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Extend Nearly Pseudo Semi-2-Absorbing Submodules (II)

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ABSTRACT

In this work we will study the concept of Extend Nearly Pseudo Semi-2-Absorbing submodules in multiplication modules and characterization of Extend Nearly Pseudo Semi-2-Absorbing ideals by of Extend Nearly Pseudo Semi-2-Absorbing submodules.

MSC.

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1. Introduction

It is known that the concept of the 2-Absorbing submodules was studied in previous years by researchers Darani and Soheilinia, where a proper submodule V of an R-module M is called 2-Absorbing submodule if whenever $abw \in V$ for $a, b \in R$ and $w \in$ M, then either $aw \in V$ or $bw \in V$ or $ab \in [V:_R \text{M}]$[1], as $[V:_R \text{M}] = \{a \in R: a\text{M} \subseteq \text{V}\}$[2]". Also, the concept of Semi-2-Absorbing submodules is one of the important generalizations in this research, where a *proper* submodule V of an R-*module* M is called Semi-2-Absorbing submodule if whenever $a^2w \in V$ for $a \in R$ and $w \in$ M, then either $aw \in V$ or $a^2 \in [V:_R \text{M}]$[3]. It is known that many concepts were circulated in previous years, such as (WN-2-Absorbing, WNS-2-Absorbing, Weakly Semi2-Absorbing, Quasi Primary-2-Absorbing, WES-2-Absorbing, WEQ-2-Absorbing and Nearly Semi-2-Absorbing) submodules; see [4, 5, 6, 7, and 8]. Also, these concepts are generalizations of Extend Nearly Pseudo Semi-2-Absorbing submodules. It is worth noting that this research is continuations of the research presented in the same journal see [9]. The multiplication module is define by an R-module M is multiplication, if every submodule Ꝁ of M is of the form $Ꝁ = IM$ for some ideal I of R. Equivalently M is a multiplication R-module if every submodule Ꝁ of M of the form $Ꝁ = [Ꝁ:_R M]M$[10]. Recall that an R-module M is faithful if $ann_R(M) = (0)$, where $ann_R(M) = \{r \in R: rw = (0)\}$[11]. Also, recall that an R-*module* M is finitely generated if $M = Rx_1 + Rx_2 + \cdots + Rx_n$ for $x_1, x_2, \ldots, x_n \in M$[12]. And an R-module M is called concellation module if $AM = BM$ for any ideals A and B of R implies that $A = B$[13]. Recall that An R-module M is a projective if for any R-epimorphism f from an R-module M on to an R-module $\bar{M}$ and for any homomorphism g from an R-module $\bar{\bar{M}}$ to

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$\bar{M}$, there exists a homomorphism h from $\bar{\bar{M}}$ to M such that $f \circ h = g$[12], Recall that a ring R is Artinian if R satisfies (DCC) is an ideals of R, that is if $\{I_\propto\}_{\propto\in\Lambda}$ is a family of ideals of R such that $I_1 \supseteq I_2 \supseteq \cdots$, then $\exists m \in Ƶ^+$ such that $In = Im$ for any $n \geq m$ [14]. Recall that a ring R is said to be local ring R if R has a unique maximal ideal[15]. The non-singular is define by an R-module M is non-singular if $Ƶ(M) = M$, where $Ƶ(M) = \{x \in M: xI = (0)$, for some essential ideal I of R$\}$[16]. And the content module is define by an R-module M is said to be content module if $(\bigcap_{i\in I} A_i)M = \bigcap_{i\in I} A_i M$ for each family of ideals A_i in R [17]. Recall that an R-module M is called a Z-regular if for each $e \in M$ there exists $f \in M' = Mom_R(M,R)$ such that $e = f(e)e$[18]. In addition, the weak cancellation can be defined as follows an R-module M is called weak cancellation if $IM = JM$, implies that $I + ann_R(M) = J + ann_R(M)$ for I, J are ideals in R[19]. All these basics helped us to present the most important propositions and new equivalents that pertain to this concept.

2. Extend Nearly Pseudo Semi-2-Absorbing Submodules in Multiplication modules.

In this paper we introduced the concept of Extend Nearly Pseudo Semi-2-Absorbing submodules in multiplication modules. As well as study the relationship between the Extend Nearly Pseudo Semi-2-Absorbing submodules with the residual of this concept.

Definition 2.1 A proper submodule Ƙ of an R-module M is said to be Extend Nearly Pseudo Semi-2-Absorbing (for short EXNPS2AB) submodule of M if whenever $\alpha^2 w \in$ Ƙ, where $\alpha \in R$, $w \in$ M implies that either $\alpha w \in$ Ƙ $+ soc(\mathrm{M}) + J(\mathrm{M})$ or $\alpha^2\mathrm{M} \subseteq$ Ƙ $+ soc(\mathrm{M}) + J(\mathrm{M})$.

And an ideal I of a ring R is called EXNPS2AB ideal of R, if I is an EXNPS2AB R-submodule of an R-module R.

Proposition 2.2 A proper submodule Ƙ of a multiplication R-module M is EXNPS2AB submodule of M if and only if $\mathcal{H}^2 V \subseteq$ Ƙ for E and G are submodules of M, implies that either $EG \subseteq$ Ƙ $+ soc(M) + J(M)$ or $\mathcal{H}^2 \subseteq$ Ƙ $+ soc(M) + J(M)$.

