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Characterizations of Almost Approximaitly Nearly Quasiprime Submodules in Some Kinds of Modules

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ABSTRACT

Our main goal in this note is to introduce several characterizations of Almost approximaitly nearly quasiprime submodule in class of multiplication modules. Moreover, we characterized Almost approximaitly nearly quasiprime submodules by theirs residual in class of multiplication modules with the help of some types of modules as projective, faithful, content and Z-regular modules. And we characterized almost approximaitly nearly quasiprime ideal B by almost approximaitly nearly quasiprime of the form BQ.

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1. Introduction

Let R be commutative ring with identity and Q be unitary left R-module. Quasiprime submodules generalization of prime submodule was first introduced by [1] in 1999. Many author interesting generalized this concept see [2, 3, 4, 5]. In the recent time quasiprime submodule generalized in [6] to Almost approximaitly nearly quasiprime submodule, where a proper submodule F of an R-module Q is called almost approximaitly nearly quasiprime (simply Alappnq-prime) submodule, if whenever $acq \in F$, for $a, c \in R$, $q \in$ Q, implies that either $aq \in F + (soc(\text{Q}) + J(\text{Q}))$ or $cq \in F + (soc(\text{Q}) + J(\text{Q}))$ [6]. Where $soc(\text{Q})$ is the socle of Q defined as intersection of all essential submodule in Q, and $J(\text{Q})$ is the Jacobson radical of Q defined to be intersection of all maximal submodules in Q[7].
Now, we recalled some concepts that we will be using in this note. An R-module Q is multiplication if each submodule F in Q has the form JQ for some ideal J of R [8], equivalent to $F = [F:_R \text{Q}]\text{Q}$ [9], we say that J is presentation ideal of Q. Every submodule F of Q has presentation ideal if and only if Q is multiplication. Let F and L are submodules of a multiplication R-module Q with $F = I\text{Q}$ and $L = J\text{Q}$ for some ideals I, J of R, the product of F and L denoted by FL defined by $FL = IJ\text{Q}$ [10]. The products of F and L is independent on presentations of F and Q, so

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the term xy for some $x, y \in$ Q represents the product Rx and Ry [11]. It is well known that if M is projective, then $J(R)M = J(M)$[12.Theo.(9.2.1)(g)], also in projective module $soc(R)M = soc(M)$ [13.pro.(3.24)]. An R-module Q is faithful if $ann_R(\mathrm{Q}) = (0)$[12]. And Q is content if $(\bigcap_{i\in I} F_i)\mathrm{Q} = \bigcap_{i\in I} F_i\,\mathrm{Q}$ [14]. If Q is content module then $J(R)\mathrm{Q} = J(\mathrm{Q})$ [15, pro. (1.1)]. Also, Q is Z-regular if for each $q \in$ Q there exists $f \in \mathrm{Q}' = Hom_R(\mathrm{Q}, R)$ such that $q = f(q)q$ [16]. If Q is Z-regular, then $soc(\mathrm{Q}) = soc(R)\mathrm{Q}$[15, pro. (3.25)]. Finally, R is good ring if $J(R)\mathrm{Q} = J(\mathrm{Q})$ [12]

2. Characterizations of Alappnq-prime Submodules in multiplication Modules.

Before we start to introduce the characterizations we need to recall the definition of Almost approximaitly nearly quasiprime submodules and some characterizations of this concept which appear in [6], that we needed in the sequel.

Definition 2.1 A proper submodule F of an R-module Q is called almost approximaitly nearly quasiprime (simply Alappnq-prime) submodule, if whenever $acq \in F$, for $a, c \in R, q \in$ Q, implies that either $aq \in F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $cq \in F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$.

Proposition 2.2 Let Q be an R-module, and F be a submodule of Q. Then F is an Alappnq-prime submodule of Q if and only if whenever $acL \subseteq F$, for $a, c \in R$ and L is submodule of Q, implies that either $aL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $cL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$.

Proof See [6, Pro. 2.6].

