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# Extend Nearly Pseudo Semi-2-Absorbing Submodules(I)

***Omar A. Abdullah[a], Haibat K. Mohammadali[b]***

*[a]Department of Mathematics, College of Computer Science and Mathematics, Univ. of Tikrit, Iraq. Email: omer.a.abdullah35383@st.tu.edu.iq*

*[b]Department of Mathematics, College of Computer Science and Mathematics, Univ. of Tikrit, Iraq . Email: H.mohammadali@tu.edu.iq*

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ABSTRACT

In this article, we will present a generalization on the (2-Absorbing, Semi-2-Absorbing, Pseudo-2-Absorbing, Pseudo Semi-2-Absorbing and Nearly-2-Absorbing) submodules. We will study the relationship between this generalization and the basic generalizations studied previously. We have provided many Propositions, Remarks, Examples and characterizations in this article.

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## 1. Introduction

2-Absorbing submodules was introduced in 2011 by Darani and Soheilinia, where a *proper* submodule G of an *R-module M* is called 2-Absorbing submodule if whenever $abh \in$ G for $a, b \in R$ and $h \in M$, then either $ah \in$ G or $bh \in$ G or $ab \in [\text{G}:_R M]$[1], as $[\text{G}:_R M] = \{a \in R: aM \subseteq \text{V}\}$[2]. And the concept of Semi-2-Absorbing submodules was introduce by Innam and Abdulrahman in 2015, where a proper submodule G of an $R$-module $M$ is called Semi-2-Absorbing submodule if whenever $a^2h \in$ G for $a \in R$ and $h \in M$, then either $ah \in$ G or $a^2 \in [\text{G}:_R M]$[3]. These two concepts are generalized in article to Extend Nearly Pseudo Semi-2-Absorbing submodules, but the converse is not true in general see Proposition 2.2 And 2.5. Many generalizations have been studied in previous years on the concept of 2-Absorbing submodules and Semi-2-Absorbing submodules, such as (WN-2-Absorbing, WVS-2-Absorbing, Weakly Semi2-Absorbing, Quasi Primary-2-Absorbing, WES-2-Absorbing, WEQ-2-Absorbing and Nearly Semi-2-Absorbing) submodules, see [4, 5, 6, 7, 8]. Also the following concepts (Pseudo-2-Absorbing, Pseudo Semi-2-Absorbing and Nearly-2-Absorbing ) submodules are generalizations of Extend Nearly Pseudo Semi-2-Absorbing submodules see Propositions 2.8, 2.11 and 2.14. Where a proper submodule G of an $R$-module $M$ is called Pseudo-2-Absorbing submodule if whenever $abh \in$ G for $a, b \in R$ and $h \in M$, then either $ah \in \text{G} + soc(M)$ or $bh \in \text{G} + soc(M)$ or $ab \in [\text{G} + soc(M):_R M]$[9], socal of an $R$-module $M$ defined to be the intersection of all essential submodule of $M$ [10], where anon-zero submodule G of an $R$-module $M$ is called essential in $M$ if $\text{G} \cap E \neq (0)$ for each non-zero submodule $E$ of $M$ [11]. And a proper submodule G of an $R$-module $M$ is called Nearly-2-Absorbing submodule if

*Corresponding author

Email addresses:

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whenever $abh \in G$ for $a, b \in R$ and $h \in M$, then either $ah \in G + J(M)$ or $bh \in G + J(M)$ or $ab \in [G + J(M):_R M]$[12], the Jacobean of an $R$-module $M$ defined to be the intersection of all maximal submodule of $M$ [13], where a submodule G of $R$-module $M$ is called maximal submodule of $M$ if whenever $K$ is a submodule of $M$ with $G \subset K$, then $K = M$ [13]. The Boolean ring play an important role in this paper, if every element $a$ of $R$ is an idempotent, then $R$ is a Boolean ring [14]. Recall that a ring $R$ is a regular ring if every element in $R$ is a regular element, that is for every element $a$ in $R$ there exist element $b$ in $R$ such that $aba = a$[14]. Recall that an $R$-module $M$ is called semi-simple, if every submodule of $M$ is a direct summand [13]". Final Recall that a submodule G of an $R$-module $M$ is called small submodule of $M$ if $G + C = M$, implies that $C = M$ for any proper submodule $C$ of $M$ [14].