Proof (⇒) Let $r^2 G \subseteq$ Ƙ for $r \in R$, G is a submodule of M. But M is a multiplication module, then $G = IM$ for some ideal I of R, it follows that $r^2 IM \subseteq$ Ƙ, then by hypothesis either $rIM \subseteq$ Ƙ $+ soc(M) + J(M)$ or $r^2 \in [$Ƙ $+ soc(M) + J(M) :_R M]$. That is either $rG \subseteq$ Ƙ $+ soc(M) + J(M)$ or $r^2 \in [$Ƙ $+ soc(M) + J(M) :_R M]$. Hence Ƙ is EXNPS2AB submodule of M.

(⇐) Let $E^2 G \subseteq$ Ƙ for E, G are submodules of a multiplication module M, it follows that $(IM)^2(JM) = I^2 JM \subseteq$ Ƙ for some ideals I, J of R. Since Ƙ is EXNPS2AB submodule of M, then we have either $IJM \subseteq$ Ƙ $+ soc(M) + J(M)$ or $I^2 \subseteq [$Ƙ $+ soc(M) + J(M) :_R M]$, that is either $EG \subseteq$ Ƙ $+ soc(M) + J(M)$ or $E^2 \subseteq$ Ƙ $+ soc(M) + J(M)$.

Proposition 2.3 A proper submodule Ƙ of a multiplication R-module M is EXNPS2AB submodule of M if and only if ${h_1}^2 h_2 \subseteq$ Ƙ for $h_1, h_2 \in M$, implies that either $h_1 h_2 \subseteq$ Ƙ $+ soc(M) + J(M)$ or ${h_1}^2 \subseteq$ Ƙ $+ soc(M) + J(M)$.

Proof (⇒) Let ${h_1}^2 h_2 \subseteq$ Ƙ for $h_1, h_2 \in M$, it follows that $(h_1)^2(h_2) \subseteq$ Ƙ. But M is a multiplication module, then $(h_1)^2 = (IM)^2 = I^2 M$ and $(h_2) = JM$ for some ideals I and J of R, then $I^2 JM \subseteq$ Ƙ, since Ƙ is EXNPS2AB submodule of M, then either $IJM \subseteq$ Ƙ $+ soc(M) + J(M)$ or $I^2 M \subseteq$ Ƙ $+ soc(M) + J(M)$. That is either $h_1 h_2 \subseteq$ Ƙ $+ soc(M) + J(M)$ or ${h_1}^2 \subseteq$ Ƙ $+ soc(M) + J(M)$.

(⇐) Clear.

The corollaries that result directly from Proposition 2.2 are as follows.

Corollary 2.4 A proper submodule Ƙ of a multiplication R-module M is EXNPS2AB submodule of M if and only if $\mathcal{H}^2 k \subseteq$ Ƙ for $\mathcal{H}$ is a submodule of M and $k \in M$, implies that either $\mathcal{H}k \subseteq$ Ƙ $+ soc(M) + J(M)$ or $\mathcal{H}^2 \subseteq$ Ƙ $+ soc(M) + J(M)$.

Corollary 2.5 A proper submodule Ƙ of a multiplication R-module M is EXNPS2AB submodule of M if and only if $y^2 G \subseteq$ Ƙ for G is a submodule of M and $y \in M$, implies that either $yG \subseteq$ Ƙ $+ soc(M) + J(M)$ or $y^2 \subseteq$ Ƙ $+ soc(M) + J(M)$.

Remark 2.6 The residual of EXNPS2AB submodule of a *module* M need not to be EXNPS2AB *ideal* of R. See the following example.

Example 2.7 Let $M = Z_{48}$, $R = Z$ and the submodule $G = \langle \overline{24} \rangle$ is EXNPS2AB submodule of M. But $[\langle \overline{24} \rangle :_R Z_{48}] = 24Z$ is not EXNPS2AB ideal of Z, since $2^2.6 \in 24Z$, for $2,6 \in Z$, implies that $2.6 = 12 \notin 24Z$ and $4 \notin 24Z$.

So the following results show that under curtained conditions it becomes true.

Lemma 2.8 [13, Prop. (3.1)] If M is a multiplication R-module, then M is concellation if and only if M is faithful finitely generated.

Lemma 2.9 [10, Coro. (2.14) (i)] Let M be faithful multiplication R-module, then $soc(R)M = soc(M)$.

Lemma 2.10 [11] Let M be faithful multiplication R-module, then $J(R)M = J(M)$.

Proposition 2.11 Let Ҡ be a proper submodule of a faithful multiplication R-module M. Then Ҡ is EXNPS2AB submodule of M if and only if $[Ҡ:_R M]$ is EXNPS2AB ideal of R.

