Journal of Al-Qadisiyah for Computer Science and Mathematics Vol. 14(4) 2022 , pp Math. 121–135

University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

On the Study Sum Graph of the Group Z_{p^n} with Some Topological Index

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ARTICLE INFO

Article history:
Received: 12 /11/2022
Rrevised form: 20 /12/2022
Accepted : 26 /12/2022
Available online: 31 /12/2022

Keywords:

group sum graph
cyclic graph
order group
Euler graph
Hamilton graph
Zagreb index
Forgotten index

ABSTRACT

In this research, we study the graph theory in the sum group via Z_{p^n} using concept of two distinct orders the sum it is always greatest that and equal the order of the group Z_{p^n}, where p is prime number. We have shown that the group sum graph of Z_{p^n} are connected, cyclic, etc. if it satisfied some properties of the graph theory and founded all the degree of graphs, we shall compute the famous of topological indices via generalized it.

MSC..

https://doi.org/10.29304/jqcm.2022.14.4.1122

1. Introduction

Let G be a graph and the set of vertices and edges of G be denoted as $V(G)$ and $E(G)$ respectively. Throughout the paper we consider only simple finite graphs. The degree of a vertex $u \in V(G)$ is denoted by $deg(u)$ of G, if their is no confusion we simply write it as $d(u)$. Two vertices x and y are called adjacent if there is an edge connecting them denoted by $x \sim y$. The connecting edge is usually denoted by xy. The degree of an edge xy is defined as the number of edges connected to the edge, (i.e.) the number of edges connected to both x and y except the edge xy. By $G \in G_+(k, a)$ we mean that G is a graph with k vertices and a edges. The notations and terminologies used but not clearly stated in this article may be found in [1]. Topological indices are the numerical values which are associated with a graph structure.

Over the years many topological indices are proposed and studied based on degree, distance and other parameters of graph. Some of them may be found in [7, 11]. Historically Zagreb indices can be considered as the first degree-

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based topological indices, which application into the many science chemical, physical and economical, etc. Various studies related to group or ring theory aspects of the Z modulo via grah theory, more see that [3-11] and the references therein. In 2018, Alwardi, A., et al., studied the entire Zagreb indices of graphs [2].

The objective of this paper is twofold. The inequality aspects of the sum two district order on Z_{p^n} modulo with order them, where p is prime number. The Topological indices have been well studied in last years, some of them may be found in [12-16, 20]. In 2013, the Zagreb indices were re-defined as first, second and third Zagreb indices by Ranjini et al. [3]. Further, be seen as a special case of the generalized inverse sum index $ISI_{(\gamma,\mu)}(G)$ of a graph G proposed by Buragohain et al. in [4].

2. Basic Concepts and Terminology

In what follows, we focus for topological index that will be needed in the subsequent considerations.

The Eccentric Connectivity index $\mathfrak{J}^c$ is defined as

$\mathfrak{J}^c = \mathfrak{J}^c(G) = \sum_{i=1}^{k} e(u)d_i$, *where* $e(u) = \max v \in V(G)d(u,v)$. [14]

We try that general of the Eccentric Connectivity index $\mathfrak{J}^c_\gamma$ is define as

$$\mathfrak{J}^c_\gamma = \mathfrak{J}^c_\gamma(G) = \sum_{i=1}^{k} e(u)d_i^\gamma, \gamma \in \mathbb{R}.$$

The first general Zagreb index (or general Zeroth-order Randic' index) Q_γ is defined as

$$Q_\gamma = Q_\gamma(G) = \sum_{i=1}^{k} d_i^\gamma = \sum_{i\sim j} \left(d_i^{\gamma-1} + d_j^{\gamma-1}\right),$$

where k is order of vertices and $\gamma \in \mathbb{R}$ [14].

The generalized Randic' index (or Connectivity index) R_γ, is defined as

$$R_\gamma = R_\gamma(G) = \sum_{i\sim j} \left(d_i d_j\right)^\gamma,$$

and $\gamma \in \mathbb{R}$ [14].

The general Sum-Connectivity index H_μ as

$$H_\mu = H_\mu(G) = \sum_{i\sim j} \left(d_i + d_j\right)^\mu,$$

and $\mu \in \mathbb{R}$ [15].

3. Main Result

In the following section, we will investigate the concept of Z_n group via graph theory by define that by order laws. Moreover, we founded the degree of special value of Z_n at p^n, where p is prime number to power positive integer number n.

Definition 3.1: Let G be a finite cyclic group. The group sum graph, denoted by $G_+(V,E)$ of G_+ is a graph with $V(G_+) = \bigcup_{x\in G} \langle x\rangle$ and two distinct vertices x and y are adjacent in G_+, denoted by $\langle x\rangle \sim \langle y\rangle$ if and only if $\mathcal{O}(x) + \mathcal{O}(y) \geq \mathcal{O}(G)$, where $\mathcal{O}(G)$ the order of the group G. (i.e.)

$V(G_+) = \bigcup_{x\in G} \langle x\rangle$,

$E(G_+) = \{xy |\ \langle x\rangle \sim \langle y\rangle$ if and ony if $\mathcal{O}(x) + \mathcal{O}(y) \geq \mathcal{O}(G)$, where $x, y \in G$ and $x \neq y\}$.

Remark 3.2:

If taking as (definition 3.1) $\mathcal{O}(x)+\mathcal{O}(y)\leq\mathcal{O}(G)$, where G is finite group of order n. We see that the graph is not connected, because, there exist at least one element a such that, $\mathcal{O}(a)=\mathcal{O}(G)$. Therefore, $\mathcal{O}(a)+\mathcal{O}(a_i)>\mathcal{O}(G), \forall\, a_i\in G, 1\leq i\leq n$, (i.e.) a is isolated-vertex, hence G is not connected.

Lemma 3.3: Every finite cyclic group hold the group sum graph are connected and cyclic graphs.

Proof: Suppose that G is finite cyclic group of order $n>2$. Since G is a cyclic group, there is at least one element, say $a\in G$, which is generated G. Thus, $\mathcal{O}(a)=\mathcal{O}(G)=n$, then $\mathcal{O}(a)+\mathcal{O}(a_i)>n$, where $\mathcal{O}(a_i)\geq 1, \forall\, i.\, a_i$ are distinct elements in $G, i=2,\dots,n$, therefore a adjacent to all $a_i, i=2,\dots,n.\ (a\sim a_i,\forall i)$, therefore any two vertices have path, then G is connected.

Now, we see that $\mathcal{O}(e)=1$ and other vertices in G have order greater than or equal 2, (i.e.) $\mathcal{O}(a_i)\geq 2, \forall\, 2\leq i\leq n$. So that, $\mathcal{O}(a_i)+\mathcal{O}(a_j)\geq 2, \forall\, 1\leq i<j\leq n$. (i.e.) G_+ is cyclic graph (by[19]).

In the following lemma, starting via Z_p which is special case.

Lemma 3.4: If Z_p be a finite group of order a prime number p, then

$G_+\left(V(Z_p),E(Z_p)\right)\cong K_p,$

where K_p is complete graph of p vertices and $\frac{p(p-1)}{2}$ edges.