## 2. Basic Properties of Extend Nearly Pseudo Semi-2-Absorbing Submodules.

In this paper, we introduced and studied the definition of Extend Nearly Pseudo Semi-2-Absorbing submodule as a new generalization of Extend Nearly Pseudo-2-Absorbing submodules. Also, this is concept is a generalizations of (2-Absorbing, Semi-2-Absorbing, Pseudo-2-Absorbing, Pseudo Semi-2-Absorbing and Nearly-2-Absorbing) submodules. Many basic properties, characterizations of this concept are given in this section.

**Definition 2.1** A proper submodule G of an $R$-module $M$ is said to be Extend Nearly Pseudo Semi-2-Absorbing (for short EXNPS2AB) submodule of $M$ if whenever $\alpha^2 h \in G$, where $\alpha \in R$, $h \in M$ implies that either $\alpha h \in G + soc(M) + J(M)$ or $\alpha^2 M \subseteq G + soc(M) + J(M)$.

And an ideal $I$ of a ring R is called EXNPS2AB ideal of R, if $I$ is an EXNPS2AB $R$-submodule of an $R$-module $R$.

**Proposition 2.2** Every 2-Absorbing submodule of an $R$-module $M$ is EXNPS2AB submodule of $M$.

**Proof** Let G be a 2-Absorbing submodule of an $R$-module $M$ and $\alpha^2 h \in G$, for $\alpha \in R, h \in M$, that is $\alpha^2 h = \alpha\alpha h \in G$ Since G is 2-Absorbing submodule of $M$, then either $\alpha h \in G \subseteq G + soc(M) + J(M)$ or $\alpha\alpha M \subseteq G \subseteq G + soc(M) + J(M)$.That is either $\alpha h \in G + soc(M) + J(M)$ or $\alpha^2 M \subseteq G + soc(M) + J(M)$. Hence G is EXNPS2AB submodule of $M$.

**Remark 2.3** Show the following example to see why the opposite of Proposition 2.2 is not always true.

**Example 2.4** Let $M = Z_{12}$, $R = Z$ and the submodule $G = \langle\bar{0}\rangle$ is EXNPS2AB submodule of $M$, since $soc(Z_{12}) = \langle\bar{2}\rangle$ and $J(Z_{12}) = \langle\bar{6}\rangle$. That is for all $\alpha \in Z$ and $m \in Z_{12}$ such that $\alpha^2 m \in \langle\bar{0}\rangle$, implies that either $\alpha m \in \langle\bar{0}\rangle + soc(Z_{12}) + J(Z_{12}) = \langle\bar{0}\rangle + \langle\bar{2}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ or $\alpha^2 \in [\langle\bar{0}\rangle + soc(Z_{12}) + J(Z_{12}):_Z Z_{12}] = 2$. That is $2^2.\bar{3} \in \langle\bar{0}\rangle$, implies that $2.\bar{3} = \bar{6} \in \langle\bar{2}\rangle$ and $2^2 = 4 \in [\langle\bar{0}\rangle + soc(Z_{12}) + J(Z_{12})_R: Z_{12}] = 2Z$. But G is not 2-Absorbing submodule of $Z_{12}$ since $2.3.\bar{2} \in G$, for $2,3 \in R$, $\bar{2} \in W$, but $2.\bar{2} = \bar{4} \notin G$ and $3.\bar{2} = \bar{6} \notin G$ and $2.3 = 6 \notin [G:_R M] = 12Z$.