Proof (⇒) Let $r^2 J \subseteq [Ҡ:_R M]$ for some ideal J of R and $r \in R$, hence $r^2(JM) \subseteq Ҡ$. But Ҡ is EXNPS2AB submodule of M, then by Corollary 2.24 In [9] either $r(JM) \subseteq Ҡ + soc(M) + J(M)$ or $r^2 \in [Ҡ + soc(M) + J(M):_R M]$. Since M is multiplication, then $Ҡ = [Ҡ:_R M]M$ and since M is faithful multiplication, then by Lemma 2.9 $soc(M) = soc(R)M$ and by Lemma 2.10 $J(M) = J(R)M$. Thus either $I(JM) \subseteq [Ҡ:_R M]M + soc(R)M + J(R)M$ or $r^2 M \subseteq [Ҡ:_R M]M + soc(R)M + J(R)M$, thus by Lemma 2.8 either $IJ \subseteq [Ҡ:_R M] + soc(R) + J(R)$ or $r^2 \in [Ҡ:_R M] + soc(R) + J(R) = \big[[Ҡ:_R M] + soc(R) + J(R): R\big]$. Hence $[Ҡ:_R M]$ is EXNPS2AB ideal of R.

(⇐) Let $M^2 L \subseteq Ҡ$ for M and L are a submodules of M. Since M is a multiplication, then $M = IM$ and $L = JM$ for some ideals I, J of R, that is $I^2 JM \subseteq Ҡ$, implies that $I^2 J \subseteq [Ҡ:_R M]$, but $[Ҡ:_R M]$ is EXNPS2AB ideal of R, then either $IJ \subseteq [Ҡ:_R M] + soc(R) + J(R)$ or $I^2 \subseteq \big[[Ҡ:_R M] + soc(R) + J(R):_R R\big] = [Ҡ:_R M] + soc(R) + J(R)$, thus either $IJM \subseteq [Ҡ:_R M]M + soc(R)M + J(R)M$ or $I^2 M \subseteq [Ҡ:_R M]M + soc(R)M + J(R)M$. Hence by Lemma 2.9 and Lemma 2.10 either $IJM \subseteq Ҡ + soc(M) + J(M)$ or $I^2 M \subseteq Ҡ + soc(M) + J(M)$, thus either $ML \subseteq Ҡ + soc(M) + J(M)$ or $M^2 \subseteq Ҡ + soc(M) + J(M)$. Hence by Proposition 2.2 Ҡ is EXNPS2AB submodule of M.

Lemma 2.12[12, Theo. (9.2.1) (g)] For any projective $R\text{-}module\ M$, we have $J(R)M = J(M)$.

Lemma 2.13[11, Prop. (3.24] For any projective $R\text{-}module\ M$, we have $soc(R)M = soc(M)$.

Proposition 2.14 Let Ҡ be a proper submodule of a multiplication projective R-module M. Then Ҡ is EXNPS2AB submodule of M if and only if $[Ҡ:_R M]$ is EXNPS2AB ideal of R.

Proof (⇒) Let $I^2 r \subseteq [Ҡ:_R M]$ for $r \in R$ and some ideal I of R, hence $I^2 rM \subseteq Ҡ$. But Ҡ is EXNPS2AB submodule of M, then by Proposition 2.20 in [9] either $IrM \subseteq Ҡ + soc(M) + J(M)$ or $I^2 \subseteq [Ҡ + soc(M) + J(M):_R M]$. Since M is multiplication, then $Ҡ = [Ҡ:_R M]M$ and since M is projective multiplication, then by Lemma 2.13 $soc(M) = soc(R)M$ and by Lemma 2.12 $J(M) = J(R)M$. Thus either $IrM \subseteq [Ҡ:_R M]M + soc(R)M + J(R)M$ or $I^2 M \subseteq [Ҡ:_R M]M + soc(R)M + J(R)M$, hence either $Ir \subseteq [Ҡ:_R M] + soc(R) + J(R)$ or $I^2 \subseteq [Ҡ:_R M] + soc(R) + J(R) = \big[[Ҡ:_R M] + soc(R) + J(R): R\big]$. Therefore by Corollary 2.22 in [9] $[Ҡ:_R M]$ is EXNPS2AB ideal of R.

(⇐) Let $M^2 m \subseteq Ҡ$ for M is a submodule of M and $m \in M$, that is $M^2(m) \subseteq Ҡ$ Since M is a multiplication, then $M = IM$ and $(m) = JM$ for some ideals I, J of R, that is $I^2 JM \subseteq Ҡ$, implies that $I^2 J \subseteq [Ҡ:_R M]$, but $[Ҡ:_R M]$ is EXNPS2AB ideal of R, then either $IJ \subseteq [Ҡ:_R M] + soc(R) + J(R)$ or $I^2 \subseteq \big[[Ҡ:_R M] + soc(R) + J(R):_R R\big] = [Ҡ:_R M] + soc(R) + J(R)$, thus either $IJM \subseteq [Ҡ:_R M]M + soc(R)M + J(R)M$ or $I^2 M \subseteq [Ҡ:_R M]M + soc(R)M + J(R)M$. Hence by Lemma 2.12 and Lemma 2.13 either $IJM \subseteq Ҡ + soc(M) + J(M)$ or $I^2 M \subseteq Ҡ + soc(M) + J(M)$, thus either $Mm \subseteq Ҡ + soc(M) + J(M)$ or $M^2 \subseteq Ҡ + soc(M) + J(M)$, hence by Corollary 2.4 Ҡ is EXNPS2AB submodule of M.

Remark 2.15[12] R is a good ring if $J(R)M = J(M)$.

