. That is either $ab \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R w] = [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R \mathcal{H}]$or $bcw \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $acw \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})))$. Hence either $ab \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R \mathcal{H}]$ that is $ab\mathcal{H} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + J(\mathcal{H})$ or $bh \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + J(\mathcal{H})$ or $ah \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + J(\mathcal{H})$. Therefore $\mathcal{F}$ is Nearly-2-Absorbing.

For the opposite see the following example:

Let $\mathcal{H} = Z_{48}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{24}\rangle$ is Nearly-2-Absorbing submodule of $\mathcal{H}$ since $soc(Z_{48}) = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{6}\rangle$ that is for all $u, v \in Z$ and $m \in Z_{48}$ such that $uvm \in \langle\overline{24}\rangle$, implies that either $um \in \mathcal{F} + (J(Z_{48})) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$ or $vm \in \mathcal{F} + (J(Z_{48})) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$. That is $2.4.\bar{3} \in \langle 24\rangle$, implies that $2.\bar{3} = \bar{6} \in \mathcal{F} + (J(Z_{48})) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$ and $4.\bar{3} = \overline{12} \in \mathcal{F} + (J(Z_{48})) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$. But $\mathcal{F}$ is not STPNQ-2-A submodule of $\mathcal{H}$, since $3.4.2.\bar{1} \in \langle\overline{24}\rangle$, but $3.2.\bar{1} \notin \langle\overline{24}\rangle + (J(Z_{48}) \cap soc(Z_{48}) = \langle\overline{24}\rangle$ and $4.2.\bar{1} \notin \langle\overline{24}\rangle + (J(Z_{48}) \cap soc(Z_{48}) = \langle\overline{24}\rangle$ and $3.4.\bar{1} \notin \langle\overline{24}\rangle + (J(Z_{48}) \cap soc(Z_{48}) = \langle\overline{24}\rangle$.

4. Any STPNQ-2-A submodule of a cyclic R-module $\mathcal{H}$ is Pseudo-2-Absorbing submodule, but not contrary.

Proof Let $abh \in \mathcal{F}$ for $a, b \in R$, $h \in \mathcal{H}$, then there exists an element $c \in R$ such that $h = cw$, hence $abh = abcw \in \mathcal{F}$. Since $\mathcal{H}$ is STPNQ-2-A , then either $abw \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcw \in \mathcal{F} + (J\mathcal{H}) \cap soc(\mathcal{H}))$or $acw \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. That is either $ab \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R w] = [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$or $bcw \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $acw \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Hence either $ab \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$ that is $ab\mathcal{H} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + soc(\mathcal{H})$ or $bh \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + soc(\mathcal{H})$ or $ah \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + soc(\mathcal{H})$. Therefore $\mathcal{F}$ is Pseudo-2-Absorbing submodule of $\mathcal{H}$.

For the opposite see the following example:

Let $\mathcal{H} = Z_{48}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{12}\rangle$ is pseudo-2-Absorbing submodule of $\mathcal{H}$ since $soc(Z_{48}) = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{6}\rangle$ that is for all $u, v \in Z$ and $m \in Z_{48}$ such that $uvm \in \langle\overline{12}\rangle$, implies that either $um \in \mathcal{F} + \left(soc(Z_{48})\right) = \langle\overline{12}\rangle + (\langle\bar{8}\rangle) = \langle\bar{4}\rangle$ or $vm \in \mathcal{F} + \left(soc(Z_{48})\right) = \langle\overline{12}\rangle + (\langle\bar{8}\rangle) = \langle\bar{4}\rangle$or $uv \in \left[\mathcal{F} + \left(soc(Z_{48})\right)_R : Z_{48}\right] = 4Z$. That is $2.2.\bar{3} \in \langle\overline{12}\rangle$, implies that $2.2 \in \left[\mathcal{F} + \left(soc(Z_{48})\right)_R : Z_{48}\right] = 4Z$. But $\mathcal{F}$ is not STPNQ-2-A submodule of $\mathcal{H}$, since $3.2.2.\bar{1} \in \langle\overline{12}\rangle$, but $3.2.\bar{1} \notin \langle\overline{12}\rangle + (J(Z_{48}) \cap soc(Z_{48}) = \langle\overline{12}\rangle$ and $2.2.\bar{1} \notin \langle\overline{12}\rangle + (J(Z_{48}) \cap soc(Z_{48}) = \langle\overline{12}\rangle$.

5. Any Quasi-2-Absorbing submodule of an R-module$\mathcal{H}$ is STPNQ-2-A submodule, but not contrary.

Proof clear.

For the converse see the following example:

Let $\mathcal{H} = Z_{36}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{12}\rangle$ is STPNQ-2-A submodule of $\mathcal{H}$, since $J(Z_{36}) = \langle\bar{6}\rangle$ and $soc(Z_{36}) = \langle\bar{6}\rangle$. That is for all $a, b, c \in Z$ and $m \in Z_{36}$ such that $abcm \in \langle\overline{12}\rangle$, implies that either $acm \in \langle\overline{12}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$ or$bcm \in \langle\overline{12}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$or $abm \in \langle\overline{12}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$, that is $2.3.2.\bar{1} \in \langle\overline{12}\rangle$ and $3.2.\bar{1} \in \langle\overline{12}\rangle + (J(Z_{36}) \cap soc(Z_{36})) = \langle\overline{12}\rangle + (\langle\bar{6}\rangle \cap \langle\bar{6}\rangle) = \langle\bar{6}\rangle$. But $\mathcal{F}$ is not Quasi-2_Absorbing submodule of $\mathcal{H}$, since $2.3.2.\bar{1} \in \langle\overline{12}\rangle$, but $3.2.\bar{1} \notin \langle\overline{12}\rangle$ and $2.2.\bar{1} \notin \langle\overline{12}\rangle$.

6. Any STPNQ-2-A submodule of an R-module $\mathcal{H}$ is Nearly-Quasi-2-Absorbing submodule, but not contrary.

Proof clear.