Proof: We prove by contradiction. (i.e.), assume that $G_+\left(V(Z_p),E(Z_p)\right)\ncong K_p$, this means, there exists a two distinct vertices u_1,u_2 such that, $u_1u_2\notin E(Z_p)$.

Now, we see that, $Z_p=\{0,1,\dots,p-1\}$. So that $\mathcal{O}(0)=1$ and $\mathcal{O}(a)=p$ for each $a\in Z_p-\{0\}$, therefore (By definition 1)

<u>**Case 1:**</u> $\mathcal{O}(0)+\mathcal{O}(a)=1+p\geq p\Rightarrow 0a\in E(Z_p), \forall\, a\in Z_p-\{0\}$.

<u>**Case 2:**</u> $\mathcal{O}(a)+\mathcal{O}(b)=p+p=2p\geq p\Rightarrow ab\in E(Z_p), \forall\, a,b\in Z_p-\{0\}, a\neq b$.

Contradiction with hypothesis. Then $G_+\left(V(Z_p),E(Z_p)\right)\cong K_p$.

Now, notice the degree different when $n\geq 2$ follow as.

Proposition 3.5: If $G_+\left(V(Z_{p^2}),E(Z_{p^2})\right)$, then

$$\deg(u)=\begin{cases}p(p-1) & \text{if } \mathcal{O}(u)\neq p^2\\ p^2-1 & \text{if } \mathcal{O}(u)=p^2\end{cases},$$

where $u\in V(Z_{p^2})$ and $p\geq 2$ prime number.

Proof: Since $Z_{p^2}=\{0,1,2,\dots,p^2-1\}$, then Z_{p^2} have three-sets distinict orders, which that

$\mathcal{O}(0)=1, \mathcal{O}(p)=\mathcal{O}(\underbrace{p,2p,3p,\dots,(p-1)p}_{(p-1)\text{-times}})=p$ and $\mathcal{O}(\underbrace{a_1,a_2,\dots a_\alpha}_{\alpha-times})=p^2$, where $a_i\in Z_{p^2}, 1\leq i\leq\alpha, \alpha=p^2-[1+(p-1)]=p(p-1)$.

Now, by (definition 1), we have

<u>**Case 1:**</u> $\mathcal{O}(0)+\mathcal{O}(p)=1+p<\mathcal{O}(Z_{p^2})=p^2\Rightarrow 0p\notin E(Z_{p^2})$

$\mathcal{O}(0)+\mathcal{O}(a_i)=1+p^2>p^2\Rightarrow a_i\in E(Z_{p^2}), \forall\, 1\leq \mathrm{i}\leq \alpha$.

<u>**Case 2:**</u> $\mathcal{O}(p)+\mathcal{O}(a_i)=p+p^2>p^2\Rightarrow\ pa_i\in E(Z_{p^2}), \forall\, 1\leq i\leq\alpha$,

which implies that, $\deg(0)=\deg(p)=\deg(2p)=\dots=\deg((p-1)p)=p(p-1)$.

<u>Case3:</u> $\mathcal{O}(a_i)+\mathcal{O}(a_j)=p^2+p^2>p^2 \Rightarrow a_ia_j \in E(Z_{p^2}), \forall\, 1 \le i,j \le \alpha,\ i \ne j$, which implies that, $\deg(a_i)=p^2-1, \forall 1 \le i \le \alpha$. (Since the graph is simple graph)

$$\deg_{u\in V(Z_{p^2})}(u) = \begin{cases} p(p-1) & \text{if } o(u) \ne p^2 \\ p^2-1 & \text{if } o(u)=p^2 \end{cases}.$$

We give simple example describe that the group sum graph of $p = 2, 3$.

Example .1: (1) $Z_{2^2}=\{0,1,2,3\}$

$\mathcal{O}(0)=1, \mathcal{O}(1,3)=4$ and $\mathcal{O}(2)=2$.

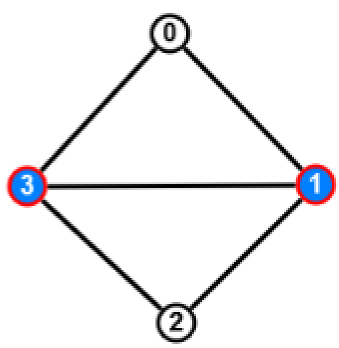
$$\deg_{u\in V(Z_4)}(u) = \begin{cases} 2 & \text{if } \mathcal{O}(u) \ne 4 \\ 3 & \text{if } \mathcal{O}(u)=4 \end{cases}$$

(2) $Z_{3^2}=\{0,1,\dots,8\}$

$\mathcal{O}(0)=1, \mathcal{O}\,\underbrace{(1,2,4,5,7,8)}_{6-times}=9$

$\mathcal{O}\,\underbrace{(3,6)}_{2-times}=3.$

$$\deg_{u\in V(Z_{3^2})}(u) = \begin{cases} 6 & \text{if } \mathcal{O}(u) \ne 9 \\ 8 & \text{if } \mathcal{O}(u)=9 \end{cases}$$



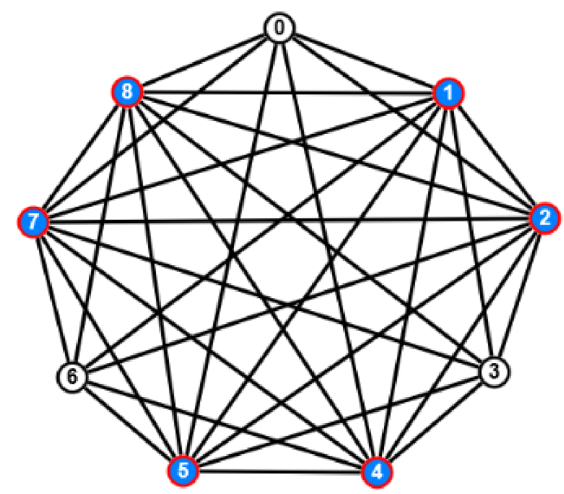


FIGURE 1B GRAPH A GRAHP

Proposition 3.6: If $G_+\left(V(Z_{p^3}), E(Z_{p^3})\right)$, then

$$\deg_{u\in V(Z_{p^3})}(u) = \begin{cases} p^2(p-1) & \text{if } \mathcal{O}(u) \ne p^3 \\ p^3-1 & \text{if } \mathcal{O}(u)=p^3 \end{cases}$$

where $p \ge 3$ is prime number.

Proof: Since $Z_{p^3}=\{0,1,\cdots,p^3-1\}$, then Z_{p^3} have four sets distinct orders which that,

$\underbrace{\mathcal{O}(0)=1}_{1-times}, \mathcal{O}(p)=\mathcal{O}\,\underbrace{(p,2p,\dots,(p^2-1)p)}_{(p^2-1)-times}=p^2$; $\mathcal{O}(p^2)=\mathcal{O}\,\underbrace{(p^2,2p^2,\dots,(p-1)p^2)}_{(p-1)-times}=p$ and $\mathcal{O}\,\underbrace{(a_1,a_2,\dots,a_\alpha)}_{\alpha-times}=p^3$,

where, $a_i \in Z_{p^3}, \mathcal{O}(a_i)=p^3, \forall 1 \le i \le \alpha, \alpha=p^3-[1+(p^2-1)]=p^2(p-1)$.