Lemma 2.16[11, Prop. (3.25] Let M be a Z-regular $R\text{-}module$, then $soc(M) = soc(R)M$.

Proposition 2.17 Let Ҡ be a proper submodule of Z-regular multiplication module M over a good ring R. Then Ҡ is EXNPS2AB submodule of M if and only if $[Ҡ:_R M]$ is EXNPS2AB ideal of R.

Proof (⇒) Let $I^2 J \subseteq [Ҡ:_R M]$ for some ideals I and J of R, then $I^2 JM \subseteq Ҡ$. But Ҡ is EXNPS2AB submodule of M, then either $IJM \subseteq Ҡ + soc(M) + J(M)$ or $I^2 \subseteq [Ҡ + soc(M) + J(M):_R M]$. Since M is multiplication, then $Ҡ = [Ҡ:_R M]M$ and since M is Z-regular multiplication module, then by Lemma 2.16 $soc(M) = soc(R)M$ and since R is a good ring, then by Remark 2.15 $J(M) = J(R)M$. Thus either $IJM \subseteq [Ꙋ:_R M]M + soc(R)M + J(R)M$ or $I^2M \subseteq [Ꙋ:_R M]M + soc(R)M + J(R)M$, then either $IJ \subseteq [Ꙋ:_R M] + soc(R) + J(R)$ or $I^2 \subseteq [Ꙋ:_R M] + soc(R) + J(R) = \left[[Ꙋ:_R M] + soc(R) + J(R): R\right]$. Hence $[Ꙋ:_R M]$ is EXNPS2AB ideal of R.

(⇐) Let $m^2L \subseteq Ꙋ$ for $m \in M$ and L is a submodule of M. Since M is a multiplication, then $(m) = IM$ and $L = JM$ for some ideals I, J of R, that is $I^2JM \subseteq Ꙋ$, implies that $I^2J \subseteq [Ꙋ:_R M]$, but $[Ꙋ:_R M]$is EXNPS2AB ideal of R, then either $IJ \subseteq [Ꙋ:_R M] + soc(R) + J(R)$ or $I^2 \subseteq \left[[Ꙋ:_R M] + soc(R) + J(R):_R R\right] = [Ꙋ:_R M] + soc(R) + J(R)$, thus either $IJM \subseteq [Ꙋ:_R M]M + soc(R)M + J(R)M$ or $I^2M \subseteq [Ꙋ:_R M]M + soc(R)M + J(R)M$. Hence by Lemma 2.16 and Remark 2.15 either $mL \subseteq Ꙋ + soc(M) + J(M)$ or $m^2 \subseteq Ꙋ + soc(M) + J(M)$. Thus by Corollary 2.5 Ꙋ is EXNPS2AB submodule of M.

Lemma 2.18 [12, Coro. (9.7.3) (a)] If $(R/J(R))$ is a semi-simple ring, then R is a good ring.

Lemma 2.19 [12, Coro. (9.7.3) (b)] If R is an Artinian ring, then R is a good ring.

The corollaries that result directly from Proposition 2.17 are as follows.

Corollary 2.20 Let M be a Z-regular multiplication R-module such that $(R/J(R))$ is a semi-simple ring, and Ꙋ be a proper submodule of M. Then Ꙋ is EXNPS2AB submodule of M if and only if $[Ꙋ:_R M]$ is EXNPS2AB ideal of R.

Corollary 2.21 Let M be a Z-regular multiplication module over Artinian ring R, and Ꙋ be a proper submodule of M. Then Ꙋ is EXNPS2AB submodule of M if and only if $[Ꙋ:_R M]$ is EXNPS2AB ideal of R.

Lemma 2.22 [21, Prop. (1.12)] If M is an R-*module* over local ring R, then $J(R)M = J(M)$.

Proposition 2.23 Let M be a Z-regular multiplication R-module over a local ring R, and Ꙋ be a proper submodule of M. Then Ꙋ is EXNPS2AB submodule of M if and only if $[Ꙋ:_R M]$ is EXNPS2AB ideal of R.

Proof Similarly of Proposition 2.17 by using Lemma 2.19.

Lemma 2.24 [16, Coro. (1.26)] Let M be is a non-singular R-*module*, then $soc(R)M = soc(M)$.

Lemma 2.25 [11, Prop. (1.11)] If M is content module, then $J(R)M = J(M)$.

Proposition 2.26 Let Ꙋ be a proper submodule of a content multiplication non-singular R-module M. Then Ꙋ is EXNPS2AB submodule of M if and only if $[Ꙋ:_R M]$ is EXNPS2AB ideal of R.

Proof (⇒) Let $a^2b \in [Ꙋ:_R M]$ for $a, b \in R$, it follows that $a^2(bM) \subseteq Ꙋ$. But Ꙋ is EXNPS2AB submodule of M, then by Corollary 2.24 in [9] either $a(bM) \subseteq Ꙋ + soc(M) + J(M)$ or $a^2 \subseteq [Ꙋ + soc(M) + J(M):_R M]$. Since M is multiplication, then $Ꙋ = [Ꙋ:_R M]M$ and since M is non-singular R-module, then by Lemma 2.24 $soc(M) = soc(R)M$ and since M is a content, then by Lemma 2.25 $J(M) = J(R)M$. Thus either $abM \subseteq [Ꙋ:_R M]M + soc(R)M + J(R)M$ or $a^2M \subseteq [Ꙋ:_R M]M + soc(R)M + J(R)M$, hence either $ab \subseteq [Ꙋ:_R M] + soc(R) + J(R)$ or $a^2 \in [Ꙋ:_R M] + soc(R) + J(R) = \left[[Ꙋ:_R M] + soc(R) + J(R): R\right]$. Hence $[Ꙋ:_R M]$ is EXNPS2AB ideal of R.