For the opposite see the following example:

Let $\mathcal{H} = Z_{48}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{24}\rangle$ is Nearly-Quasi-2-Absorbing submodule of M since $soc(Z_{48}) = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{6}\rangle$ that is for all $u, v, h \in Z$ and $m \in Z_{48}$ such that $uvhm \in \langle\overline{24}\rangle$, implies that either $uhm \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$ or $vhm \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$ or $uvm \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$. That is $2.4.3.\bar{1} \in \langle 24\rangle$, implies that $2.3.\bar{1} = \bar{6} \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$ and $4.3.\bar{1} = \overline{12} \in \mathcal{F} + \left(J(Z_{48})\right) = \langle\overline{24}\rangle + (\langle\bar{6}\rangle) = \langle\bar{6}\rangle$. But $\mathcal{F}$ is not STPNQ-2-A submodule of $\mathcal{H}$, since $3.4.2.\bar{1} \in \langle\overline{24}\rangle$, but $3.2.\bar{1} \notin \langle\overline{24}\rangle + (J(Z_{48}) \cap soc(Z_{48}) = \langle\overline{24}\rangle$ and $4.2.\bar{1} \notin \langle\overline{24}\rangle + (J(Z_{48}) \cap soc(Z_{48}) = \langle\overline{24}\rangle$ and $3.4.\bar{1} \notin \langle\overline{24}\rangle + (J(Z_{48}) \cap soc(Z_{48}) = \langle\overline{24}\rangle$.

7. Any STPNQ-2-A submodule of an R-module $\mathcal{H}$ is Pseudo-Quasi-2-Absorbing submodule, but not contrary.

Proof clear.

For the opposite see the following example:

Let $\mathcal{H} = Z_{48}$, $R = Z$ and the submodule $\mathcal{F} = \langle\overline{12}\rangle$ is pseudo-2-Absorbing submodule of $\mathcal{H}$ since $soc(Z_{48}) = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{6}\rangle$ that is for all $u, v, h \in Z$ and $m \in Z_{48}$ such that $uvhm \in \langle\overline{12}\rangle$, implies thateither $uhm \in \mathcal{F} + \left(soc(Z_{48})\right) = \langle\overline{12}\rangle + (\langle\bar{8}\rangle) = \langle\bar{4}\rangle$ or $vhm \in \mathcal{F} + \left(soc(Z_{48})\right) = \langle\overline{12}\rangle + (\langle\bar{8}\rangle) = \langle\bar{4}\rangle$ or $uvm \in \mathcal{F} + \left(soc(Z_{48})\right) = \langle\overline{12}\rangle + (\langle\bar{8}\rangle) = \langle\bar{4}\rangle$. That is $2.2.3.\bar{1} \in \langle\overline{12}\rangle$, implies that $2.2.\bar{1} \in \mathcal{F} + \left(soc(Z_{48})\right) = \langle\overline{12}\rangle + (\langle\bar{8}\rangle) = \langle\bar{4}\rangle$. But $\mathcal{F}$ is not STPNQ-2-A submodule of $\mathcal{H}$, since $3.2.2.\bar{1} \in \langle\overline{12}\rangle$, but $3.2.\bar{1} \notin \langle\overline{12}\rangle + (J(Z_{48}) \cap soc(Z_{48}) = \langle\overline{12}\rangle$ and $2.2.\bar{1} \notin \langle\overline{12}\rangle + (J(Z_{48}) \cap soc(Z_{48}) = \langle\overline{12}\rangle$.

The following proposition gives characterization of STPNQ-2-A submodules.

Proposition 2.3 A proper submodule $\mathcal{F}$ of $\mathcal{H}$ is STPNQ-2-A submodule of $\mathcal{H}$ if and only if for any $a, b, c \in R$, $[\mathcal{F} :_{W} abc]$ $[\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ab] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ac] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} bc]$.

Proof (⇒) Let $x \in [\mathcal{F} :_{\mathcal{H}} abc]$, then $abcx \in \mathcal{F}$. Since $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$, then either $acx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $abx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus either $x \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ac]$ or $x \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_{\mathcal{H}} bc]$ or $x \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ab]$. It means that $x \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ac] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} bc] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ab]$. Therefore $[\mathcal{F} :_{\mathcal{H}} abc] \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ab] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ac] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(W)):_{\mathcal{H}} bc]$.

(⇐) Let $abcx \in \mathcal{F}$ for $a, b, c \in R$, $x \in \mathcal{H}$, then $x \in [\mathcal{F} :_{\mathcal{H}} abc]$. By our hypothesis $x \in [\mathcal{F} :_{\mathcal{H}} abc] \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ab] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ac] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} bc]$. It means that either $x \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ac]$ or $x \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} bc]$ or $x \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{H}} ab]$. That is either $acx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $abx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore $\mathcal{F}$ STPNQ-2-A submodule of $\mathcal{H}$.

Proposition 2.4 A proper submodule $\mathcal{F}$ of $\mathcal{H}$ is STPNQ-2-A submodule of $\mathcal{H}$ if and only if for any $a, b \in R$ and $x \in \mathcal{H}$ such that $abx \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, then $[\mathcal{F} :_R abx] \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R ax] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R bx]$.

Proof (⇒) Let $t \in [\mathcal{F} :_R abx]$, then $abtx \in \mathcal{F}$. Since $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ and $abx \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, it follows that either $atx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $btx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus either $t \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R ax]$ or $t \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R bx]$. Hence $t \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R ax] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R bx]$. then $[\mathcal{F} :_R abx] \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R ax] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R bx]$.

(⇐) Let $abcx \in \mathcal{F}$ for $a, b, c \in R$, $x \in \mathcal{H}$ and let $abx \notin \mathcal{F} + (J(W) \cap soc(W))$. Hence $\in [\mathcal{F} :_R abx] \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R ax] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R bx]$. It follows that either $c \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R ax]$ or $c \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R bx]$. That is either $acx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore $\mathcal{F}$ STPNQ-2-A submodule of $\mathcal{H}$.