**Proposition 2.5** Every Semi-2-Absorbing submodule of an $R$-module $M$ is EXNPS2AB submodule of $M$.

**Proof** Direct.

**Remark 2.6** Show the following example to see why the opposite of Proposition 2.5 is not always true.

**Example 2.7** Let $M = Z_{48}$, $R = Z$ and the submodule $G = \langle\overline{24}\rangle$ is EXNPS2AB submodule of $M$, since $soc(Z_{48}) = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{6}\rangle$. That is for all $\alpha \in Z$ and $m \in Z_{48}$ such that $\alpha^2 m \in \langle\overline{24}\rangle$, implies that either $\alpha m \in \langle\overline{24}\rangle + soc(Z_{48}) + J(Z_{48}) = \langle\overline{24}\rangle + \langle\bar{8}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ or $\alpha^2 \in [\langle\overline{24}\rangle + soc(Z_{48}) + J(Z_{48}):_Z Z_{48}] = 2Z$. That is $2^2.\bar{6} \in \langle\overline{24}\rangle$, implies that $2.\bar{6} = \overline{12} \in \langle\bar{2}\rangle$ and $2^2 = 4 \in [\langle\overline{24}\rangle + soc(Z_{48}) + J(Z_{48})_R: Z_{48}] = 2Z$. But $G = \langle\overline{24}\rangle$ is not Semi-2-Absorbing since $2^2.\bar{6} \in \langle\overline{24}\rangle$ for $2 \in Z$ and $\bar{6} \in Z_{48}$, implies that $2.\bar{6} = \overline{12} \notin \langle\overline{24}\rangle$ and $2^2 = 4 \notin [\langle\overline{24}\rangle:_Z Z_{48}] = 24Z$.

**Proposition 2.8** Every Pseudo-2-Absorbing submodule of an $R$-module $M$ is EXNPS2AB submodule of $M$.

**Proof** Let G be a Pseudo-2-Absorbing submodule of an $R$-module $M$ and $\alpha^2 h \in G$, for $\alpha \in R, h \in M$, that is $\alpha^2 h = \alpha\alpha h \in G$. Since G is Pseudo-2-Absorbing submodule of $M$, then either $\alpha h \in G + soc(M) \subseteq G + soc(M) + J(M)$ or $\alpha\alpha M \subseteq G + soc(M) \subseteq G + soc(M) + J(M$. That is either $\alpha h \in G + soc(M) + J(M)$ or $\alpha^2 M \subseteq G + soc(M) + J(M)$. Hence G is EXNPS2AB submodule of $M$.

**Remark 2.9** Show the following example to see why the opposite of Proposition 2.8 is not always true.

**Example 2.10** Let $M = Z_{48}$, $\mathrm{R} = Z$ and the submodule $\mathrm{G} = \langle \bar{8} \rangle$. It's clear that G is EXNPS2AB submodule of $M$, but G is not Pseudo-2-Absorbing submodule of $Z_{48}$, since $2.2.\bar{2} \in \langle \bar{8} \rangle$, for $2 \in Z$ and $\bar{2} \in Z_{48}$, implies that $2.\bar{2} = \bar{4} \notin \mathrm{G} + soc(Z_{48}) = \langle \bar{8} \rangle + \langle \bar{8} \rangle = \langle \bar{8} \rangle$ and $2.2 = 4 \notin [\langle \bar{8} \rangle + \langle \bar{8} \rangle :_{\mathrm{R}} Z_{48}] = 8Z$.

**Proposition 2.11** Every Pseudo Semi-2-Absorbing submodule of an $R$-module $M$ is EXNPS2AB submodule of $M$.

**Proof** Clear.

**Remark 2.12** Show the following example to see why the opposite of Proposition 2.10 is not always true.

**Example 2.13** Same example 2.10.

**Proposition 2.14** Every Nearly-2-Absorbing submodule of an $R$-module $M$ is EXNPS2AB submodule of $M$.

**Proof** Let G be a Nearly-2-Absorbing submodule of an $R$-module $M$ and $\alpha^2 h \in \mathrm{G}$, for $\alpha \in R, h \in M$, that is $\alpha^2 h = \alpha\alpha h \in \mathrm{G}$ Since G is Nearly-2-Absorbing submodule of $M$, then either $\alpha h \in \mathrm{G} + J(M) \subseteq \mathrm{G} + soc(M) + J(M)$ or $\alpha\alpha M \subseteq \mathrm{G} + J(M) \subseteq \mathrm{G} + soc(M) + J(M)$.That is either $\alpha h \in \mathrm{G} + soc(M) + J(M)$ or $\alpha^2 M \subseteq \mathrm{G} + soc(M) + J(M)$. Hence G is EXNPS2AB submodule of $M$.