(⇐) Let $a^2L \subseteq Ꙋ$ for $a \in R$, L is a submodule of M. Since M is a multiplication, then $L = JM$ for some ideal J of R, that is $a^2JM \subseteq Ꙋ$, implies that $a^2J \subseteq [Ꙋ:_R M]$. But $[Ꙋ:_R M]$is EXNPS2AB ideal of R, then either $aJ \subseteq [Ꙋ:_R M] + soc(R) + J(R)$ or $a^2 \in \left[[Ꙋ:_R M] + soc(R) + J(R):_R R\right] = [Ꙋ:_R M] + soc(R) + J(R)$, thus either $aJM \subseteq [Ꙋ:_R M]M + soc(R)M + J(R)M$ or $a^2M \subseteq [Ꙋ:_R M]M + soc(R)M + J(R)M$. Since M is multiplication, then $Ꙋ = [Ꙋ:_R M]M$ and since M is non-singular R-*module*, then by Lemma2.24 $soc(M) = soc(R)M$ and since M is a content, then by Lemma 2.25 $J(M) = J(R)M$, then either $aL \subseteq Ꙋ + soc(M) + J(M)$ or $a^2 \in [Ꙋ + soc(M) + J(M):_R M]$. Thus by Corollary 2.24 in [9] Ꙋ is EXNPS2AB submodule of M.

By Proof of Proposition 2.26 and using Remark 2.15 we get the following.

Proposition 2.27 Let Ꙋ be a proper submodule of non-singular multiplication R-module M over a good ring R. Then Ꙋ is EXNPS2AB submodule of M if and only if $[Ꙋ:_R M]$ is EXNPS2AB ideal of R.

As a direct application of Proposition 2.26, we get the following corollaries.

Corollary 2.28 Let Ꙋ be a proper submodule of non-singular multiplication R-module M over Artinian ring R. Then Ꙋ is EXNPS2AB submodule of M if and only if $[Ꙋ:_R M]$ is EXNPS2AB ideal of R.

Corollary 2.29 Let M be a non-singular multiplication R-module such that($R/J(R)$)is a semi-simple ring, and Ҡ be a proper submodule of M. Then Ҡ is EXNPS2AB submodule of M if and only if $[Ҡ:_R M]$ is EXNPS2AB ideal of R.

Corollary 2.30 Let Ҡ be a proper submodule of non-singular multiplication R-module M over local ring R. Then Ҡ is EXNPS2AB submodule of M if and only if $[Ҡ:_R M]$ is EXNPS2AB ideal of R.

Lemma 2.31 [17, Coro. (15)] Let M be finitely generated multiplication R-module with $JM \neq M$ for all maximal ideal J of R, then $J(M) = J(R)M$.

Proposition 2.32 Let M be finitely generated multiplication non-singular R-module with $IM \neq M$ for all *maximal* ideal I of R, and Ҡ be a proper submodule of M. Then Ҡ is EXNPS2AB submodule of M if and only if $[Ҡ:_R M]$ is EXNPS2AB ideal of R.

Proof Clear.

3. More Result of Extend Nearly Pseudo Semi-2-Absorbing Submodules in Multiplication Modules.

In this part we studied more result of Extend Nearly Pseudo Semi-2-Absorbing submodules in multiplication modules. And we got the most important results.

Lemma 3.1[20, Coro of Theo. (9)] Let M be a finitely generated multiplication R-module I and J are ideals of R. Then $IM \subseteq JM$ if and only if $I \subseteq J + ann_R(M)$.

Proposition 3.2 Let M be a finitely generated multiplication projective R-module, and B is an ideal of R with $ann_R(M) \subseteq B$. Then B is EXNPS2AB ideal of R if and only if BM is EXNPS2AB submodule of M.

Proof (⇒) Let $M^2m \subseteq BM$, for M is a submodule of M and $m \in M$, that is $M^2(m) \subseteq BM$. Since M is a multiplication, then $M^2 = I^2M$ and $(m) = JM$ for some ideals I, J of R, that is $I^2JM \subseteq BM$. But M is a finitely generated multiplication R-module then by Lemma 3.1 $I^2J \subseteq B + ann_R(M)$, but $ann_R(M) \subseteq B$, implies that $B + ann_R(M) = B$, thus $I^2J \subseteq B$. Now, by assumption B is EXNPS2AB ideal of R, then either $IJ \subseteq B + (soc(R) + J(R))$ or $I^2 \subseteq [B + (soc(R) + J(R)):_R R] = B + (soc(R) + J(R))$, it follows that either $IJM \subseteq BM + soc(R)M + J(R)M$ or $I^2M \subseteq BM + soc(R)M + J(R)M$. Since M is a projective then by Lemma 2.12 and Lemma 2.13 $\big(soc(M) + J(M)\big) = (soc(R)M + J(R)M)$, it follows that either $M(m) \subseteq BM + \big(soc(M) + J(M)\big)$ or $M^2 \subseteq \big[BM + \big(soc(M) + J(M)\big):_R M\big]$. Hence by Corollary 2.4 BM is EXNPS2AB submodule of M.