Proposition 2.3 Let Q be an R-module, and F be a submodule of Q. Then F is an Alappnq-prime submodule of Q if and only if whenever $IJL \subseteq F$, for I, J are ideals of R and L is submodule of Q, implies that either $IL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $JL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$.

Proof See [6, Cor. 2.7].

Now, we are ready to introduce characterizations of Alappnq-prime submodule in multiplication modules.

Proposition 2.4 Let Q be a multiplication R-module, and F is a proper submodule of Q. Then F is an Alappnq-prime submodule of Q if and only if whenever $KHL \subseteq F$ for K, H and L are submodules in Q, implies that either $KL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $HL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$.

Proof ($\Longrightarrow$) Let $KHL \subseteq F$ for K, H and L are submodules of Q. Since Q is a multiplication, then $K = I\mathrm{Q}$, $H = J\mathrm{Q}$ and $L = C\mathrm{Q}$ for some ideals I, J and C in R. That is $IJ(C\mathrm{Q}) \subseteq F$. But F is an Alappnq-prime submodule of Q, so by proposition 2.3 we have either $IC\mathrm{Q} \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $JC\mathrm{Q} \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$. It follows that either $KL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $HL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$.

($\Longleftarrow$) Suppose $IJL \subseteq F$ for L is a submodule of Q, and I, J are ideals in R. Since Q is a multiplication, then $KHL = IJL \subseteq F$, so by hypothesis we have either $KL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $HL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$.That is either $IL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $JL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$. Therefore by proposition 2.3 F is an Alappnq-prime submodule of Q.

The following corollaries flow directly from proposition 2.4.

Corollary 2.5 Let Q be a multiplication R-module, and F is a proper submodule of Q. Then F is an Alappnq-prime submodule of Q if and only if whenever $KHq \subseteq F$ for K, F are submodules of Q and $q \in$ Q, implies that either $Kq \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $Hq \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$.

Corollary 2.6 Let Q be a multiplication R-module, and F is a proper submodule of Q. Then F is an Alappnq-prime submodule of Q if and only if whenever $q_1q_2L \subseteq F$ for $q_1, q_2 \in$ Q, and L is submodule of Q, implies that either $q_1L \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $q_2L \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$.

Corollary 2.7 Let Q be a multiplication R-module, and F is a proper submodule of Q. Then F is an Alappnq-prime submodule of Q if and only if whenever $q_1q_2q_3 \subseteq F$ for $q_1, q_2, q_3 \in$ Q, implies that either $q_1q_3 \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $q_2q_3 \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$.

Now, we introduce many characterizations of Alappnq-prime submodules by their residuals.

Proposition 2.8 A proper submodule F of projective multiplication R-module Q is an Alappnq-prime submodule of Q if and only if $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R.

Proof ($\Longrightarrow$) Let $IJC \subseteq [F:_R \mathrm{Q}]$ for I, J and C are ideals in R, implies that $IJC\mathrm{Q} \subseteq F$. Since Q is multiplication, then $IJC\mathrm{Q} = FKL$ by taking $F = I\mathrm{Q}$, $K = J\mathrm{Q}$ and $L = C\mathrm{Q}$ are submodules in Q, hence $FKL \subseteq F$. But F is an Alappn submodule of multiplication R-module Q, it follows by proposition 2.4 that either $FL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $KL \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$. Again since Q is multiplication, then $F = [F:_R \mathrm{Q}]\mathrm{Q}$, and since Q is projective then $soc(\mathrm{Q}) = soc(R)\mathrm{Q}$ and $J(\mathrm{Q}) = J(R)\mathrm{Q}$. Thus either $IC\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$ or $JC\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$, it follows that either $IC \subseteq [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$ or $JC \subseteq [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$. Thus by proposition 2.3 $[F:_R \mathrm{Q}]$ is Alappnq-prime ideal of R.