**Remark 2.15** Show the following example to see why the opposite of Proposition 2.14 is not always true.

**Example 2.16** The submodule $\mathrm{G} = \langle \overline{30} \rangle$ in $Z_{60}$ and $R = Z$ is EXNPS2AB submodule of $M$, but G is not Nearly-2-Absorbing submodule of $Z_{60}$, since $2.3.\bar{5} \in \langle \overline{30} \rangle$, for $2,3 \in Z$ and $\bar{5} \in Z_{60}$, implies that $2.\bar{5} = \overline{10} \notin \mathrm{G} + J(Z_{60}) = \langle \overline{30} \rangle + \langle \overline{30} \rangle = \langle \overline{30} \rangle$ and $3.\bar{5} = \overline{15} \notin \langle \overline{30} \rangle$ and $2.3 = 6 \notin [\langle \overline{30} \rangle + \langle \overline{30} \rangle :_R Z_{48}] = 30Z$.

**Remark 2.17** It is not necessary for the intersection of two EXNPS2AB submodules of a module to be so.

**Example 2.18** The submodules $3Z$ and $4Z$ of the $Z$-module $Z$ are EXNPS2AB, but $3Z \cap 4Z = 12Z$ is not EXNPS2AB submodule of $Z$, because $2^2.3 \in 12Z$ for $2,3 \in Z$, hence $2.3 = 6 \notin 12Z + soc(Z) + J(Z) = 12Z$ and $2^2 = 4 \notin [12Z + soc(Z) + J(Z) :_Z Z] = 12Z$.

Now, we will give the most important results related to the concept of EXNPS2AB submodules.

**Proposition 2.19** A proper submodule G of an $R$-*module* $M$ is EXNPS2AB submodule of $M$ if and only if for any $\alpha \in R$ such that $\alpha^2 \notin [\mathrm{G} + soc\,(M) + J(M) :_R M]$, then $[\mathrm{G} :_M \alpha^2] \subseteq [\mathrm{G} + soc\,(M) + J(M) :_M \alpha]$.

**Proof (⇒)** Suppose that G is EXNPS2AB submodule of $M$ and let $e \in [\mathrm{G} :_M \alpha^2]$, then $\alpha^2 e \in \mathrm{G}$. Since G is EXNPS2AB submodule of $M$ and $\alpha^2 \notin [\mathrm{G} + soc\,(M) + J(M) :_R M]$, it follows that $\alpha e \in \mathrm{G} + soc\,(M) + J(M)$. Thus $e \in [\mathrm{G} + soc(M) + J(M) :_M \alpha]$. Therefore $[\mathrm{G} :_M \alpha^2] \subseteq [\mathrm{G} + soc(M) + J(M) :_M \alpha]$.

**(⇐)** Let $\alpha^2 e \in \mathrm{G}$ for $\alpha \in R$, $e \in M$ and let $\alpha^2 \notin [\mathrm{G} + soc(M) + J(M) :_R M]$. But $e \in [\mathrm{G} :_M \alpha^2] \subseteq [\mathrm{G} + soc\,(M) + J(M) :_M \alpha]$. It follows that $e \in [\mathrm{G} + soc\,(M) + J(M) :_M \alpha]$, that is $\alpha e \in \mathrm{G} + soc\,(M) + J(M)$. Hence G EXNPS2AB submodule of $M$.

**Proposition 2.20** Let G be *a proper* submodule of an $R$-module $M$. Then G is EXNPS2AB submodule of $M$ if and only if $I^2 L \subseteq \mathrm{G}$ for $I$ is an ideal of $R$ and $L$ is a submodule of $M$, implies that either $IL \subseteq \mathrm{G} + soc(M) + J(M)$ or $I^2 \subseteq [\mathrm{G} + soc(M) + J(M) :_R M]$.

**Proof (⇒)** Let $I^2 L \subseteq \mathrm{G}$ for $I$ is an ideal of $R$ and $L$ is a submodule of $M$, with $I^2 \nsubseteq [\mathrm{G} + soc(M) + J(M) :_R M]$ and G is EXNPS2AB submodule of $M$. To prove that $IL \subseteq \mathrm{G} + soc(M) + J(M)$. Let $x \in IL$, implies that $x = r_1 x_1 + r_2 x_2 + \cdots + r_k x_k$ for $r_i \in I$ and $x_i \in L$, $i = 1,2, \dots, k$, it follows that ${r_i}^2 x_i \in I^2 L \subseteq \mathrm{G}$. That is ${r_i}^2 x_i \in \mathrm{G}$. But G is EXNPS2AB submodule of an $R$-*module* $M$, then $r_i x_i \in \mathrm{G} + soc(M) + J(M)$ and ${r_i}^2 \notin [\mathrm{G} + soc(M) + J(M) :_R M]$ for each $i = 1,2,3, \dots, k$, thus $r_1 x_1 + r_2 x_2 + \cdots + r_k x_k \in \mathrm{G} + soc(M) + J(M)$, that is $x \in \mathrm{G} + soc(M) + J(M)$. Hence $IL \subseteq \mathrm{G} + soc(M) + J(M)$.