We see that, $\mathcal{O}(p^2)=p$ is suborder of $\mathcal{O}(p)=p^2$ and $p<p^2, \forall \ge 3$.

It is sufficient for taking $\mathcal{O}(p)=p^2$ in the proof by (definition 3.1), we have

<u>Case 1:</u> $\mathcal{O}(0)+\mathcal{O}(p)=1+p^2<\mathcal{O}(Z_{p^3})=p^3 \Rightarrow 0p \notin E(Z_{p^3})$

$$\mathcal{O}(0) + \mathcal{O}(a_i) = 1 + p^3 < p^3 \Rightarrow 0a_i \in E(Z_{p^3}), \forall\, 1 \le i \le \alpha.$$

<u>Case 2:</u> $\mathcal{O}(p) + \mathcal{O}(a_i) = p^2 + p^3 > p^3 \Rightarrow pa_i \in E(Z_{p^3}), \forall 1 \le i \le \alpha$, which implies that,

$\deg(0) = \deg(p) = \deg(2p) = \cdots = \deg((p^2-1)p) = p^2(p-1)$.

<u>Case 3:</u> $\mathcal{O}(a_i) + \mathcal{O}(a_j) = p^3 + p^3 > p^3 \Rightarrow a_i a_j \in E(Z_{p^3}), \forall\, 1 \le i, j \le \alpha,\ i \ne j$, which implies that, $\deg(a_i) = p^3 - 1, \forall 1 \le i \le \alpha$. [Since the graph is simple graph]

$$\underset{u \in V(Z_{p^3})}{\deg(u)} = \begin{cases} p^2(p-1) & \text{if } \mathcal{O}(u) \ne p^3 \\ p^3 - 1 & \text{if } \mathcal{O}(u) = p^3 \end{cases}$$

Example .2: $Z_{3^3} = \{0, 1, \dots, 26\}$

$\underbrace{\mathcal{O}(0)}_{1-times} = 1, \mathcal{O}\underbrace{(3, 6, 12, 15, 21, 24)}_{6-times} = 9, \mathcal{O}\underbrace{(9,18)}_{2-times} = 3$ and

$\mathcal{O}\underbrace{(1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26)}_{18-times} = 27$.

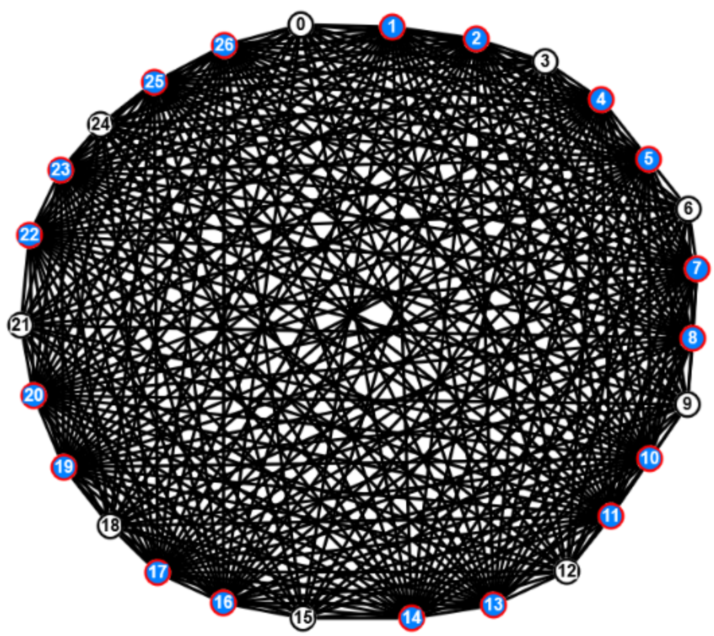


FIGURE 2

$$\underset{u \in V(Z_{3^3})}{\deg(u)} = \begin{cases} 18 & \text{if } \mathcal{O}(u) \ne 27 \\ 26 & \text{if } \mathcal{O}(u) = 27 \end{cases}$$

Remark 3.7: In the previous example we notice that this graph to express by this formal, $K_{18} + S$, where any $v \in K_{18}$ have $\mathcal{O}(v) = 27$ and $t \in S$ have $\mathcal{O}(u) \ne 27$.

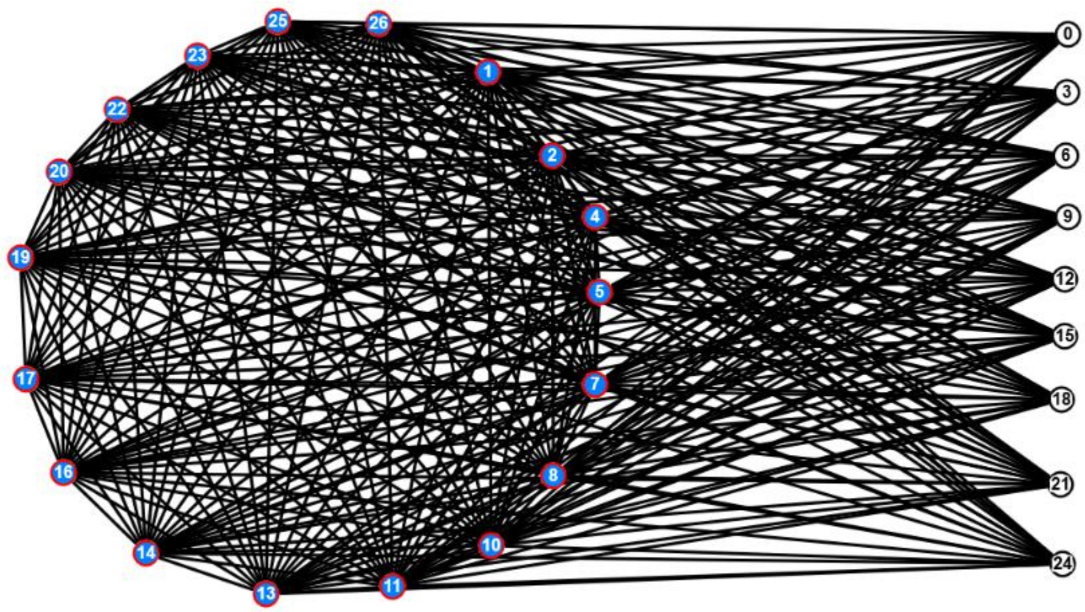


FIGURE 3

In general, we can to express for the graph $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$ by the formal $K_{p^{n-1}(p-1)} + S$, where any $v \in K_{p^{n-1}(p-1)}$ have $\mathcal{O}(v) = p^n$ and $t \in S$ have $\mathcal{O}(u) \neq p^n$, $p \geq 3$ is prime number and $n \geq 2$ is positive integer number.

In the following theorem, we try generalized the power of p.