(⇐) Let $r^2I \subseteq B$, for I is an ideal of R and $r \in R$, implies that $r^2(IM) \subseteq BM$. But BM is EXNPS2AB submodule of M, then by Corollary 2.24 in [9] either $r(IM) \subseteq BM + (soc(M) + J(M))$ or $r^2M \subseteq BM + (soc(M) + J(M))$. But M is a projective then $\big(soc(M) + J(M)\big) = (soc(R)M + J(R)M)$. Thus either $rIM \subseteq BM + soc(R)M + J(R)M$ or $r^2M \subseteq BM + soc(R)M + J(R)M$, it follows that either $rI \subseteq B + soc(R) + J(R)$ or $r^2 \in B + soc(R) + J(R) = [B + soc(R) + J(R):_R R]$. Hence by Corollary 2.24 in [9] B is EXNPS2AB ideal of R.

Proposition 3.3 Let M be a faithful finitely generated multiplication R-module and B is an ideal of R. Then B is EXNPS2AB ideal of R if and only if BM is EXNPS2AB submodule of M.

Proof (⇒) Let $y^2G \subseteq BM$, for $y \in M$ and G is a submodule of M, it follows that $(y^2)G \subseteq BM$. Since M is a multiplication, then $(y)^2 = I^2M$ and $G = JM$for some ideals I, J of R, that is $I^2JM \subseteq BM$. But M is a finitely generated multiplication R-module then by Lemma 3.1 $I^2J \subseteq B + ann_R(M)$ and since M is faithful, then $ann_R(M) = (0)$, implies that $B + ann_R(M) = B$, hence $I^2J \subseteq B$. But B is EXNPS2AB ideal of R then either $IJ \subseteq B + (soc(R) + J(R))$ or $I^2 \subseteq [B + (soc(R) + J(R)):_R R] = B + (soc(R) + J(R))$ .Thus either $IJM \subseteq BM + (soc(R)M + J(R)M)$ or $I^2M \subseteq BM + (soc(R)M + J(R)M)$. Since M is a faithful multiplication, then by Lemma 2.9 $soc(R)M = soc(M)$ and by Lemma 2.10 $J(R)M = J(M)$. Hence either $IJM \subseteq BM + (soc(M) + J(M))$ or $I^2M \subseteq BM + (soc(M) + J(M))$. That is either $yG \subseteq BM + (soc(M) + J(M))$ or $y^2 \subseteq BM + (soc(M) + J(M))$. Therefore by Corollary 2.5 BM is EXNPS2AB submodule of M.

(⇐) Let $r^2s \in B$, for $r, s \in R$, implies that $r^2(sM) \subseteq BM$. Since BM is EXNPS2AB submodule of M, then either $r(sM) \subseteq BM + (soc(M) + J(M))$ or $r^2 \in [BM + \big(soc(M) + J(M)\big):_R M]$. That is either $rsM \subseteq BM + (soc(M) + J(M))$ or $r^2M \subseteq BM + (soc(M) + J(M))$. But M is a faithful multiplication, then either $rsM \subseteq BM + (soc(R)M + J(R)M)$ or $r^2M \subseteq BM + (soc(R)M + J(R)M)$, it follows that either $rs \in B + (soc(R) + J(R))$ or $r^2 \in B + soc(R) + J(R) = [B + soc(R) + J(R):_R R]$. Hence B is EXNPS2AB ideal of R.

Proposition 3.4 Let M be a finitely generated non-singular multiplication module over good ring R and B is an ideal of R with $ann_R(M) \subseteq B$. Then B is EXNPS2AB ideal of R if and only if BM is EXNPS2AB submodule of M.

Proof (⇒) Let $M^2G \subseteq BM$, for M, G are a submodules of M. Since M is a multiplication, then $M^2 = I^2M$ and $G = JM$ for some ideals I, J of R, that is $I^2JM \subseteq BM$. But M is a finitely generated multiplication R-module then by Lemma 3.1 $I^2J \subseteq B + ann_R(M)$, since $ann_R(M) \subseteq B$, implies that $B + ann_R(M) = B$, implies that $I^2J \subseteq B$. But B is EXNPS2AB ideal of R then either $IJ \subseteq B + (soc(R) + J(R))$ or $I^2 \subseteq [B + (soc(R) + J(R)):_R R] = B + (soc(R) + J(R))$. Thus either $IJM \subseteq BM + (soc(R)M + J(R)M)$ or $I^2M \subseteq BM + (soc(R)M + J(R)M)$. Since M is non-singular multiplication, then by Lemma 2.24 $soc(R)M = soc(M)$ and since R is good ring, then by Remark 2.15 $J(R)M = J(M)$. Hence either $IJM \subseteq BM + (soc(M) + J(M))$ or $I^2M \subseteq BM + (soc(M) + J(M))$. That is either $MG \subseteq BM + (soc(M) + J(M))$ or $M^2 \subseteq BM + (soc(M) + J(M))$. Therefore by Proposition 2.2 BM is EXNPS2AB submodule of M.