**(⇐)** Suppose that $r^2 y \in \mathrm{G}$ for $r \in R$ and $y \in M$, implies that $\langle r^2 \rangle \langle x \rangle \subseteq \mathrm{G}$. Thus by our assumption we have either $\langle r \rangle \langle x \rangle \subseteq \mathrm{G} + soc(M) + J(M)$ or $\langle r^2 \rangle \subseteq [\mathrm{G} + soc(M) + J(M) :_R M]$. That is $rx \in \langle r \rangle \langle x \rangle \subseteq \mathrm{G} + soc(M) + J(M)$ or $r^2 \in \langle r^2 \rangle \subseteq [\mathrm{G} + soc(M) + J(M) :_R M]$, that is either $rx \in \mathrm{G} + soc(M) + J(M)$ or $r^2 \in [\mathrm{G} + soc(M) + J(M) :_R M]$. Hence G is EXNPS2AB submodule of $M$.

As a direct consequence of Proposition 2.20 we get the following corollaries.

**Corollary 2.21** Let G be a proper submodule of an $R$-module $M$. Then G is EXNPS2AB submodule of $M$ if and only if $I^2M \subseteq$ G for $I$ is an ideal of $R$, implies that either $IM \subseteq$ G $+ soc(M) + J(M)$ or $I^2 \subseteq [$G$ + soc(M) + J(M):_R M]$.

**Corollary 2.22** Let G be a proper submodule of an $R$-module $M$. Then G is EXNPS2AB submodule of $M$ if and only if $I^2y \subseteq$ G for $I$ is an ideal of $R$ and $y \in M$, implies that either $Iy \subseteq$ G $+ soc(M) + J(M)$ or $I^2 \subseteq [$G$ + soc(M) + J(M):_R M]$.

**Proposition 2.23** A proper submodule G of an $R$-module $M$ is EXNPS2AB submodule of $M$ if and only if $[$G$:_M r^2] \subseteq [$G$+soc(M) + J(M):_M r]$ for $r \in R$ such that $r^2 \notin [$G$+soc(M) + J(M):_R M]$.

**Proof (⇒)** Let $y \in [$G$:_M r^2]$, then $yr^2 \in$ G for $r \in R$ and $y \in M$. Since G is EXNPS2AB submodule of $M$ and $r^2 \notin [$G$+soc(M) + J(M):_R M]$, then $ry \in$ G $+ soc(M) + J(M)$. Hence $y \in [$G$+soc(M) + J(M):_M r]$. That is $[$G$:_M r^2] \subseteq [$G$+soc(M) + J(M):_M r]$.

**(⇐)** Let $r^2y \in$ G for $r \in R$, $y \in M$, implies that $y \in [$G$:_M r^2] \subseteq [$G$+soc(M) + J(M):_M r]$, hence $y \in [$G$+soc(M) + J(M):_M r]$, that is $ry \in$ G $+ soc(M) + J(M)$. Therefore G is EXNPS2AB submodule of $M$.

As a direct consequence of Proposition 2.23 we get the following corollaries.

**Corollary 2.24** Let G be a proper submodule of an $R$-module $M$. Then G is EXNPS2AB submodule of $M$ if and only if $r^2L \subseteq$ G for $r \in R$, $L$ is a submodule of $M$, implies that either $rL \subseteq$ G $+ soc(M) + J(M)$ or $r^2 \in [$G$ + soc(M) + J(M):_R M]$.

**Corollary 2.25** Let G be a proper submodule of an $R$-module $M$. Then G is EXNPS2AB submodule of $M$ if and only if $r^2M \subseteq$ G for $r \in R$ and $L$ is a submodule of $M$, implies that either $rM \subseteq$ G $+ soc(M) + J(M)$ or $r^2 \in [$G$ + soc(M) + J(M):_R M]$.