Proposition 2.5 A proper submodule $\mathcal{F}$ of $\mathcal{H}$ is STPNQ-2-A submodule of $\mathcal{H}$ if and only if $abcL \subseteq \mathcal{F}$, for $a, b, c \in R$ and L is a submodule of $\mathcal{H}$, implies that either $acL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $abL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$

Proof (⇒) Let $abcL \subseteq \mathcal{F}$, for $a, b, c \in R$ and L is a submodule of $\mathcal{H}$. Suppose that $abL \nsubseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, $acL \nsubseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and $bcL \nsubseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Then there is $e_1, e_2, e_3 \in L$ such that $abe_1 \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, $ace_2 \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and $bce_3 \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Now, $abce_1 \in \mathcal{F}$ and since $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ with $abe_1 \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, then either $bce_1 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $ace_1 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Also since $abce_2 \in \mathcal{F}$ and $ace_2 \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, then either $bce_2 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $abe_2 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Again $abce_3 \in \mathcal{F}$ and since $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ with $bce_3 \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, then either $ace_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$or $abe_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Now, $abc(e_1 + e_2 + e_3) \in \mathcal{F}$ and $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$, implies that either $ab(e_1 + e_2 + e_3) \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $ac(e_1 + e_2 + e_3) \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bc(e_1 + e_2 + e_3) \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. If $ab(e_1 + e_2 + e_3) = abe_1 + abe_2 + abe_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. But $abe_2 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and $abe_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, then $abe_1 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ which is incongruent. If $ac(e_1 + e_2 + +e_3) = ace_1 + ace_2 + ace_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. But $ace_1 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and $ace_3 \in \mathcal{F} + (J(W) \cap soc(\mathcal{H}))$, then $ace_2 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ which is contradiction. If $bc(e_1 + e_2 + e_3) = bce_1 + bce_2 + bce_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. But $bce_1 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and $bce_2 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, then $bce_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ which is contradiction. Hence $acL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $abL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

(⇐) Let $abcn \in \mathcal{F}$ for $a, b, c \in R$, $\in \mathcal{H}$, then $abc(n) \subseteq \mathcal{F}$, hence by hypothesis either $ac(n) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bc(n) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $ab(n) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. That is either $acn \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcn \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $abn \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$.

Proposition 2.6 A proper submodule $\mathcal{F}$ of $\mathcal{H}$ is STPNQ-2-A submodule of $\mathcal{H}$ if and only if for any $a, b, t \in R$ and A is a submodule of $\mathcal{H}$ with $abA \nsubseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, then $[\mathcal{F} :_R abA] \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R aA] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R bA]$.

Proof (⇒) Let $c \in [\,\mathcal{F} :_R abA\,]$, then $abcA \subseteq \mathcal{F}$. Since $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ and $abA \nsubseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, it follows that either $acA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus either $c \in [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R aA\,]$or $c \in [\mathcal{F} + (J(\mathbb{W}) \cap soc(\mathbb{W})) :_R bA]$. Hence $c \in [\mathcal{F} + (J(\mathbb{W}) \cap soc(\mathbb{W})) :_R aA] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R bA]$, then $[\,\mathcal{F} :_R abA\,] \subseteq [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R aA\,] \cup [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R bA\,]$.

(⇐) Let $abcA \subseteq \mathcal{F}$ for $a, b, c \in R$, A is a submodule of $\mathcal{H}$ with $abA \nsubseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Hence $c \in [\,\mathcal{F} :_R abA\,] \subseteq [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R aA] \cup [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R bA]$. It follows that either $c \in [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R aA]$ or $c \in [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R bA]$. That is either $acA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore $\mathcal{F}$ STPNQ-2-A submodule of $\mathcal{H}$.

Proposition 2.7 Let $\mathcal{H}$ be module and $\mathcal{F}$ be a proper submodule of $\mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ ifandonlyif for every submodule A of $\mathcal{H}$ and for every ideals I_1, I_2, I_3 of R such that $I_1 I_2 I_3 A \subseteq \mathcal{F}$ implies that either $I_1 I_2 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1 I_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2 I_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Proof (⇒) Let $I_1 I_2 I_3 A \subseteq \mathcal{F}$, where I_1, I_2, I_3 are ideals of R and A is a submodule of $\mathcal{H}$, with $I_1 I_2 A \nsubseteq [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R \mathcal{H}\,]$. To prove that $I_1 I_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2 I_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Suppose that $I_1 I_3 A \nsubseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and $I_2 I_3 A \nsubseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, that is there exist $a_1, a_2, a_3 \in A$ and a nonzero $r \in I_1$, $s \in I_2$ and $t \in I_3$ such that $rsa_1 \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and $rta_2 \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and $sta_3 \notin \mathcal{F} + (J(\mathbb{W}) \cap soc(\mathbb{W}))$. Now, $rsta_1 \in \mathcal{F}$ and $rsa_1 \notin \mathcal{F} + (J(\mathbb{W}) \cap soc(\mathbb{W}))$, implies that either $rta_1 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sta_1 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Also $rsta_2 \in \mathcal{F}$ and $rta_2 \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, implies that either $rsa_2 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$or $sta_2 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Again, $rsta_3 \in \mathcal{F}$ and $sta_3 \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, implies that either $rta_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rsa_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Now, $rst(a_1 + a_2 + a_3) \in \mathcal{F}$ and $\mathcal{F}$ is STPNQ-2-A , then either $rs(a_1 + a_2 + a_3) \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rt(a_1 + a_2 + a_3) \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $st(a_1 + a_2 + a_3) \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. If $rs(a_1 + a_2 + a_3) = rsa_1 + rsa_2 + rsa_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and rsa_2, $rsa_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, hence $rsa_1 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ which is a contradiction. If $rt(a_1 + a_2 + a_3) = rta_1 + rta_2 + rta_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and rta_1, $rta_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, hence $rta_2 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ which is a contradiction. If $st(a_1 + a_2 + a_3) = sta_1 + sta_2 + sta_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and sta_1, $sta_2 \in \mathcal{F} + (J(\mathbb{W}) \cap soc(\mathbb{W}))$, hence $sta_3 \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ which is a contradiction. Thus either $I_1 I_2 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1 I_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2 I_3 A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

(⇐) Suppose that $abcA \subseteq \mathcal{F}$, where $a, b, c \in R$, A is a submodule of $\mathbb{W}$ then $\langle a\rangle\langle b\rangle\langle c\rangle A \subseteq \mathcal{F}$, so by hypothesis, either $\langle a\rangle\langle b\rangle A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $\langle a\rangle\langle c\rangle A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $\langle b\rangle\langle c\rangle A \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Hence either $abA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $acA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Then by proposition 2.5 $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$.