(⇐) Let $I^2s \subseteq B$, for some ideal I of R and $s \in R$, implies that $I^2(sM) \subseteq BM$. Since BM is EXNPS2AB submodule of M, then by Proposition 2.20 in [9] either $I(sM) \subseteq BM + (soc(M) + J(M))$ or $I^2 \subseteq [BM + \left(soc(M) + J(M)\right):_R M]$. That is either $IsM \subseteq BM + (soc(M) + J(M))$ or $I^2M \subseteq BM + (soc(M) + J(M))$. But M is finitely generated non-singular multiplication module over good ring R, then either $IsM \subseteq BM + (soc(R)M + J(R)M)$ or $I^2M \subseteq BM + (soc(R)M + J(R)M)$, thus either $Is \subseteq B + (soc(R) + J(R))$ or $I^2 \subseteq B + soc(R) + J(R) = [B + soc(R) + J(R):_R R]$. Hence by Corollary 2.4 B is EXNPS2AB ideal of R.

Corollary 3.5 Let M be a finitely generated non-singular multiplication module over Artinia ring R and B is an ideal of R with $ann_R(M) \subseteq B$. Then B is EXNPS2AB ideal of R if and only if BM is EXNPS2AB submodule of M.

Proposition 3.6 Let M be a finitely generated multiplication Z-regular module over a good ring R,and B is an ideal of R with $ann_R(M) \subseteq B$. Then B is EXNPS2AB ideal of R if and only if BM is EXNPS2AB submodule of M.

Proof (⇒) Let $A^2Ҡ \subseteq BM$, for $A, Ҡ$ are a submodules of M. Since M is a multiplication, then $A^2 = I^2M$ and $Ҡ = JM$ for some ideals I, J of R, that is $I^2JM \subseteq BM$. But M is a finitely generated multiplication R-*module* then by Lemma 3.1 $I^2J \subseteq B + ann_R(M)$, since $ann_R(M) \subseteq B$, implies that $B + ann_R(M) = B$, implies that $I^2J \subseteq B$. But B is EXNPS2AB ideal of R then by Proposition 2.20 in [9] either $IJ \subseteq B + (soc(R) + J(R))$ or $I^2 \subseteq [B + (soc(R) + J(R)):_R R] = B + (soc(R) + J(R))$. Thus either $IJM \subseteq BM + (soc(R)M + J(R)M)$ or $I^2M \subseteq BM + (soc(R)M + J(R)M)$. Since M is Z-regular multiplication, then by Lemma 2.16 $soc(R)M = soc(M)$ and since R is good ring, then by Remark 2.15 $J(R)M = J(M)$. Hence either $IJM \subseteq BM + (soc(M) + J(M))$ or $I^2M \subseteq BM + (soc(M) + J(M))$. That is either $AҠ \subseteq BM + (soc(M) + J(M))$ or $A^2 \subseteq BM + (soc(M) + J(M))$. Therefore by Proposition 2.2 BM is EXNPS2AB submodule of M.

(⇐) Let $I^2r \subseteq B$, for I is an ideals of R and $r \in R$, implies that $I^2(rM) \subseteq BM$. Since BM is EXNPS2AB *submodule* of M, then by Proposition 2.20 in [9] either $I(rM) \subseteq BM + (soc(M) + J(M))$ or $I^2 \subseteq [BM + \left(soc(M) + J(M)\right):_R M]$. That is either $IrM \subseteq BM + (soc(M) + J(M))$ or $I^2M \subseteq BM + (soc(M) + J(M))$. But M is finitely generated multiplication Z-regular *module* over a good ring R, then either $IrM \subseteq BM + (soc(R)M + J(R)M)$ or $I^2M \subseteq BM + (soc(R)M + J(R)M)$, it follows that either $Ir \subseteq B + (soc(R) + J(R))$ or $I^2 \subseteq B + soc(R) + J(R) = [B + soc(R) + J(R):_R R]$. Hence by Corollary 2.4 B is EXNPS2AB ideal of R.

Corollary 3.7 Let M be a finitely generated multiplication Z-regular module over an Artinian ring R, and B is an ideal of R with $ann_R(M) \subseteq B$. Then B is EXNPS2AB ideal of R if and only if BM is EXNPS2AB submodule of M.

Directly from Proposition 3.6 and using Lemma 2.22 we will get the following result.

Proposition 3.8 Let M be a finitely generated multiplication Z-regular module over local ring R, and B is an ideal of R with $ann_R(M) \subseteq B$. Then B is EXNPS2AB ideal of R if and only if BM is EXNPS2AB submodule of M.

Now, from Proposition 2.11 and Proposition 3.3 we get the following.

Proposition 3.9 Let M be a faithful finitely generated multiplication R-*module* and Ҡ be a proper submodule of M, Consequently, the following claims are equal:

1. Ҡ is EXNPS2AB submodule of M.

2. $[Ҡ:_R M]$ is EXNPS2AB ideal of R.

3. $Ҡ = BM$ for some EXNPS2AB ideal B of R.

Lemma 3.10 [13, Prop. (3.9)] If M is a multiplication R-$module$, then M is $finitely\ generated$ if and only if M is weak cancellation.