Theorem 3.8: In generalized, if $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$, then

$$\underset{u\in V(Z_{p^n})}{\deg(u)} = \begin{cases} p^{n-1}(p-1) & \text{if } \mathcal{O}(u) \neq p^n \\ p^n - 1 & \text{if } \mathcal{O}(u) = p^n \end{cases},$$

where $p \geq 3$ is prime number and $n \geq 2$ is positive integer number.

Proof: We prove that, by mathematical induction, when $n = 2,3$ is satisfied in (Proposition 3.5, 3.6).

Now, assume that is hold whenever n and to prove that is satisfy, when $n+1$. Since $Z_{p^n} = \{0, 1, \dots, p^n - 1\}$ where, $\mathcal{O}(0) = 1, \mathcal{O}(p) = \mathcal{O}\,\underbrace{(p, 2p, \dots, (p^{n-1}-1)p)}_{(p^{n-1}-1)-times} = p^{n-1}, \mathcal{O}\,\underbrace{(a_1, a_2, \dots, a_\alpha)}_{\alpha\text{-times}} = p^n$, where $a_i \in Z_{p^n}, \mathcal{O}(a_i) = p^n, \forall\, 1 \leq i \leq \alpha, \alpha = p^{n-1}(p-1)$.

$$\underset{u\in V(Z_{p^n})}{\deg(u)} = \begin{cases} p^{n-1}(p-1) & \text{if } \mathcal{O}(u) \neq p^n \\ p^n - 1 & \text{if } \mathcal{O}(u) = p^n \end{cases}, \text{ where } p \geq 3 \text{ and } n \geq 2.$$

Now, we prove that, $Z_{p^{n+1}} = \{0, 1, \dots, p^{n+1} - 1\}$.

Then $\mathcal{O}(0) = 1, \mathcal{O}(p) = \underbrace{(p, 2p, \dots, (p^{n-1}-1)p)}_{(p^n-1)\text{-times}} = p^n, \mathcal{O}\,\underbrace{(a_1, a_2, \dots, a_\alpha)}_{\alpha\text{-times}} = p^{n+1}$,

where $a_i \in Z_{p^{n+1}}, \mathcal{O}(a_i) = p^{n+1}, \forall\, 1 \leq i \leq \alpha,\ \alpha = p^{n+1} - [1 + (p^n - 1)] = p^n(p-1)$.

Now, by (definition 3.1), we have

<u>**Case 1:**</u> $\mathcal{O}(0) + \mathcal{O}(p) = 1 + p^n < p^{n+1} = \mathcal{O}(Z_{p^{n+1}}) \Rightarrow 0p \notin E(Z_{p^{n+1}})$,

$\mathcal{O}(0) + \mathcal{O}(a_i) = 1 + p^{n+1} > p^{n+1} \Rightarrow\ 0a_i \in E(Z_{p^{n+1}}), \forall 1 \leq i \leq \alpha$.

<u>**Case2:**</u> $\mathcal{O}(p) + \mathcal{O}(a_i) = p^n + p^{n+1} > p^{n+1} \Rightarrow pa_i \in E(Z_{p^{n+1}}), \forall 1 \leq i \leq \alpha$, which implies that, $\deg(0) = \deg(p) = \deg(2p) = \cdots = \deg\left((p^n-1)p\right) = p^n(p-1)$.

Case3: $\mathcal{O}(a_i) + \mathcal{O}(a_j) = p^{n+1} + p^{n+1} > p^{n+1} \Rightarrow a_i, a_j \in E(Z_{p^{n+1}}),\ \forall\, 1 \le i, j \le \alpha, i \ne j$, which implies that, $\deg(a_i) = p^{n+1} - 1, \forall\, 1 \le i \le \alpha$. [Since the graph is simple graph]

$$\underset{u \in V(Z_{p^n})}{\deg(u)} = \begin{cases} p^{n-1}(p-1) & \text{if } \mathcal{O}(u) \ne p^n \\ p^n - 1 & \text{if } \mathcal{O}(u) = p^n, \text{where } p \ge 3, n \ge 2 \end{cases}.$$

Remarks 3.9

1-The Sum graph $G+(Z_{p^n})$ is Hamilton graph because all its vertices are of degree $\ge p^n/2$ (by Dirak) .

2- The Sum graph $G+(Z_{p^n})$ is Euler graph because all its vertices are of even degree.

4. Topological index of Z_{p^n}

W Notice in this section, we will compute of the famous some generalized topological index with special cases.

Remark 4.1: The first general Zagreb index (or general Zeroth-order Randic' index) Q_γ is defined as

$$Q_\gamma = Q_\gamma(G) = \sum_{i=1}^{k} d_i^\gamma = \sum_{i \sim j} (d_i^{\gamma-1} + d_j^{\gamma-1}),$$

where k is order of vertices.

Theorem 4.2: If $G_+\left(V(Z_{p^2}), E(Z_{p^2})\right)$, then the first general Zagreb index is

$Q_\gamma = p(p-1)[p^\gamma(p-1)^{\gamma-1} + (p^2-1)^\gamma]$.

Proof: The graph of $G_+\left(V(Z_{p^2}), E(Z_{p^2})\right)$ have degree p^2 (Proposition 3.5), where $p \ge 2$ a prime number. we get

$$Q_\gamma = Q_\gamma(Z_{p^2}) = \sum_{i=1}^{p^2} d_i^\gamma = \underbrace{[p(p-1)]^\gamma + [p(p-1)]^\gamma + \cdots + [p(p-1)]^\gamma}_{p-\text{ times}} + \underbrace{(p^2-1)^\gamma + (p^2-1)^\gamma + \cdots + (p^2-1)^\gamma}_{p(p-1)-\text{ times}}$$

$$= p[p(p-1)]^\gamma + p(p-1)[(p^2-1)]^\gamma$$

$$= p(p-1)[p^\gamma(p-1)^{\gamma-1} + (p^2-1)^\gamma]$$

(1) if $\gamma = 1 \Rightarrow Q_1 = \sum_{i=1}^{p^2} d_i = 2\ell \Rightarrow \ell = \frac{1}{2}Q_1 = \frac{1}{2}p(p-1)[p + p^2 - 1]$

(2) if $\gamma = 2 \Rightarrow Q_2 = \sum_{i=1}^{p^2} d_i^2 = \sum_{i \sim j} d_i + d_j = M_1 = p(p-1)[p^2(p-1) + (p^2-1)^2]$

(3) if $\gamma = 3 \Rightarrow Q_3 = \sum_{i=1}^{p^2} d_i^3 = \sum_{inj}^{2} d_i^2 + d_j^2 = F_1 = p(p-1)[p^3(p-1)^2 + (p^2-1)^3]$

Remark 4.3:

(1) Let $u \in V(G)$, where G_+ is a finite simple graph of order ($n \ne p$), where p is a prime number.

$$e(u) = \begin{cases} 1 & \text{if } \mathcal{O}(u) = |Z_n| \\ 2 & \text{if } \mathcal{O}(u) \ne |Z_n| \end{cases}.$$

(2) The general of the Eccentric Connectivity index $\mathfrak{J}_\gamma^c$ is define as

$$\Im_\gamma^c = \Im_\gamma^c(G) = \sum_{i=1}^{o} e(u)d_i^\gamma, \gamma \in \mathbb{R}.$$

Theorem 4.4: The Eccentric connectivity index of the graph $G_+\left(V(Z_{p^2}), E(Z_{p^2})\right)$ is $\Im_\gamma^c(Z_p^2) = p(p-1)[2p^\gamma(p-1)^{\gamma-1} + (p^2-1)^\gamma]$.