Proposition 2.8 Let $\mathcal{H}$ be module and $\mathcal{F}$ be a proper submodule of $\mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ if and only if for any $r, s \in R$ and I is an ideal of R and $x \in \mathbb{W}$ with $rsIx \subseteq \mathcal{F}$ implies that either $rsx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rIx \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sIx \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Proof (⇒) Let $rsIx \subseteq \mathcal{F}$ for $r, s \in R$ and I is an ideal of R and $x \in \mathcal{H}$, it follows that $I \subseteq [\mathcal{F} :_R rsx]$. If $rsx \in \mathcal{F} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, hence $rsx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, then we are done. Suppose that $rsx \notin \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, then by proposition 2.4 $[\,\mathcal{F} :_R rsx\,] \subseteq [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R rx\,] \cup [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R sx\,]$. But $rsIx \subseteq \mathcal{F}$,then$I \subseteq [\mathcal{F} :_R rsx] \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R rx\,] \cup [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R sx\,]$,hence $I \subseteq [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R rx\,] \cup [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}) :_R sx\,]$, it follows that either $I \subseteq [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathbb{W})) :_R rx\,]$ or $I \subseteq [\,\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R sx\,]$, thus either $rIx \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sIx \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

(⇐) Let $rstx \in \mathcal{F}$ for $r, s, t \in R$ and $x \in \mathcal{H}$, that is $rs\langle t\rangle x \subseteq \mathcal{F}$. It follows by hypothesis either $rsx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $r\langle t\rangle x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $s\langle t\rangle x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Hence either $rsx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rtx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $stx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$.

From the proposition 2.7 and proposition 2.8 we get the following corollaries.

Corollary 2.9 Let $\mathcal{H}$ be module and $\mathcal{F}$ be a proper submodule of $\mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ if and only if for every ideals I_1, I_2, I_3 of R and $x \in \mathbb{W}$ such that $I_1 I_2 I_3 x \subseteq \mathcal{F}$ implies that either $I_1 I_2 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1 I_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2 I_3 x \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Corollary 2.10 Let $\mathcal{H}$ be module and $\mathcal{F}$ be a proper submodule of $\mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ ifandonlyif for each $r \in R$ and any ideals I, J of R and every submodule A of $\mathbb{W}$ with $rIJA \subseteq \mathcal{F}$ implies that either $rIA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rJA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $IJA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Corollary 2.11 Let $\mathcal{H}$ be module and $\mathcal{F}$ be a proper submodule of $\mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ if and only if for any $r, s \in R$ and any ideal I of R and every submodule A of $\mathcal{H}$ with $rsIA \subseteq \mathcal{F}$ implies that either $rsA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$or $rIA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sIA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Corollary 2.12 Let $\mathcal{H}$ be an R-module and $\mathcal{F}$ be a proper submodule of $\mathcal{H}$. Then $\mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ if and only if for every ideals I, J of R and $x \in \mathbb{W}$, with $IJx \nsubseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$,$[\mathcal{F}:_R IJx] \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R Ix] \cup [\mathcal{F} + soc(\mathcal{H}) + J(\mathcal{H}):_R Jx]$.

Before proving the following proposition, we need the following lemmas .

Lemma 2.13 [7 , Ex .(12) (c)] An R _ module $\mathcal{H}$ is a semi simple if and only if for each submodule $\mathcal{F}$ of $\mathcal{H}$, $soc\left(\frac{\mathcal{H}}{\mathcal{F}}\right) = \frac{soc(\mathcal{H})+\mathcal{F}}{\mathcal{F}}$.

Lemma 2.14 [7, Ex(12), P. 239] Let $\mathcal{F}$ be a submodule of a semi simple R _ module $\mathcal{H}$ then $J\left(\frac{\mathcal{H}}{\mathcal{F}}\right) = \frac{J(\mathcal{H})+\mathcal{F}}{\mathcal{F}}$.

Proposition 2.15 Let $\mathcal{H}$ is a semi simple R-module $\mathcal{F}$ and A are submdules for $\mathcal{H}$such that $A \subseteq \mathcal{F}$, and $\mathcal{F}$ is a proper submodule of $\mathcal{H}$. If A and $\frac{\mathcal{F}}{A}$ are STPNQ-2-A submodules of $\mathcal{H}$ and $\frac{\mathcal{H}}{A}$ respectively, then $\mathcal{F}$ is STPNQ-2-A submodules of $\mathcal{H}$.