**Proposition 2.26** Let $M$ be an $R$-module and G be a proper submodule of $M$. Then G $+ soc(M) + J(M)$ is EXNPS2AB submodule of $M$ if and only if $[$G$ + soc(M) + J(M):_R r^2y] = [$G$ + soc(M) + J(M):_R ry]$ for each $y \in M$ or $r^2 \in [E + soc(M) + J(M):_R M]$.

**Proof (⇒)** Assume that $r^2 \notin [E + soc(M) + J(M):_R M]$ and let $a \in [E + soc(M) + J(M):_R r^2y]$, then $r^2ay \in E + soc(M) + J(M)$. But $E + soc(M) + J(M)$ is EXNPS2AB submodule of $M$ and $r^2 \notin [E + soc(M) + J(M):_R M]$, then $ray \in (E + soc(M) + J(M)) + soc(M) + J(M) = E + soc(M) + J(M)$. That is $a \in [E + soc(M) + J(M):_R ry]$. Thus $[E + soc(M)+J(M):_R r^2y] \subseteq [E + soc(M)+J(M):_R ry]$. It is clear that $[E + soc(M)+J(M):_R ry] \subseteq [E + soc(M) + J(M):_R r^2y]$, hence $[E + soc(M) + J(M):_R r^2y] = [E + soc(M)+J(M):_R ry]$.

**(⇐)** Let $r^2y \in E + soc(M) + J(M)$, for $r \in R$, $y \in M$, then by hypothesis $[E + soc(M) + J(M):_R r^2y] = [E + soc(M) + J(M):_R ry]$ or $r^2 \in [E + soc(M) + J(M):_R M]$. If $[E + soc(M) + J(M):_R r^2y] = [E + soc(M) + J(M):_R ry]$ and $r^2y \in E + soc(M) + J(M)$ then $[E + soc(M) + J(M):_R r^2y] = R$, it follows that $[E + soc(M)+J(M):_R ry] = R$, hence $ry \in E + soc(M) + J(M) \subseteq E + soc(M) + J(M) + soc(M) + J(M)$, so $ry \in E + soc(M) + J(M) + soc(M) + J(M)$ or $r^2M \subseteq E + soc(M) + J(M) + soc(M) + J(M)$. That is $E + soc(M) + J(M)$ is EXNPS2AB submodule of $M$.

Now, we need to recall the following lemma.

**Lemma 2.27 [16, EX.(12)P.239]**

**1)** Let G is submodule of an $R$-module $M$ with G is a direct summand of $M$, then $J(\frac{M}{G}) = \frac{J(M)+G}{G}$.

**2)** An $R$-module $M$ is a semi-simple if and only if for each submodule G of $M$ $soc(\frac{M}{G}) = \frac{soc(M)+G}{G}$.

**Proposition 2.28** Let G be EXNPS2AB submodule of an $R$-module $M$ and L is a submodule of $M$ with L⊆ G , then $\frac{G}{L}$ is EXNPS2AB submodule of an $R$-module $\frac{M}{L}$.

**Proof** Let G be EXNPS2AB submodule of $M$ and $\alpha^2(e + L) = \alpha^2 e + L \in G/L$ for $\alpha \in R$ and $e + L \in M/L$, $e \in M$, implies that $\alpha^2 e \in$ G. Since G is EXNPS2AB submodule of $M$, then either $\alpha e \in$ G $+ soc(M) + J(M)$ or $\alpha^2 M \subseteq$ G $+ soc(M) + J(M)$. Hence either $\alpha(e + L) \in$ G $+ soc(M) + J(M)/L$ or $\alpha^2 M/L \subseteq$ G $+ soc(M) + J(M)/L$, then either $\alpha(e + L) \in G/L + G + soc(M)/L + G + J(M)/L \subseteq G/L + soc(M/L) + J(M/L)$ or $\alpha^2 M/L \subseteq G/L + G + soc(M)/L + G + J(M)/L \subseteq G/L + soc(M/L) + J(M/L)$. Hence G/L is EXNPS2AB submodule of $M/L$.

**Proposition 2.29** Let $M$ is a semi simple $R$-module, G and $K$ are submdules for $M$ such that $K \subseteq$ G and G is a proper submodule of $M$. If $K$ and G$/K$ are EXNPS2AB submodules of $M$ and $M/K$ respectively, then G is EXNPS2AB submodule of $M$.