Proof: From (Remark 4.3) and by Proposition (3.5), we get

$$\Im_\gamma^c(Z_p^2) = 2\underbrace{[[p(p-1)]^\gamma + [p(p-1)]^\gamma + \cdots + [p(p-1)]^\gamma]}_{p-times} +$$

$$1\underbrace{[[p^2-1]^\gamma + \cdots + [p^2-1]^\gamma]}_{p(p-1)-times}$$

$$= p(p-1)[2p^\gamma(p-1)^{\gamma-1} + (p^2-1)^\gamma]$$

Remark 4.5: The generalized Randic' index (or Connectivity index) R_γ, is defined as

$$R_\gamma = R_\gamma(G) = \sum_{i\sim j} (d_i d_j)^\gamma$$

Theorem 4.6: If $G_+\left(V(Z_p^2), E(Z_{p^2})\right)$, then

$$R_\gamma = (p^2-1)^\gamma \sum_{i=1}^{p(p-1)} \left(p(p-1)(p^2+p-1) - i(p^2-1)\right)^\gamma.$$

Proof: Suppose that $S = \{a_1, q_2, \cdots, q_\alpha\}, \alpha = p(p-1), S \subseteq Z_p^2$, where $a_i \in Z_{p^2}, \mathcal{O}(a_i) = p^2, 1 \le i \le \alpha$.

We see that, every $a_i \in S, 1 \le i \le \alpha$ is adjacent to all vertices belong to Z_{p^2} except itself. (Since the graph is simple graph) $M_2(Z_{p^2}) = R_1$

$$R_1 = d(a_1)\left[\sum_{i=1}^{p^2} d(i_1) - d(a_1)\right] + d(a_2)\left[\sum_{i=1}^{p^2} d(i) - \left(d(a_1) + d(a_2)\right) + \overset{\alpha\text{-times}}{\cdots} + d(a_\alpha)\left[\sum_{i=1}^{p^2} d(i) - \sum_{i=1}^{\alpha} d(a_i)\right]\right.$$

Now, since $d(a_1) = d(a_2) = \cdots = d(a_\alpha) = d(a) = (p^2-1)$ and $\sum_{i=1}^{p^2} d(i) = 2\ell$

$$R_1 = d(a)\underbrace{[(2\ell - d(a)) + (2\ell - 2d(a)) + \cdots + (2\ell - \alpha d(a))]}_{\alpha-times}$$

$$= d(a)\left[\alpha(2\ell) - d(a)\sum_{i=1}^{\alpha} i\right]$$

$$= d(a)\left[\alpha(2\ell) - d(a)\frac{\alpha(\alpha+1)}{2}\right], \left(\sum_{i=1}^{\alpha} i = \frac{\alpha(\alpha+1)}{2}\right)$$

$$= \alpha d(a)\left[2\ell - d(a)\frac{(\alpha+1)}{2}\right]$$

$$= p(p-1)(p^2-1)\left[[p(p-1)(p^2+p-1)] - (p^2-1)\frac{(p(p-1)+1)}{2}\right]$$

So, in general, we get

$$R_\gamma = (d(a))^\gamma \underbrace{[(2\ell - d(a))^\gamma + (2\ell - 2d(a))^\gamma + \cdots + (2\ell - \alpha d(a))^\gamma]}_{\alpha-times}$$

$$= (d(a))^\gamma \sum_{i=1}^{\alpha} (2\ell - id(a))^\gamma$$

$$R_\gamma = (p^2-1)^\gamma \sum_{i=1}^{p(p-1)} \left(p(p-1)(p^2+p-1) - i(p^2-1)\right)^\gamma$$

In Particular,

(1) $R_{-1} = \sum_{i\sim j} \frac{1}{d_i d_j} = \frac{1}{(p^2-1)} \sum_{i=1}^{p(p-1)} \left(p(p-1)(p^2+p-1) - i(p^2-1)\right)^{-1}$

(2) $X(Z_{p^2}) = R_{-\frac{1}{2}} = \sum_{i\sim j} \frac{1}{\sqrt{d_i d_j}} = \frac{1}{\sqrt{p^2-1}} \sum_{i=1}^{p(p-1)} \frac{1}{\sqrt{p(p-1)(p^2+p-1) - i(p^2-1)}}$

Remark 4.7: In [15] Zhou Trinajstic defined general Sum-Connectivity index H_μ as

$$H_\mu = H_\mu(G) = \sum_{i\sim j} (d_i + d_j)^\mu$$

Theorem 4.8: If $G_+\left(V(Z_{p^2}), E(Z_{p^2})\right)$, then $H_\mu = \sum_{i\sim j} (d_i + d_j)^\mu$

$$= p(p-1)\left[2^{\mu-1}(p(p-1)-1)(p^2-1)^\mu + p\left(\underbrace{(p^2-1) + p(p-1)}_{2p^2-p-1}\right)^\mu\right]$$

Proof: Suppose that $S = \{a_1, a_2, \dots a_\alpha\}, \alpha = p(p-1), S \subseteq Z_{p^2}, a_i \in Z_{p^2}$, where $\mathcal{O}(a_i) = p^2, \forall\, 1 \le i \le \alpha$.

Since, every a_i is adjacent to all vertices belong to $Z_{p^2}, \forall\, 1 \le i \le \alpha$ except itself. Then

$$\begin{aligned} H_\mu = {} & \underbrace{\left(d(a_1) + d(a_1)\right)^\mu + \left(d(a_1) + d(a_2)\right)^\mu + \cdots + \left(d(a_1) + d(a_\alpha)\right)^\mu}_{\alpha-times} - \left(d(a_1) + d(a_1)\right)^\mu \\ & + \underbrace{\left(d(a_1) + d(0)\right)^\mu + \cdots + \left(d(a_1) + d\left((p-1)p\right)\right)^\mu}_{p-times} \\ & + \underbrace{\left(d(a_2) + d(a_1)\right)^\mu + \left(d(a_2) + d(a_2)\right)^\mu + \cdots + \left(d(a_2) + d(a_\alpha)\right)^\mu}_{\alpha-times} \\ & - \left[\left(d(a_2) + d(a_1)\right)^\mu + \left(d(a_2) + d(a_2)\right)^\mu\right] + \underbrace{\left(d(a_2) + d(0)\right)^\mu + \cdots + \left(d(a_2) + d\left((p-1)p\right)\right)^\mu}_{p-times} \\ & + \underbrace{\left(d(a_\alpha) + d(a_1)\right)^\mu + \cdots + \left(d(a_\alpha) + d(a_\alpha)\right)^\mu}_{\alpha-times} - \left[\left(d(a_\alpha) + d(a_1)\right)^\mu + \cdots + \left(d(a_\alpha) + d(a_\alpha)\right)^\mu\right] \\ & + \underbrace{\left(d(a_\alpha) + d(0)\right)^\mu + \cdots + \left(d(a_\alpha) + d\left((p-1)p\right)\right)^\mu}_{p-times} \end{aligned}$$