Proof Let $I_1I_2I_3m \subseteq \mathcal{F}$, for I_1, I_2, I_3 are ideals of R, $m \in \mathcal{H}$. So $I_1I_2I_3(m + A) = I_1I_2I_3m + A \subseteq \frac{\mathcal{F}}{A}$. If $I_1I_2I_3m \subseteq A$ and A is STPNQ-2-A submodules of $\mathbb{W}$,implies that by corollary (2.9) either $I_1I_2m \subseteq A + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_1I_3m \subseteq A + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I_2I_3m \subseteq A + (J(\mathcal{H}) \cap soc(\mathcal{H})) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, hence $\mathcal{F}$ is STPNQ-2-A submodules for $\mathcal{H}$. So, we may assume that $I_1I_2I_3m \nsubseteq A$. It follows that $I_1I_2I_3(m + A) \subseteq \frac{\mathcal{F}}{A}$, but $\frac{\mathcal{F}}{A}$ is STPNQ-2-A submodules of $\frac{\mathcal{H}}{A}$, again by corollary (2.9) either $I_1I_2(m + A) \subseteq \frac{\mathcal{F}}{A} + \left(J\left(\frac{\mathcal{H}}{A}\right) \cap soc\left(\frac{\mathcal{H}}{A}\right)\right)$ or $I_1I_3(m + A) \subseteq \frac{\mathcal{F}}{A} + \left(J\left(\frac{\mathcal{H}}{A}\right) \cap soc\left(\frac{\mathcal{H}}{A}\right)\right)$ or $I_2I_3(m + A) \subseteq \frac{\mathcal{F}}{A} + \left(J\left(\frac{\mathcal{H}}{A}\right) \cap soc\left(\frac{\mathcal{H}}{A}\right)\right)$. Since $\mathcal{H}$ is a semi simple then by lemmas (2.10, 2.11) either $I_1I_2(m + A) \subseteq \frac{\mathcal{F}}{H} + \left(\frac{A+J(\mathcal{H})}{A} \cap \frac{A+soc(\mathcal{H})}{A}\right)$ or $I_1I_3(m + A) \subseteq \frac{\mathcal{F}}{A} + \left(\frac{H+J(\mathcal{H})}{H} \cap \frac{H+soc(\mathcal{H})}{H}\right)$ or $I_2I_3(m + A) \subseteq \frac{N}{H} + \left(\frac{H+J(\mathcal{H})}{H} \cap \frac{H+soc(\mathcal{H})}{H}\right)$. But $A \subseteq \mathcal{F}$,it follows that $A + soc(\mathcal{H}) \subseteq \mathcal{F} + soc(\mathcal{H})$ and $A + J(\mathcal{H}) \subseteq \mathcal{F} + J(\mathcal{H})$, hence $\frac{\mathcal{F}}{A} + \left(\frac{A+J(\mathcal{H})}{A} \cap \frac{A+soc(\mathcal{H})}{A}\right) \subseteq \frac{\mathcal{F}}{A} + \left(\frac{\mathcal{F}+J(\mathcal{H})}{A} \cap \frac{\mathcal{F}+soc(\mathcal{H})}{A}\right) = \frac{\mathcal{F}+(J(\mathcal{H}) \cap soc(\mathcal{H}))}{A}$. Thus either $I_1I_2(m + A) \subseteq \frac{\mathcal{F}+(J(\mathcal{H}) \cap soc(\mathcal{H}))}{A}$ or $I_1I_3(m + A) \subseteq \frac{\mathcal{F}+(J(\mathcal{H}) \cap soc(\mathcal{H}))}{A}$ or $I_1I_3(m + A) \subseteq \frac{\mathcal{F}+(J(\mathcal{H}) \cap soc(\mathcal{H}))}{A}$, it follows that either $I_1I_2m \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$or $I_1I_3m \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$or $I_2I_3m \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Hence by corollary (2.9) $\mathcal{F}$ is STPNQ-2-A submodules of $\mathcal{H}$.

Remark 2.16 The junction of two STPNQ-2-A submodules of an R_module $\mathcal{H}$ need not be an STPNQ-2-A submodule.

The following example explain:

Consider the Z_module Z_{60} and the submodules $\mathcal{L}=\langle\bar{5}\rangle$ and $\mathbb{Q}=\langle\bar{6}\rangle$ are STPNQ-2-A submodules of the Z_module Z_{60} (because $\langle\bar{5}\rangle$ and $\langle\bar{6}\rangle$ are 2-Absorbing of Z_{60}), but $\mathcal{L} \cap \mathbb{Q} = \langle\overline{30}\rangle$ is not STPNQ_2_Absorbing, since $2.3.5.\bar{1} \in \langle\overline{30}\rangle$, but $2.5.\bar{1} \notin \langle\overline{30}\rangle + \big(J(Z_{60}) \cap soc(Z_{60})\big) = \langle\overline{30}\rangle$ and $3.5.\bar{1} \notin \langle\overline{30}\rangle + \big(J(Z_{60}) \cap soc(Z_{60})\big) = \langle\overline{30}\rangle$ and $2.3.\bar{1} \notin \langle\overline{30}\rangle + \big(J(Z_{60}) \cap soc(Z_{60})\big) = \langle\overline{30}\rangle$.

The above remark is fulfilled under the condition.

But before that we need the following lemma.

Lemma 2.17 [7 , Lemma. (2.3.15)] Let $\mathcal{L}$, $\mathbb{Q}$ and $\mathcal{B}$ be submodule of an R_module $\mathcal{H}$ with $\mathbb{Q} \subseteq \mathcal{B}$. Then $(\mathcal{L} + \mathbb{Q}) \cap \mathcal{B} = (\mathcal{L} \cap \mathcal{B}) + \mathbb{Q} = (\mathcal{L} \cap \mathcal{B}) + (\mathbb{Q} \cap \mathcal{B})$.

Proposition 2.18 Let $\mathcal{L}$ and $\mathcal{F}$ be a proper submodules of an R-module $\mathcal{H}$, with $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq \mathcal{L}$ or $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq \mathcal{F}$. If $\mathcal{L}$ and $\mathcal{F}$ are STPNQ-2-A submodules of $\mathcal{H}$, then $\mathcal{L} \cap \mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$..

Proof

Let $rsIx \subseteq \mathcal{L} \cap \mathcal{F}$, for $r, s \in R$, $x \in \mathcal{H}$ and I is an ideal of R, it follows that $rsIx \subseteq \mathcal{L}$ and $rsIx \subseteq \mathcal{F}$. But both $\mathcal{L}$ and $\mathcal{F}$ are STPNQ-2-A submodules of $\mathcal{H}$, then by proposition (2.8) we have either $rsx \in \mathcal{L} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rIx \subseteq \mathcal{L} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sIx \subseteq \mathcal{L} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ and $rsx \in \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rIx \subseteq \mathcal{F} + (J(\mathcal{H}) \cap$