($\Longleftarrow$) Let $KHL \subseteq F$ for K, H, L are submodules of Q. Since Q is a multiplication, then $K = I\mathrm{Q}$, $H = J\mathrm{Q}$, and $L = C\mathrm{Q}$ for some ideals I, J, C in R, that is $IJC\mathrm{Q} \subseteq F$, implies that $IJC \subseteq [F:_R \mathrm{Q}]$, but $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R, it follows by proposition 2.3 that either $IC \subseteq [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$ or $JC \subseteq [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$. Hence either $IC\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$ or $JC\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$. For Q is a multiplication and projective then either $IC\mathrm{Q} \subseteq F + \big(soc(\mathrm{Q}) + J(\mathrm{Q})\big)$ or $JC\mathrm{Q} \subseteq F + \big(soc(\mathrm{Q}) + J(\mathrm{Q})\big)$. That is either $KL \subseteq F + \big(soc(\mathrm{Q}) + J(\mathrm{Q})\big)$ or $HL \subseteq F + \big(soc(\mathrm{Q}) + J(\mathrm{Q})\big)$. Thus by proposition 2.4 F is Alappnq-prime submodule of Q.

Before we introduced the next characterization, we must recall this lemma

Lemma 2.9 Let Q be a faithful multiplication R module, then:

1. $soc(R)\mathrm{Q} = soc(\mathrm{Q})$ [8, Cor. 2.14(i)].
2. $J(R)\mathrm{Q} = J(\mathrm{Q})$ [15, Rem. P. 14].

Proposition 2.20 A proper submodule F of faithful multiplication R-module Q is an Alappnq-prime submodule of Q if and only if $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R.

Proof ($\Longrightarrow$)Let $ace \in [F:_R \mathrm{Q}]$ for $a, c, e \in R$, implies that $ac(e\mathrm{Q}) \subseteq F$. But F is an Alappnq-prime submodule of Q, it follows by proposition 2.2 that either $a(e\mathrm{Q}) \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $c(e\mathrm{Q}) \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$. Since Q is multiplication, then $F = [F:_R \mathrm{Q}]\mathrm{Q}$, and since Q is faithful multiplication, then by lemma 2.9 we have $soc(\mathrm{Q}) = soc(R)\mathrm{Q}$ and $J(\mathrm{Q}) = J(R)\mathrm{Q}$. Thus either $ae\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$ or $ce\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$, it follows that either $ae \in [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$ or $ce \in [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$. Therefore$[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R.

($\Longleftarrow$) Let $q_1 q_2 q_3 \subseteq F$ for $q_1, q_2, q_3 \in \mathrm{Q}$. Since Q is a multiplication then $q_1 = Rq_1 = I_1\mathrm{Q}$, $q_2 = Rq_2 = I_2\mathrm{Q}$, and $q_3 = Rq_3 = I_3\mathrm{Q}$ for some ideals I_1, I_2, I_3 of R, that is $I_1 I_2 I_3 \mathrm{Q} \subseteq F$, implies that $I_1 I_2 I_3 \subseteq [F:_R \mathrm{Q}]$, but $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R, it follows by proposition 2.3 that either $I_1 I_3 \subseteq [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$ or $I_2 I_3 \subseteq [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$. Hence either $I_1 I_3 \mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$ or $I_2 I_3 \mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$. For Q is faithful multiplication we have either $I_1 I_3 \mathrm{Q} \subseteq F + \big(soc(\mathrm{Q}) + J(\mathrm{Q})\big)$ or $I_2 I_3 \mathrm{Q} \subseteq F + \big(soc(\mathrm{Q}) + J(\mathrm{Q})\big)$. That is either $q_1 q_3 \subseteq F + \big(soc(\mathrm{Q}) + J(\mathrm{Q})\big)$ or $q_2 q_3 \subseteq F + \big(soc(\mathrm{Q}) + J(\mathrm{Q})\big)$. Thus by corollary 2.7 F is an Alappnq-prime submodule of Q.

Proposition 2.21 A proper submodule F of a content multiplication Z-regular R-module Q is an Alappnq-prime submodule of Q if and only if $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R.