Now, since $d(a_1) = d(a_2) = \cdots = d(a_\alpha) = d(a) = (p^2-1)$

$$d(0) = d(p) = \cdots = d((p-1)p) = p(p-1) = \alpha$$

$$\begin{aligned} H_\mu = {} & [(\alpha-1)(2d(a))^\mu + p(d(a)+\alpha)^\mu] + [(\alpha-2)(2d(a))^\mu + p(d(a)+\alpha)^\mu] + \cdots \\ & + [(\alpha-(\alpha))(2d(a))^\mu + p(d(a)+\alpha)^\mu] \end{aligned}$$

$$H_\mu = \sum_{i=1}^{\alpha} (\alpha - i)(2d(a))^\mu + \alpha p(d(a)+\alpha)^\mu$$

Since $\sum_{i=1}^{\alpha} (\alpha - i) = \frac{\alpha(\alpha-1)}{2}$

$$H_\mu = \alpha\left[\frac{(\alpha-1)}{2}(2d(a))^\mu + p(d(a)+\alpha)^\mu\right]$$

$$H_\mu = p(p-1)\left[2^{\mu-1}(p(p-1)-1)(p^2-1)^\mu + p\left(\underbrace{(p^2-1)+p(p-1)}_{2p^2-p-1}\right)^\mu\right]$$

In particular,

(1) if $\mu = 1 \Rightarrow H_1 = M_1 = Q_2 = p(p-1)[p^2(p^2-1) + p(p-1)(p^2+p-1) - (p^2-1)(p(p-1)+1)]$.

(2) if $\mu = -1 \Rightarrow 2H_{-1} = H = \sum_{i\sim j} \frac{2}{d_i+d_j} = 2p(p-1)\left[\frac{1}{4}\frac{(p(p-1)-1)}{(p^2-1)} + \frac{p}{2p^2-p-1}\right]$

(3) if $\mu = -\frac{1}{2} \Rightarrow \lambda(Z_{p^2}) = H_{-\frac{1}{2}} = \sum_{i\sim j} \frac{1}{\sqrt{d_i+d_j}} = p(p-1)\left[\frac{(p(p-1)-1)}{(\sqrt{2})^3\sqrt{p^2-1}} + \frac{p}{\sqrt{2p^2-p-1}}\right]$.

Another method for compute the H_1 of Z_{p^2}.

Theorem 4.9: If $G_+\left(V(Z_{p^2}), E(Z_{p^2})\right)$, then $M_1(Z_{p^2}) = H_1 = \sum_{i\sim j}(d_i + d_j) = p(p-1)[p^2(p^2-1) + p(p-1)(p^2+p-1) - (p^2-1)(p(p-1)+1)]$.

Proof: Suppose that $S = \{a_1, a_2, \dots, a_\alpha\}, \alpha = p(p-1), S \subseteq Z_{p^2}, \alpha_i \in Z_p^2$, where $O(a_i) = p^2, \forall\ 1 \le i \le \alpha$.

Since, every a_i is adjacent to all vertices belong to $Z_{p^2}, \forall\ 1 \le i \le \alpha$ except itself. Then

$$H_1 = \sum_{i=1}^{p^2}\left(d(a_1)+d(i)\right) - \left(d(a_1)+d(a_1)\right) + \sum_{i=1}^{p^2}\left(d(a_2)+d(i)\right) - \left[\left(d(a_2)+d(a_1)\right) + \left(d(a_2)+d(a_2)\right)\right] + \overset{\alpha-times}{\overbrace{\cdots}} + \sum_{i=1}^{p^2}\left(d(a_\alpha)+d(i)\right) - \left(\sum_{i=1}^{\alpha} d(a_\alpha) + \sum_{i=1}^{\alpha} d(a_i)\right)$$

Now, since $d(a_1) = d(a_2) = \cdots = d(a_\alpha) = d(0) = p^2 - 1,\ \sum_{i=1}^{p^2} d(i) = 2\ell$

$$H_1 = \left[\sum_{i=1}^{p^2} d(a) + \sum_{i=1}^{p^2} d(i) - 2d(a)\right] + \left[\sum_{i=1}^{p^2} d(a) + \sum_{i=1}^{p^2} d(i) - 2\left(2d(a)\right)\right] + \overset{\alpha-times}{\overbrace{\cdots}} + \left[\sum_{i=1}^{p^2} d(a) + \sum_{i=1}^{p^2} d(i) - 2\left(\alpha d(a)\right)\right]$$

$$H_1 = [p^2 d(a) + 2\ell - 2d(a)] + [p^2 d(a) + 2\ell - 2(2d(a))] + \overset{\alpha-times}{\overbrace{\cdots}} + [p^2(d(a)) + 2\ell - 2(\alpha d(a))]$$

$$H_1 = \alpha[p^2 d(a) + 2\ell] - 2d(a)\sum_{i=1}^{\alpha} i$$

$$= \alpha[p^2 d(a) + 2\ell] - 2d(a)\frac{\alpha(\alpha+1)}{2}$$

$$H_1 = M_1 = \alpha[p^2 d(a) + 2\ell - d(a)(\alpha+1)]$$

$$= p(p-1)[p^2(p^2-1) + p(p-1)(p^2+p-1) - (p^2-1)(p(p-1)+1)].$$

Theorem 4.10: If $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$, then the first general Zegrab index is

$$Q_\gamma = p[(p^{n-1}(p-1))^\gamma + p^{n-2}(p-1)(p^n-1)^\gamma].$$

Proof: The graph of $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$ have degree by (Theorem 3.8), where $p \geq 3$ is a prime number and $n \geq 2$ is positive integer number. We get

$$\begin{aligned} Q_\gamma = Q_\gamma(Z_{p^n}) &= \textstyle\sum_{i=1}^{p^n} d_i^\gamma \\ &= \underbrace{[p^{n-1}(p-1)]^\gamma + [p^{n-1}(p-1)]^\gamma + \cdots + [\rho^{n-1}(p-1)]^\gamma}_{p-t_{\text{imes}}} + \underbrace{(p^n-1)^\gamma + (p^n-1)^\gamma + \cdots + (p^n-1)^\gamma}_{p^{n-1}(p-1)-\text{times}} \\ &= p[p^{n-1}(\mathrm{p}-1)]^\gamma + p^{n-1}(p-1)[(p^n-1)]^\gamma \\ &= p[(p^{n-1}(p-1))^\gamma + p^{n-2}(p-1)(p^n-1)^\gamma] \end{aligned}$$