$soc(\mathcal{H}))$ or $sIx \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ .Thus either $rsx \in (\mathcal{L} + (J(\mathcal{H}) \cap soc(\mathcal{H}))) \cap (\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})))$ or $rIx \subseteq (\mathcal{L} + (J(\mathcal{H}) \cap soc(\mathcal{H}))) \cap (\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sIx \subseteq (\mathcal{L} + (J(\mathcal{H}) \cap soc(\mathcal{H}))) \cap (\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})))$. If $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq \mathcal{F}$, then $\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}) = \mathcal{F}$. Hence either $rsx \in \mathcal{F} \cap (\mathcal{L} + (J(\mathcal{H}) \cap soc(\mathcal{H})))$ or $rIx \subseteq \mathcal{F} \cap (\mathcal{L} + (J(\mathcal{H}) \cap soc(\mathcal{H})))$ or $sIx \subseteq \mathcal{F} \cap (\mathcal{L} + (J(\mathcal{H}) \cap soc(\mathcal{H})))$. Therefore by lemma 2.17 we get either $rsx \in (\mathcal{L} \cap \mathcal{F}) + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $rIx \subseteq (\mathcal{L} \cap \mathcal{F}) + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $sIx \subseteq (\mathcal{L} \cap \mathcal{F}) + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Hence by proposition 2.8 $\mathcal{L} \cap \mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$. In similar way $\mathcal{L} \cap \mathcal{F}$ is STPNQ-2-A submodule of $\mathcal{H}$ if $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq \mathcal{L}$.

3.The Relations Of STPNQ-2-A Submodules With 2-Absorbing Submodules And Other Form Of Submodules.

In this section we introduce the relations Of STPNQ-2-A submodules with 2-Absorbing submodules and other form of submodules.

The opposite of Remarks and Examples 2.2 (2) is true under certain conditions.

But before that we need the following lemma.

Lemma 3.1 [7, prop . (9.14) (c)] If $\mathcal{H}$ is a semi-simple R-module, then $J(\mathcal{H}) = 0$.

Proposition 3.2 Let $\mathcal{H}$ be a cyclic semi-simple R_module, A is a propersubmodule of $\mathcal{H}$. Then A is STPNQ-2-A of $\mathcal{H}$ if and only if A is 2_Absorbing submodule of $\mathcal{H}$.

Proof (⇒) Let $aby \in A$ for $a, b \in R$, $y \in \mathcal{H}$, then ∃an element $c \in R$ such that $y = ch$, hence $aby = abch \in A$. Since A is STPNQ-2-Absorbing, then either $abh \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bch \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $ach \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. That is either $ab \in [A + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R h] = [A + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R \mathcal{H}]$ or $by \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $ay \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$, Since G is semi-simple, then by lemma (3.1) $J(\mathcal{H}) = 0$, so $(J(\mathcal{H}) \cap soc(\mathcal{H})) = (0) \cap soc(\mathcal{H}) = (0)$. Thus either $ay \in A$ or $by \in A$ or $ab \in [A :_R \mathcal{H}]$. Hence A is a 2_Absorbing submodule of $\mathcal{H}$.

(⇐) Direct.

Before proving the following proposition, we need the following lemma.

Lemma 3.3 [10] If $\mathcal{H}$ is a regular R-module, then $J(\mathcal{H}) = 0$.

Proposition 3.4 Let $\mathcal{H}$ be a cyclic regular R_module, and A is a propersubmodule of $\mathcal{H}$. Then A is STPNQ-2-A of $\mathcal{H}$ if and only if A is 2_Absorbing submodule of $\mathcal{H}$.

Proof Follows as in proposition 3.2 and use lemma 3.3.

The opposite of Remarks and Examples 2.2 (3) is true under certain conditions.

Proposition 4.4 Let $\mathcal{H}$ be an R_module, and $A \subset \mathcal{H}$ with $soc(\mathcal{H}) = \mathcal{H}$. Then A is Nearly_2_Absorbing if and only if A STPNQ-2-A submodule of $\mathcal{H}$.

Proof (⇒) Let $abcy \in A$ for $a, b, c \in R$, $y \in \mathcal{H}$, that is $ab(cy) \in A$. Since A is Nearly-2-Absorbing, then either $a(cy) \in A + J(\mathcal{H})$ or $b(cy) \in A + J(\mathcal{H})$ or $ab\mathcal{H} \subseteq A + J(\mathcal{H})$, that is $aby \in A + J(\mathcal{H})$. Thus either $acy \in A + J(\mathcal{H})$ or $bcy \in A + J(\mathcal{H})$ or $by \in A + J(\mathcal{H})$. But $J(\mathcal{H}) \subseteq \mathcal{H}$, so $J(\mathcal{H}) \cap \mathcal{H} = J(\mathcal{H})$, that is either $acy \in A + J(\mathcal{H}) \cap \mathcal{H}$ or $bcy \in A + J(\mathcal{H}) \cap \mathcal{H}$ or $aby \in A + J(\mathcal{H}) \cap \mathcal{H}$. Since $soc(\mathcal{H}) = \mathcal{H}$ it follows that either $ay \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $by \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $aby \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ Thus A is STPNQ-2-A submodule of $\mathcal{H}$.

(⇐) Direct.

Before proving the following proposition, we need the following lemma.

Lemma 3.5 [4,prop (2.8)] Let A be a Nearly 2-Absorbingsubmodule of an R-module $\mathcal{H}$ with $J(\mathcal{H}) \subseteq A$. Then A is 2-Absorbing submodule.

Proposition 3.6 Let $\mathcal{H}$ be an R_module, and $A \subset \mathcal{H}$ with $J(\mathcal{H}) \subseteq A$. Then the following are Valente:

1. A is STPNQ-2-A submodule of $\mathcal{H}$.

2. A is Nearly_2_Absorbingsubmodule of $\mathcal{H}$.

3. A is 2_Absorbingsubmodule of $\mathcal{H}$.

Proof (1) ⇒ (2) Direct Remarks and Examples 2.2 (3).

(2) ⇒ (3) Direct by lemma 3.5.

(3) ⇒ (1) Direct Remarks and Examples 2.2 (2).

The converse of Remarks and Examples2.2 (4) is true under certain conditions.

But before that we need the following lemma.

Lemma 3.7 [20,lemma (2.3)] If an R-module H is an injective, then $J(H) = H$.

Proposition 3.8 Let $\mathcal{H}$ be an injective R_module, and $\subset \mathcal{H}$. Then A is Pseudo_2_Absorbing if and only if A is STPNQ-2-A submodule of $\mathcal{H}$.