Proof ($\Longrightarrow$) Let $acI \in [F:_R \mathrm{Q}]$ for $a, c \in R$ and I is an ideal of R, implies that $acI\mathrm{Q} \subseteq F$. But F is an Alappnq-prime submodule of Q, it follows by proposition 2.2 that either $aI\mathrm{Q} \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $cI\mathrm{Q} \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$. Since Q is multiplication, then $F = [F:_R \mathrm{Q}]\mathrm{Q}$, and since Q is content R-module then $J(\mathrm{Q}) = J(R)\mathrm{Q}$ and Q is Z-regular then $soc(\mathrm{Q}) = soc(R)\mathrm{Q}$. Thus either $aI\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$ or $cI\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$, we have either $aI \subseteq [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$ or $cI \subseteq [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$. Hence $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R.

($\Longleftarrow$) Let $KHq \subseteq F$ for K, H are submodules in Q, and $q \in \mathrm{Q}$. Since Q is a multiplication, then $K = I\mathrm{Q}$, $H = J\mathrm{Q}$ and $q = Rq = C\mathrm{Q}$ for some ideals I, J, C in R, that is $IJC\mathrm{Q} \subseteq F$, implies that $IJC \subseteq [F:_R \mathrm{Q}]$, but $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R, it follows by proposition 2.3 that either $IC \subseteq [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$ or $JC \subseteq [F:_R \mathrm{Q}] + \big(soc(R) + J(R)\big)$. Hence either $IC\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$ or $JC\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$. Since Q is content Z-regular then either $IC\mathrm{Q} \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $JC\mathrm{Q} \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$. That is either $Kq \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $Hq \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$. Thus by corollary 2.5 F is an Alappnq-prime submodule of Q.

Proposition 2.22 Let Q be a Z-regular multiplication module over a good ring R, and F be a proper submodule of Q. Then F is an Alappnq-prime submodule of Q if and only if $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R.

Proof (⟹) Let $ace \in [F:_R \mathrm{Q}]$ for $a, c, e \in R$, implies that $ac(e\mathrm{Q}) \subseteq F$. But F is an Alappnq-prime submodule of Q, it follows by proposition 2.2 that either $a(e\mathrm{Q}) \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $c(e\mathrm{Q}) \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$. Since Q is multiplication, then $F = [F:_R \mathrm{Q}]\mathrm{Q}$, and since Q Z-regular then $soc(\mathrm{Q}) = soc(R)\mathrm{Q}$ and R is a good ring $J(\mathrm{Q}) = J(R)\mathrm{Q}$. Thus either $ae\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$ or $ce\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$, it follows that either $ae \in [F:_R \mathrm{Q}] + (soc(R) + J(R))$ or $ce \in [F:_R \mathrm{Q}] + (soc(R) + J(R))$. Therefore $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R.

(⟸) Let $q_1 q_2 L \subseteq F$ for $q_1, q_2 \in \mathrm{Q}$, and L is a submodule of Q. Since Q is a multiplication, then $q_1 = Rq_1 = I\mathrm{Q}$, $q_2 = Rq_2 = J\mathrm{Q}$, and $H = A\mathrm{Q}$, for some ideals I, J and A in R, that is $IJA\mathrm{Q} \subseteq F$, implies that $IJA \subseteq [F:_R \mathrm{Q}]$, but $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R, it follows by proposition 2.3 that either $IA \subseteq [F:_R \mathrm{Q}] + (soc(R) + J(R))$ or $JA \subseteq [F:_R \mathrm{Q}] + (soc(R) + J(R))$. Hence either $IA\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$ or $JA\mathrm{Q} \subseteq [F:_R \mathrm{Q}]\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$, it follows that either $IA\mathrm{Q} \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $JA\mathrm{Q} \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ [for Q is Z- regular multiplication over good ring]. That is either $q_1 L \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $q_2 L \subseteq F + (soc(\mathrm{Q}) + J(\mathrm{Q}))$. Thus by corollary 2.6 F is Alappnq-prime submodule of Q.

It is known that Artinian ring is good ring [12]. We get the following direct result of proposition 2.22.

Corollary 2.13 Let Q be a Z-regular multiplication module over Artinian ring R, and F be a proper submodule of Q. Then F is an Alappnq-prime submodule of Q if and only if $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R.