In particular, if

(1) If $\gamma = 1 \Rightarrow Q_1 = \sum_{i=1}^{p^n} d_i = 2\ell \Rightarrow \ell = \frac{1}{2}Q_1$

$$= \frac{1}{2} \cdot p[p^{n-1}(p-1)p^{n-2}(p-1)(p^n-1)]$$

$$= \frac{p}{2}[p^n - p^{n-1} + p^{n-2}(p-1)(p^n-1)]$$

(2) If $\gamma = 2 \Rightarrow Q_2 = \sum_{i=1}^{p^n} d_i^2 = \sum_{i\sim j} d_i + d_j = M_1$

$$= p[(p^{n-1}(p-1))^2 + p^{n-2}(p-1)(p^n-1)^2].$$

(3) If $\gamma = 3 \Rightarrow Q_3 = \sum_{i=1}^{p^n} d_i^3 = \sum_{i=j} d_i^2 + d_j^2 = F_1$

$$= p[(p^{n-1}(p-1))^3 + p^{n-2}(p-1)(p^n-1)^3]$$

Theorem 4.11: The Eccentric connectivity index of the graph $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$ is $\mathfrak{I}_\gamma^c = p[2(p^{n-1}(p-1))^\gamma + p^{n-2}(p-1)(p^n-1)^\gamma]$.

Proof: From Remark (4.3) and by Theorem (3.8), we get

$$\begin{aligned} \mathfrak{I}_\gamma^c(Z_{p^n}) &= 2\underbrace{[[p^{n-1}(p-1)]^\gamma + \cdots + [\rho^{n-1}(p-1)]^\gamma]}_{p-t_{\text{imes}}} + \underbrace{[(p^n-1)^\gamma + \cdots + (p^n-1)^\gamma]}_{p^{n-1}(p-1)-\text{times}} \\ &= 2p[p^{n-1}(\mathrm{p}-1)]^\gamma + p^{n-1}(p-1)[(p^n-1)]^\gamma \\ &= p[2(p^{n-1}(p-1))^\gamma + p^{n-2}(p-1)(p^n-1)^\gamma]. \end{aligned}$$

Theorem 4.12: If $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$, then

$$R_\gamma = (p^n-1)^\gamma \sum_{i=1}^{p^{n-1}(p-1)} [p^{n-1}(p-1)(p^n + p^{n-1} - 1) - i(p^n-1)]^\gamma.$$

Proof: Suppose that $S = \{a_1, a_2, \cdots a_\alpha\}, \alpha = p^{n-1}(p-1),\ S \subseteq Z_{p^n}$, where $a_i \in Z_{p^n}, \mathcal{O}(a_i) = p^n, 1 \leq i \leq \alpha$ we see that, every $a_i \in \mathrm{S}, 1 \leq i \leq \alpha$ is adjacent to all vertices belong to Z_{p^n} except itself. (Since the graph is a simple graph).

$$R_1 = d(a_1)\left[\sum_{i=1}^{p^n} d_i - d(a_1)\right] + d(a_2)\left[\sum_{i=1}^{p^n} d_i - \left(d(a_1)+d(a_2)\right)\right] + \overset{\alpha-times}{\cdots} + d(a_\alpha)\left[\sum_{i=1}^{p^n} d_i - \sum_{i=1}^{\alpha} d(a_i)\right]$$

Now, since $d(a_1) = d(a_2) = \cdots = d(a_\alpha) = d(a) = p^n - 1$ and $\sum_{i=1}^{p^n} d_i = 2\ell$

$$\Rightarrow R_1 = d(a)\left[\underbrace{\left(2\ell - d(a)\right) + \left(2\ell - 2d(a)\right) + \cdots + \left(2\ell - \alpha d(a)\right)}_{\alpha\text{-times}}\right] \quad \dots (2)$$

$$= d(a)\left[\alpha 2\ell - \sum_{i=1}^{\alpha} i\, d(a)\right] = d(a)\left[\alpha 2\ell - d(a)\frac{\alpha(\alpha+1)}{2}\right]$$

$$R_1 = \alpha d(a)\left[2\ell - d(a)\frac{(\alpha+1)}{2}\right]$$

$$= p^{n-1}(p-1)(p^n-1)\left[\underbrace{(p^{n-1}(p-1)(p^n+p^{n-1}-1)}_{2\ell} - (p^n-1)\left(\frac{p^{n-1}(p-1)+1}{2}\right)\right]$$

So, in general, we get by eq. (2)

$$R_\gamma = (d(a))^\gamma\left[\underbrace{\left(2\ell - d(a)\right)^\gamma + \left(2\ell - 2d(a)\right)^\gamma + \cdots + \left(2\ell - \alpha d(a)\right)^\gamma}_{\alpha-times}\right]$$

$$= (d(a))^\gamma \sum_{i=1}^{\alpha}\left(2\ell - i d(a)\right)^\gamma$$

$$R_\gamma = (p^n-1)^\gamma \sum_{i=1}^{p^{n-1}(p-1)} [p^{n-1}(p-1)(p^n+p^{n-1}-1) - i(p^n-1)]^\gamma$$

In particular,

(1) $R_{-1} = \sum_{i\sim j}\frac{1}{d_i d_j} = \frac{1}{(p^n-1)}\cdot\sum_{i=1}^{p^{n-1}(p-1)}\cdot\frac{1}{p^{n-1}(p-1)(p^n+p^{n-1}-1)-i(p^n-1)}$

(2) $\lambda(Z_{p^n}) = R_{-1/2} = \sum_{i\sim j}\frac{1}{\sqrt{d_i d_j}} = \frac{1}{\sqrt{p^n-1}}\cdot\sum_{i=1}^{p^{n-1}(p-1)}\frac{1}{\sqrt{p^{n-1}(p-1)(p^n+p^{n-1}-1)-i(p^n-1)}}$

Theorem 4.13: If $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$, then

$$H_\mu = \sum_{i\sim j}\left(d_i + d_j\right)^\mu$$

$$= p^{n-1}(p-1)\left[\frac{p^{n-1}(p-1)-1}{2}\left(2(p^n-1)\right)^\mu + p^{n-1}(2p^n - p^{n-1} - 1)^\mu\right], \mu \in \mathbb{R}.$$

Proof: Suppose that $S = \{a_1, \dots, a_\alpha\}, \alpha = p^{n-1}(p-1), S \subseteq Z_{p^n}, a_i \in Z_{p^n}$, where $O(a_i) = p^n, \forall\ 1 \le i \le \alpha$.