Proof (⇒) Let $\alpha bcy \in A$ for $\alpha, b, c \in$ R, $y \in \mathcal{H}$, that is $\alpha b(cy) \in A$. Since A is Pseudo-2-Absorbing, then either $\alpha(cy) \in A + soc(\mathcal{H})$ or $b(cy) \in A + soc(\mathcal{H})$ or $\alpha b\mathcal{H} \subseteq A + soc(\mathcal{H})$, that is $aby \in A + soc(\mathcal{H})$. Therefore either $\alpha cy \in A + soc(\mathcal{H})$ or $bcy \in A + soc(\mathcal{H})$ or $aby \in A + soc(\mathcal{H})$. But $soc(\mathcal{H}) \subseteq \mathcal{H}$, so $\mathcal{H} \cap soc(\mathcal{H}) = soc(\mathcal{H})$, that is either $acy \in A + \mathcal{H} \cap soc(\mathcal{H})$ or $bcy \in A + \mathcal{H} \cap soc(\mathcal{H})$ or $aby \in A + \mathcal{H} \cap soc(\mathcal{H})$. Since $\mathcal{H}$ is an injective, then by lemma 3.7 $J(\mathcal{H}) = \mathcal{H}$, it follows that either $acy \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcy \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $aby \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Thus A is STPNQ-2-A submodule of $\mathcal{H}$.

(⇐) Direct.

Before proving the following proposition, we need the following lemma.

Lemma 3.9 [5,remark (1.2)] It is clear that every 2-Absorbingsubmodule of an R-module $\mathcal{H}$ is Pseudo 2-Absorbingsubmodule.

Proposition 3.10 Let $\mathcal{H}$ be an R_module, and $A \subset \mathcal{H}$ with $soc(\mathcal{H}) \subseteq A$ and $J(\mathcal{H}) \subseteq A$. Then the following are Valente:

1. A is 2_Absorbingsubmodule of $\mathcal{H}$.

2. A is Pseudo_2_Absorbingsubmodule of $\mathcal{H}$.

3. A is STPNQ-2-A submodule of $\mathcal{H}$.

4. A is Nearly_2_Absorbingsubmodule of $\mathcal{H}$.

Proof (1) ⇒ (2) Direct by lemma 3.9.

(2) ⇒ (3) Let $\alpha bcy \in A$ for $\alpha, b, c \in$ R, $y \in \mathcal{H}$, that is $\alpha b(cy) \in A$. Since A is Pseudo-2-Absorbing, then either $\alpha(cy) \in A + soc(\mathcal{H})$ or $b(cy) \in A + soc(\mathcal{H})$ or $\alpha b\mathcal{H} \subseteq A + soc(\mathcal{H})$, that is $aby \in A + soc(\mathcal{H})$.Thus either $\alpha cy \in A + soc(\mathcal{H})$ or $bcy \in A + soc(\mathcal{H})$ or $aby \in A + soc(\mathcal{H})$. But $soc(\mathcal{H}) \subseteq A$ then $A + soc(\mathcal{H}) = A$, and. Thus we have either $acy \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcy \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $aby \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. That is A is STPNQ-2-A submodule of $\mathcal{H}$.

(3) ⇒ (4) Direct Remarks and Examples 2.2 (3).

(4) ⇒ (1) Direct by lemma 3.5.

The opposite of Remarks and Examples 2.2 (5) is true under certain conditions

Proposition 3.11 Let $\mathcal{H}$ be an R_module, and A is a proper submodule of $\mathcal{H}$ with $J(\mathcal{H}) \subseteq A$ and $oc(\mathcal{H}) \subseteq A$. Then A is STPNQ_2_ Absorbing submodule of $\mathcal{H}$ if and only if A is Quasi 2_Absorbing submodule of $\mathcal{H}$.

Proof (⇒) Let A be a STPNQ-2-A submodule of an R-module $\mathcal{H}$ and $abcy \in A$, for $a,b,c \in R$, $y \in \mathcal{H}$. Since A is STPNQ-2-A submodule of $\mathcal{H}$, then either $acy \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcm \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $aby \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. But $J(\mathcal{H}) \subseteq A$ and $soc(\mathcal{H}) \subseteq A$, it follows that $J(\mathcal{H}) \cap soc(\mathcal{H}) \subseteq A$, that is $J(\mathcal{H}) \cap soc(\mathcal{H}) = A$. Thus either $acy \in A$ or $bcy \in A$ or $aby \in A$. Hence A is a Quasi 2_Absorbing submodule of $\mathcal{H}$.

(⇐) Direct.

The opposite of Remarks and Examples 2.2 (6) is true under certain conditions.

Proposition 3.12 Let $\mathcal{H}$ be an R_module with $J(\mathcal{H}) = soc(\mathcal{H})$, and $A \subset \mathcal{H}$. Then A is Nearly Quasi _2_Absorbing if and only if A is STPNQ-2-A submodule of $\mathcal{H}$.

Proof Direct by taking $J(\mathcal{H}) = J(\mathcal{H}) \cap soc(\mathcal{H})$.

Proposition 3.13 Let $\mathcal{H}$ be a multiplication R_module, and A is a proper submodule of $\mathcal{H}$ with $J(\mathcal{H}) \subseteq A$. Then the following are Valente:

1. A is STPNQ-2-A submodule of $\mathcal{H}$.
2. A is Nearly Quasi _2_Absorbingsubmodule of $\mathcal{H}$.
3. A is Quasi 2_Absorbingsubmodule of $\mathcal{H}$.
4. A is 2_Absorbingsubmodule of $\mathcal{H}$.

Proof (1) ⇒ (2) Direct Remarks and Examples 2.2 (3).

(2) ⇒ (3) Let $abcy \in A$, for $a,b,c \in R$, $y \in \mathcal{H}$. Since A is Nearly Quasi-2-Absorbing submodule of $\mathcal{H}$, then either $acy \in A + J(\mathcal{H})$ or $bcm \in A + J(\mathcal{H})$ or $aby \in A + J(\mathcal{H})$. But $J(\mathcal{H}) \subseteq A$ then $A + J(\mathcal{H}) = A$, that is either $acy \in A$ or $bcy \in A$ or $aby \in A$. That is A is Quasi 2_Absorbing submodule of $\mathcal{H}$.