3. Characterizations of Alappnq-prime Ideals By Alappnq-prime Submodules.

We start this section by recalling the following lemma which appears in [17].

Lemma 3.1 Let Q be finitely generated multiplication R-module, I and J are ideals in R. Then $I\mathrm{Q} \subseteq J\mathrm{Q}$ if and only if $I \subseteq J + ann_{\mathrm{R}}(\mathrm{Q})$.

Proposition 3.2 Let Q be a finitely generated multiplication projective R-module, and B is an ideal of R with $ann_R(\mathrm{Q}) \subseteq B$. Then B is an Alappnq-prime ideal of R if and only if $B\mathrm{Q}$ is an Alappnq-prime submodule of Q.

Proof (⟹) Let $q_1 q_2 q_3 \subseteq B\mathrm{Q}$ for $q_1, q_2, q_3 \in \mathrm{Q}$. Since Q is a multiplication then $q_1 = Rq_1 = I_1\mathrm{Q}$, $q_2 = Rq_2 = I_2\mathrm{Q}$, and $q_3 = Rq_3 = I_3\mathrm{Q}$ for some ideals I_1, I_2, I_3 of R, that is $I_1 I_2 I_3 \mathrm{Q} \subseteq B\mathrm{Q}$. But Q is a finitely generated multiplication R-module then by lemma 3.1 $I_1 I_2 I_3 \subseteq B + ann_R(\mathrm{Q})$. For $ann_R(\mathrm{Q}) \subseteq B$, implies that $B + ann_R(\mathrm{Q}) = B$, thus $I_1 I_2 I_3 \subseteq B$. Now, by assumption B is an Alappnq-prime ideal of R it follows by proposition 2.3 that either $I_1 I_3 \subseteq B + (soc(R) + J(R))$ or $I_2 I_3 \subseteq B + (soc(R) + J(R))$, it follows that either $I_1 I_3 \mathrm{Q} \subseteq B\mathrm{Q} + soc(R)\mathrm{Q} + J(R)\mathrm{Q}$ or $I_2 I_3 \mathrm{Q} \subseteq B\mathrm{Q} + soc(R)\mathrm{Q} + J(R)\mathrm{Q}$. Since Q is a projective then $soc(R)\mathrm{Q} = soc(\mathrm{Q})$ and $J(R)\mathrm{Q} = J(\mathrm{Q})$, it follows that either $q_1 q_3 \subseteq B\mathrm{Q} + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $q_2 q_3 \subseteq B\mathrm{Q} + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ Hence by corollary 2.7 $B\mathrm{Q}$ is an Alappnq-prime submodule of Q.

(⟸) Let $acI \subseteq B$, for $a, c \in R$ and I is an ideal of R, implies that $ac(I\mathrm{Q}) \subseteq B\mathrm{Q}$. But $B\mathrm{Q}$ is an Alappnq-prime submodule of Q, it follows by proposition 2.2 that either $a(I\mathrm{Q}) \subseteq B\mathrm{Q} + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or $c(I\mathrm{Q}) \subseteq B\mathrm{Q} + (soc(\mathrm{Q}) + J(\mathrm{Q}))$. But Q is a projective then $(soc(\mathrm{Q}) + J(\mathrm{Q})) = (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$. Thus either $a(I\mathrm{Q}) \subseteq B\mathrm{Q} + soc(R)\mathrm{Q} + J(R)\mathrm{Q}$ or $c(I\mathrm{Q}) \subseteq B\mathrm{Q} + soc(R)\mathrm{Q} + J(R)\mathrm{Q}$, it follows that either $aI \subseteq B + soc(R) + J(R)$ or $cI \subseteq B + soc(R) + J(R)$. Hence by proposition 2.2 B is an Alapp-quasiprime ideal of R.

Proposition 3.3 Let Q be a faithful finitely generated multiplication R-module, and B be an Alappnq-prime ideal of R. Then $B\mathrm{Q}$ is an Alappnq-prime submodule of Q.