Since every a_i is adjacent to all vertices belong to $Z_{p^n}, \forall\ 1 \le i \le \alpha$, except itself, then

$$H_\mu = \underbrace{\left(d(a_1)+d(a_1)\right)^\mu + \left(d(a_1)+d(a_2)\right)^\mu + \cdots + \left(d(a_1)+d(a_\alpha)\right)^\mu}_{\alpha-times} - \left(d(a_1)+d(a_1)\right)^\mu + \left(d(a_1)+d(0)\right)^\mu + \cdots$$
$$+ \left(d(a_1) + d\left(p^{n-1}(p-1)\right)\right)^\mu$$

$$+\underbrace{\left(d(a_2)+d(a_1)\right)^{\mu}+\left(d(a_2)+d(a_2)\right)^{\mu}+\cdots+\left(d(a_2)+d(a_\alpha)\right)^{\mu}}_{\alpha-times}-\left[\left(d(a_2)+d(a_2)\right)^{\mu}+\left(d(a_2)+d(a_2)\right)^{\mu}\right]$$

$$+\underbrace{\left(d(a_2)+d(0)\right)^{\mu}+\cdots+\left(d(a_2)+d\left(p^{n-1}(p-1)\right)\right)^{\mu}}_{p^{n-1}-times}+\left(d(a_\alpha)+d(a_1)\right)^{\mu}+\cdots$$

$$+\left(d(a_\alpha)+d(a_\alpha)\right)^{\mu}-\left[\left(d(a_\alpha)+d(a_1)\right)^{\mu}+\left(d(a_\alpha)+d(a_2)\right)^{\mu}+\cdots+\left(d(a_\alpha)+d(a_\alpha)\right)^{\mu}\right]+\left(d(a_\alpha)+d(0)\right)^{\mu}+\cdots$$
$$+\left(d(a_\alpha)+d\left(p^{n-1}(p-1)\right)\right)^{\mu}$$

Now, since $d(a_1)=d(a_2)=\cdots d(a_\alpha)=d(a)=p^n-1$

$$d(0)=d(p)=\cdots d(p^{n-1}(p-1))=p^{n-1}(p-1)=\alpha$$

$$H_{\mu}=[(\alpha-1)(2d(a))^{\mu}+p^{n-1}(d(a)+\alpha)^{\mu}]+[(\alpha-2)(2d(a))^{\mu}+p^{n-1}(2d(a))^{\mu}]+\cdots$$
$$+[(\alpha-\alpha)(2d(a))^{\mu}+\rho^{n-1}(d(a)+\alpha)^{\mu}]$$

$$H_{\mu}=(2d(a))^{\mu}\sum_{i=1}^{\alpha}(\alpha-i)+\alpha p^{n-1}(d(a)+\alpha)^{\mu}$$

Since $\sum_{i=1}^{\alpha}(\alpha-i)=\frac{\alpha(\alpha-1)}{2}$

$$H_{\mu}=\alpha\left[\frac{(\alpha-1)}{2}(2d(a))^{\mu}+p^{n-1}(d(a)+\alpha)^{\mu}\right]$$

$$H_{\mu}=p^{n-1}(p-1)\left[\frac{p^{n-1}(p-1)-1}{2}\left(2(p^n-1)\right)^{\mu}+p^{n-1}\left((p^{n-1})+p^{n-1}(p-1)\right)^{\mu}\right]$$

$$=p^{n-1}(p-1)\left[\frac{p^{n-1}(p-1)-1}{2}\left(2(p^n-1)\right)^{\mu}+p^{n-1}(2p^n-p^{n-1}-1)^{\mu}\right]$$

In particular

(1) If $\mu=1\Longrightarrow H_1=M_1=Q_2=p^{n-1}(p-1)\left[\frac{p^{n-1}(p-1)-1}{2}\left(2(p^n-1)\right)+p^{n-1}(2p^n-p^{n-1}-1)\right]$

$$=p^{n-1}(p-1)[p^{n-1}(p-1)(p^n-1)-1+p^{n-1}(2p^n-p^{n-1}-1)]$$

(2) If $\mu=-1\Rightarrow 2H_{-1}=H=\sum_{i\sim j}\frac{2}{d_i+d_j}$

$$=2\left[p^{n-1}(p-1)\left[\frac{(p^{n-1}(p-1)-1)}{4(p^n-1)}+\frac{p^{n-1}}{(2p^n-p^{n-1}-1)}\right]\right]$$

(3) If $\mu={}^{-1}/_{2}\Rightarrow\lambda(Z_{p^n})=H_{-1/2}=\sum_{i\sim j}\frac{1}{\sqrt{d_i+d_j}}$

$$=p^{n-1}(p-1)\left[\frac{p^{n-1}(p-1)-1}{(\sqrt{2})^3\sqrt{p^n-1}}+\frac{p^{n-1}}{\sqrt{(2p^n-p^{n-1}-1)}}\right].$$

Theorem 4.14: If $G_+\left(V(Z_{p^n}),E(Z_{p^n})\right)$, then

$$M_1(Z_{p^n}) = H_1 = \sum_{i\sim j} (d_i + d_j)$$

$$= p^{n-1}(p-1)[p^n(p^n-1) + p^{n-1}(p-1)(p^n + p^{n-1} - 1) - (p^n-1)(p^{n-1}(p-1))]$$

Proof: Suppose that $S = \{a_1, \dots, a_\alpha\}, \alpha = p^{n-1}(p-1), S \subseteq Z_{p^n}, a_i \in Z_{p^n}$, where $\mathcal{O}(a_i) = p^n, \forall\ 1 \le i \le \alpha$.
Since, every a_i adjacent to all vertices belong to $Z_{p^n}, \forall\ 1 \le i \le \alpha$ except to itself, then:

$$H_1 = \sum_{i=1}^{p^n} (d(a_1) + d(i)) - (d(a_1) + d(a_1)) + \sum_{i=1}^{p^n} (d(a_2) + d(i)) - [d(a_2) + d(a_1) + d(a_2) + d(a_2)] + \cdots + \sum_{i=1}^{p^n} (d(a_\alpha) + d(i)) - \left(\sum_{i=1}^{\alpha} d(a_\alpha) + \sum_{i=1}^{\alpha} d(a_i)\right)$$

Now, since $d(a_1) = d(a_2) = \cdots = d(a_\alpha) = d(a) = p^n - 1, \sum_{i=1}^{p^n} d(i) = 2\ell$

$$H_1 = \left[\sum_{i=1}^{p^n} d(a) + \sum_{i=1}^{p^n} d(i) - 2d(a)\right] + \left[\sum_{i=1}^{p^n} d(a) + \sum_{i=1}^{p^n} d(i) - 2(2d(a))\right] + \cdots + \left[\sum_{i=1}^{p^n} d(a) + \sum_{i=1}^{p^n} d_i - 2(\alpha d(a))\right]$$

$$H_1 = [p^n d(a) + 2\ell - 2d(a)] + [p^n d(a) + 2\ell - 2(2d(a))] + \cdots + [p^n(d(a)) + 2\ell - 2(\alpha d(a))]$$

$$H_1 = \alpha[p^n d(a) + 2\ell] - 2d(a)\sum_{i=1}^{\alpha} i$$

$$= \alpha[p^n d(a) + 2\ell] - 2d(a)\frac{\alpha(\alpha+1)}{2}$$

$$= \alpha[p^n d(a) + 2\ell - d(a)(\alpha+1)]$$

$$= p^{n-1}(p-1)[p^n(p^n-1) + p^{n-1}(p-1)(p^n + p^{n-1} - 1) - (p^n-1)(p^{n-1}(p-1))]$$

5. Conclusion

The study uncovers some explicit properties of graphs of a group Z_{p^n} such as (The Eccentric Connectivity index) , First , and second Zagreb indices , Sum-Connectivity index , Randic' index (or Connectivity index) and sum particular special cases indices .

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