Proposition 3.11 Let M be a finitely generated multiplication projective R-module and Ꝁ be a proper submodule of M with $ann_R(M) \subseteq [Ꝁ:_R M]$. Consequently, the following claims are equal:

1. Ꝁ is EXNPS2AB submodule of M.

2. $[Ꝁ:_R M]$ is EXNPS2AB ideal of R.

3. $Ꝁ = BM$ for some EXNPS2AB ideal B of R with $ann_R(M) \subseteq B$.

Proof (1⇔2) It follows by Proposition 2.14.

(2⇒3) Since $[Ꝁ:_R M]$ is EXNPS2AB ideal of R and $ann_R(M) \subseteq [0:_R M] \subseteq [Ꝁ:_R M]$, then by Proposition 3.2 $[Ꝁ:_R M]M$ is EXNPS2AB submodule of M. Since M is a multiplication, then $Ꝁ = [Ꝁ:_R M]M = BM$, where $B = [Ꝁ:_R M]$ is EXNPS2AB ideal of R.

(3⇒1) Since $Ꝁ = BM$ for some EXNPS2AB ideal B of R such that $ann_R(M) \subseteq B$. From other hand M is a multiplication, then $Ꝁ = [Ꝁ:_R M]M$, but M is a finitely generated, then by Lemma 3.10 M is weak cancellation, it follows that $[Ꝁ:_R M] + ann_R(M) = B + ann_R(M)$, but $ann_R(M) \subseteq B$, and $ann_R(M) \subseteq [Ꝁ:_R M]$ implies that $ann_R(M) + B = B$ and $[Ꝁ:_R M] + ann_R(M) = [Ꝁ:_R M]$. Thus $B = [Ꝁ:_R M]$, but B is EXNPS2AB ideal of R, hence $[Ꝁ:_R M]$ is EXNPS2AB ideal of R. Therefore by Proposition 2.14 we have Ꝁ is EXNPS2AB submodule of M.

Proposition 3.12 Let M be a non-singular finitely generated multiplication module over a good ring R and Ꝁ be a proper submodule of M with $ann_R(M) \subseteq [Ꝁ:_R M]$. Consequently, the following claims are equal:

1. Ꝁ is EXNPS2AB submodule of M.

2. $[Ꝁ:_R M]$ is EXNPS2AB ideal of R.

3. $Ꝁ = BM$ for some EXNPS2AB ideal B of R with $ann_R(M) \subseteq B$.

Proof Clear.

Proposition 3.13 Let M be a non-singular finitely generated multiplication module over an Artinian ring R, and Ꝁ be a proper submodule of M with $ann_R(M) \subseteq [Ꝁ:_R M]$. Consequently, the following claims are equal:

1. Ꝁ is EXNPS2AB submodule of M.

2. $[Ꝁ:_R M]$ is EXNPS2AB ideal of R.

3. $Ꝁ = BM$ for some EXNPS2AB ideal B of R with $ann_R(M) \subseteq B$.

Proof Direct.

Proposition 3.14 Let M be a non-singular finitely generated multiplication module over a local ring R and Ꝁ be a proper submodule of M with $ann_R(M) \subseteq [Ꝁ:_R M]$. Consequently, the following claims are equal:

1. Ꝁ is EXNPS2AB submodule of M.

2. $[Ꝁ:_R M]$ is EXNPS2AB ideal of R.

3. $Ꝁ = BM$ for some EXNPS2AB ideal B of R with $ann_R(M) \subseteq B$.

Proof (1⇔2) It follows by Corollary 2.30.

(2⇔3) Follows in the same way as the Proof of Proposition 3.11.

From Proposition 2.17 and Proposition 3.6 we get.

Proposition 3.15 Let M be a finitely generated multiplication Z-regular module over a good ring R, and Ҟ be a proper submodule of M with $ann_R(M) \subseteq [Ҟ:_R M]$. Consequently, the following claims are equal:

1. Ҟ is EXNPS2AB submodule of M.

2. $[Ҟ:_R M]$ is EXNPS2AB ideal of R.

3. $Ҟ = BM$ for some EXNPS2AB ideal B of R with $ann_R(M) \subseteq B$.

Proposition 3.16 Let M be a finitely generated multiplication Z-regular module over an Artinian ring R and Ҟ be a proper submodule of M with $ann_R(M) \subseteq [Ҟ:_R M]$**.** Consequently, the following claims are equal:

1. Ҟ is EXNPS2AB submodule of M.

2. $[Ҟ:_R M]$ is EXNPS2AB ideal of R.

3. $Ҟ = BM$ for some EXNPS2AB ideal B of R with $ann_R(M) \subseteq B$.

Proof Direct.

Proposition 3.17 Let M be a finitely generated multiplication Z-regular module over a local ring R and Ҟ be a proper submodule of M with $ann_R(M) \subseteq [Ҟ:_R M]$**.** Consequently, the following claims are equal:

1. Ҟ is EXNPS2AB submodule of M.

2. $[Ҟ:_R M]$ is EXNPS2AB ideal of R.

3. $Ҟ = BM$ for some EXNPS2AB ideal B of R with $ann_R(M) \subseteq B$.

Proof (1⇔2) It follows by Proposition 2.23.

(2⇔3) Follows in the same way as the Proof of Proposition 3.8.

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