(3) ⇒ (4) Let $\alpha by \in A$ for $\alpha, b \in$ R, $y \in \mathcal{H}$, since $\mathcal{H}$ is a multiplication then $y = I\mathcal{H}$ for some ideal I of R, hence $aby = abI\mathcal{H} \subseteq A$. Since A is Quasi -2-Absorbing, then either $\alpha I\mathcal{H} \subseteq A$ or $bI\mathcal{H} \subseteq A$ or $\alpha b\mathcal{H} \subseteq A$. That is either $\alpha y \in A$ or $by \in A$ or $ab \in [A:_R \mathcal{H}]$. That is A is 2_Absorbing submodule of $\mathcal{H}$.

(4) ⇒ (1) Direct Remarks and Examples 2.2 (2).

The opposite of Remarks and Examples 2.2 (7) is true under certain conditions.

Proposition 3.14 Let $\mathcal{H}$ be an R_module with $soc(\mathcal{H}) \subseteq J(\mathcal{H})$, and $A \subset \mathcal{H}$ with. Then A is Pseudo Quasi _2_Absorbing if and only if A is STPNQ-2-A submodule of $\mathcal{H}$.

Proof (⇒) Let $abcy \in A$, for $a,b,c \in R$, $y \in \mathcal{H}$. Since A is Pseudo Quasi-2-Absorbing submodule of $\mathcal{H}$, then either $acy \in A + soc(\mathcal{H})$ or $bcy \in A + soc(\mathcal{H})$ or $aby \in A + soc(\mathcal{H})$. But $soc(\mathcal{H}) \subseteq J(\mathcal{H})$, then $soc(\mathcal{H}) \cap J(\mathcal{H}) = soc(\mathcal{H})$, so either $acy \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcy \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $aby \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. That is A is STPNQ-2-A submodule of $\mathcal{H}$.

(⇐) Direct.

Before proving the following proposition, we need the following lemma.
Lemma 3.15 [5, Remarks and Examples(2.1.2)(4)] Every pseudo 2-Absorbingsubmodule of an R-module $\mathcal{H}$ is a pseudo Quasi 2-Absorbingsubmodule of an R-module $\mathcal{H}$.

Proposition 3.16 Let $\mathcal{H}$ be a multiplication R_module, and A a proper submodule of $\mathcal{H}$ with $soc(\mathcal{H}) \subseteq J(\mathcal{H})$ and $(\mathcal{H}) \subseteq A$. Then the following are Valente:

1. A is 2_Absorbingsubmodule of $\mathcal{H}$.

2. A is Pseudo_2_Absorbingsubmodule of $\mathcal{H}$.

3. A is Pseudo Quasi _2_Absorbingsubmodule of $\mathcal{H}$.

4. A is STPNQ-2-A submodule of $\mathcal{H}$.

5. A is Nearly_2_Absorbingsubmodule of $\mathcal{H}$.

6. A is Nearly Quasi _2_Absorbingsubmodule of $\mathcal{H}$.

7. A is Quasi 2_Absorbingsubmodule of $\mathcal{H}$.

Proof (1) ⇒ (2) Direct by lemma 3.9.

(2) ⇒ (3) Direct by lemma 3.15.

(3) ⇒ (4) Let $abcy \in A$, for $a,b,c \in R$, $y \in \mathcal{H}$. Since A is Pseudo Quasi-2-Absorbing submodule of $\mathcal{H}$, then either $acy \in A + soc(\mathcal{H})$ or $bcy \in A + soc(\mathcal{H})$ or $aby \in A + soc(\mathcal{H})$. But $soc(\mathcal{H}) \subseteq J(\mathcal{H})$, then $soc(\mathcal{H}) \cap J(\mathcal{H}) = soc(\mathcal{H})$, so either $acy \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $bcy \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $aby \in A + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. That is A is STPNQ-2-A submodule of $\mathcal{H}$.

(4) ⇒ (5) Direct Remarks and Examples 2.2 (3).

(5) ⇒ (6) Let $\alpha bcy \in A$ for $\alpha, b, c \in R$, $y \in \mathcal{H}$, that is $\alpha b(cy) \in A$. Since A is Nearly-2-Absorbing, then either $\alpha(cy) \in A + J(\mathcal{H})$ or $b(cy) \in A + J(\mathcal{H})$ or $\alpha b\mathcal{H} \subseteq A + soc(\mathcal{H})$, that is $aby \in A + J(\mathcal{H})$.Thus either $\alpha cy \in A + J(\mathcal{H})$ or $bcy \in A + J(\mathcal{H})$ or $aby \in A + J(\mathcal{H})$. That is Nearly Quasi _2_Absorbing submodule of $\mathcal{H}$.

(6) ⇒ (7) Let $abcy \in A$, for $a,b,c \in R$, $y \in \mathcal{H}$. Since A is Nearly Quasi-2-Absorbing submodule of $\mathcal{H}$, then either $acy \in A + J(\mathcal{H})$ or $bcm \in A + J(\mathcal{H})$ or $aby \in A + J(\mathcal{H})$. But $J(\mathcal{H}) \subseteq A$ then $A + J(\mathcal{H}) = A$, that is either $acy \in A$ or $bcy \in A$ or $aby \in A$. That is A is Quasi 2_Absorbing submodule of $\mathcal{H}$.

(7) ⇒ (1) Let $\alpha by \in A$ for $\alpha, b \in R$, $y \in \mathcal{H}$, since $\mathcal{H}$ is a multiplication then $y = I\mathcal{H}$ for some ideal I of R, hence $aby = abI\mathcal{H} \subseteq A$. Since A is Quasi -2-Absorbing, then either $\alpha I\mathcal{H} \subseteq A$ or $bI\mathcal{H} \subseteq A$ or $\alpha b\mathcal{H} \subseteq A$. That is either $\alpha y \in A$ or $by \in A$ or $ab \in [A:_R \mathcal{H}]$. That is A is 2_Absorbing submodule of $\mathcal{H}$.

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