Proof (⟹) Let $q_1 q_2 K \subseteq B\mathrm{Q}$, for $q_1, q_2 \in \mathrm{Q}$ and K is a submodule of Q. Since Q is a multiplication, then $q_1 = Rq_1 = I\mathrm{Q}$, $q_2 = Rq_2 = J\mathrm{Q}$ and $K = C\mathrm{Q}$ for some ideals I, J and C in R, that is $IJC\mathrm{Q} \subseteq B\mathrm{Q}$. But Q is a finitely generated multiplication R-module then by lemma 3.1 $IJC \subseteq B + ann_R(\mathrm{Q})$, since Q is faithful then $ann_R(\mathrm{Q}) = 0$, implies that $IJC \subseteq B$. But B is an Alappnq-prime ideal of R it follows by proposition 2.3 that either $IC \subseteq B + (soc(R) + J(R))$ or $JC \subseteq B + (soc(R) + J(R))$. Thus either $IC\mathrm{Q} \subseteq B\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$ or $JC\mathrm{Q} \subseteq B\mathrm{Q} + (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$. But Q is faithful multiplication, then $(soc(\mathrm{Q}) + J(\mathrm{Q})) = (soc(R)\mathrm{Q} + J(R)\mathrm{Q})$. Hence either $IC\mathrm{Q} \subseteq B\mathrm{Q} + (soc(\mathrm{Q}) + J(\mathrm{Q}))$ or

$JCQ \subseteq BQ + \big(soc(Q) + J(Q)\big)$. That is either $q_1K \subseteq BQ + (soc(Q) + J(Q))$ or $q_2K \subseteq BQ + (soc(Q) + J(Q))$. Therefore by corollary 2.6 BQ is an Alappnq-prime submodule of Q.

($\Longleftarrow$) Let $ace \in B$, for $a, c, e \in R$, implies that $ac(eQ) \subseteq BQ$. But BQ is an Alappnq-prime submodule of Q, it follows by proposition 2.2 that either $a(eQ) \subseteq BQ + (soc(Q) + J(Q))$ or $c(eQ) \subseteq BQ + (soc(Q) + J(Q))$. Since Q is faithful multiplication then either $a(eQ) \subseteq BQ + soc(R)Q + J(R)Q$ or $c(eQ) \subseteq BQ + soc(R)Q + J(R)Q$, it follows that either $ae \in B + soc(R) + J(R)$ or $ce \in B + soc(R) + J(R)$. Therefore B is an Alappnq-prime ideal of R.

Proposition 3.4 Let Q be a finitely generated multiplication Z-regular module over a good ring R, and B is an ideal of R with $ann_R(Q) \subseteq B$. Then B is an Alappnq-prime ideal of R if and only if BQ is an Alappnq-prime submodule of Q.

Proof ($\Longrightarrow$) Let $KHq \subseteq BQ$, for $q \in Q$ and K, H are submodules of Q. Since Q is a multiplication, then $K = IQ$, $H = JQ$ and $q = Rq = AQ$ for some ideals I, J and A in R, that is $IJAQ \subseteq BQ$. But Q is a finitely generated multiplication R-module then by lemma 3.1 $IJA \subseteq B + ann_R(Q)$, since $ann_R(Q) \subseteq B$, implies that $B + ann_R(Q) = B$ implies that $IJA \subseteq B$. But B is an Alappnq-prime ideal of R it follows by proposition 2.3 that either $IA \subseteq B + (soc(R) + J(R))$ or $JA \subseteq B + (soc(R) + J(R))$. Thus either $IAQ \subseteq BQ + (soc(R)Q + J(R)Q)$ or $JAQ \subseteq BQ + (soc(R)Q + J(R)Q)$. Since Q is Z-regular then $soc(R)Q = soc(Q)$ and R is good ring then $J(R)Q = J(Q)$. Hence either $IAQ \subseteq BQ + (soc(Q) + J(Q))$ or $JAQ \subseteq BQ + \big(soc(Q) + J(Q)\big)$. That is either $Kq \subseteq BQ + (soc(Q) + J(Q))$ or $Hq \subseteq BQ + (soc(Q) + J(Q))$. Therefore by corollary 2.5 BQ is an Alappnq-prime submodule of Q.

($\Longleftarrow$) Let $IJC \subseteq B$, for I, J and C are ideals in R, implies that $IJ(CQ) \subseteq BQ$. Since BQ is an Alappnq-prime submodule of Q, it follows by proposition 2.3 that either $I(CQ) \subseteq BQ + (soc(Q) + J(Q))$ or $J(CQ) \subseteq BQ + (soc(Q) + J(Q))$. But Q is Z-regular and R is good ring then $\big(soc(Q) + J(Q)\big) = (soc(R)Q + J(R)Q)$. Thus either $I(CQ) \subseteq BQ + soc(R)Q + J(R)Q$ or $J(CQ) \subseteq BQ + soc(R)Q + J(R)Q$, it follows that either $IC \subseteq B + soc(R) + J(R)$ or $JC \subseteq B + soc(R) + J(R)$. Hence by proposition 2.3 B is an Alappnq-prime ideal of R.

Corollary 3.5 Let Q be a finitely generated multiplication Z-regular module over Artinian ring R, and B is an ideal in R with $ann_R(Q) \subseteq B$. Then B is an Alappnq-prime ideal of R if and only if BQ is an Alappnq-prime submodule of Q.

Proposition 3.6 Let Q be a finitely generated multiplication projective R-module, and $F \subset Q$ with $ann_R(Q) \subseteq [F:_R Q]$ then the sentences that follow are comparable:

1. F is an Alappnq-prime submodule of Q.
2. $[F:_R Q]$ is an Alappnq-prime ideal of R.
3. $F = BQ$ for some Alappnq-prime ideal B of R with $ann_R(Q) \subseteq B$.

Proof (1) $\Leftrightarrow$ (2) It follows by proposition 2.8.

(2) $\Longrightarrow$ (3) Since Q is a multiplication, then $F = [F:_R Q]Q = BQ$, where $B = [F:_R Q]$ is an Alappnq-prime ideal of R with $ann_R(Q) = [0:_R Q] \subseteq [F:_R Q] = B$, implies that $ann_R(Q) \subseteq B$.

(3) $\Longrightarrow$ (2) It is given $F = BQ$ for some Alappnq-prime ideal B of R with $ann_R(Q) \subseteq B$. Since Q is a multiplication, then $F = [F:_R Q]Q$, thus we have $[F:_R Q]Q = BQ$. But Q is a finitely generated, so Q is weak cancellation [18], it follows that $[F:_R Q] + ann_R(Q) = B + ann_R(Q)$. But $ann_R(Q) \subseteq B$, and $ann_R(Q) \subseteq [F:_R Q]$, implies that $ann_R(Q) + B = B$ and $[F:_R Q] + ann_R(Q) = [F:_R Q]$. Thus $[F:_R Q] = B$, but B is an Alappnq-prime ideal of R, hence $[F:_R Q]$ is an Alappnq-prime ideal of R.

Proposition 3.7 Let Q be a faithful finitely generated multiplication R-module, and $F \subset Q$, then the sentences that follow are comparable:

1. F is an Alappnq-prime submodule of Q.
2. $[F:_R Q]$ is an Alappnq-prime ideal of R.
3. $F = BQ$ for some Alappnq-prime ideal B of R.

Proof (1) $\Leftrightarrow$ (2) It follows by proposition 2.20.

(2) $\Longrightarrow$ (3) Since Q is a multiplication, then $F = [F:_R Q]Q$, put $B = [F:_R Q]$ is an Alappnq-prime ideal of R and $F = BQ$.

(3) $\Longrightarrow$ (2) We have $F = B\mathrm{Q}$ for some Alappnq-prime ideal B of R. Since Q is a multiplication, then $F = [F:_R \mathrm{Q}]\mathrm{Q}$. Thus $[F:_R \mathrm{Q}]\mathrm{Q} = B\mathrm{Q}$. But Q is a faithful finitely generated multiplication, then Q is cancellation [18], it follows that $[F:_R \mathrm{Q}] = B$, hence $[F:_R \mathrm{Q}]$ is an Alappnq-prime ideal of R.

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