**Proof** Suppose $K$ and G$/K$ are EXNPS2AB submodules for $M$ and $M/K$ respectively, and let $I^2 u \subseteq$ G, for $I$ is an ideals of $R$ and $u \in M$. So $I^2(u+K) = I^2 u + K \subseteq$ G$/K$. If $I^2 u \subseteq K$ and $K$ is EXNPS2AB submodules of $M$, implies that by Corollary 2.22 either $Iu \subseteq K + (soc(M) + J(M)) \subseteq$ G $+ (soc(M) + J(M))$ or $I^2 M \subseteq K + (soc(M) + J(M)) \subseteq$ G $+ (soc(M) + J(M))$, hence G is EXNPS2AB submodules for $M$. Now, we may assume that $I^2 u \nsubseteq K$. It follows that $I^2(u+K) \subseteq$ G$/K$ , but G$/K$ is EXNPS2AB submodules of $M/K$ , again by Corollary 2.24 either $I(u+K) \subseteq$ G$/K + \big(soc(M/K)\big) + \big(J(M/K)\big)$ or $I^2 M/K \subseteq$ G$/K + \big(soc(M/K)\big) + \big(J(M/K)\big)$. Since $M$ is a semi simple then by Lemma 2.27 either $I(u+K) \subseteq$ G$/K + (soc(M)+K/K) + (J(M)+K/K)$ or $I^2 M/K \subseteq$ G$/K + (soc(M)+K/K) + (J(M)+K/K)$. But $K \subseteq$ G, it follows that $K + soc(M) \subseteq$ G $+ soc(M)$ and $K + J(M) \subseteq$ G $+ J(M)$, hence G$/K + (soc(M)+K/K) + (J(M)+K/K) \subseteq$ G$/K + (soc(M)+$ G$/K) + (J(M)+$ G$/K) =$ G $+ soc(M) + J(M)/K$ , thus we have either $I(u+K) \subseteq$ G $+ soc(M) + J(M)/K$ or $I^2 M/K \subseteq$ G $+ soc(M) + J(M)/K$it follows that either $Iu \subseteq$ G $+ (soc(M) + J(M))$ or $I^2 M \subseteq$ G $+ (soc(M) + J(M))$. Hence by Corollary 2.22 G is EXNPS2AB submodules of $M$.

Under the certain condition the intersection of two EXNPS2AB submodules is EXNPS2AB submodule.

**Lemma 2.30 [13, lemma (2.3.15)]** Let $A, B$ and $C$ are submodules of an $R$-*module* $M$ with$B \subseteq C$, then$(A+B) \cap C = (A \cap C) + B = (A \cap C) + (B \cap C)$.

**Lemma 2.31 [17, EX (12.5). p. 242]** A submodule G of an $R$-*module* $M$ is maximal and essential if and only if $soc(M) \subseteq$ G.

**Proposition 2.32** Let $M$ be an $R$-module either $E$ or G is maximal essential submodule of $M$ and $E$ not contained in G. If $E$ and G are EXNPS2AB submodules of $M$, then G $\cap E$ is EXNPS2AB submodule of $M$.

**Proof** Clear that G $\cap E$ is a proper submodule of $M$. Now, let $I^2 L \subseteq E \cap$ G, for some ideal $I$ of $R$ and $L$ is a submodule of $M$ it follows' that $I^2 L \subseteq E$ and $I^2 L \subseteq$ G. But both $E$ and G are EXNPS2AB submodules of $M$, then by Proposition 2.20 we have either $IL \subseteq E + soc(M) + J(M)$ or $I^2 M \subseteq E + soc(M) + J(M)$ and $IL \subseteq$ G $+ soc(M) + J(M)$ or $I^2 M \subseteq$ G $+ soc(M) + J(M)$ . Thus either $IL \subseteq (E + soc(M) + J(M)) \cap ($G $+ soc(M) + J(M))$ or $I^2 M \subseteq (E + soc(M) + J(M)) \cap ($G $+ soc(M) + J(M))$. Since either $E$ or G is maximal essential submodule of $M$, then either $soc(M) \subseteq E$ or $soc(M) \subseteq$ G. Suppose that G is maximal essential submodule of $M$, so that by Lemma 2.31 $soc(M) \subseteq$ G and since G is maximal submodule of $M$, then $J(M) \subseteq$ G. It follows that G $+ soc(M) + J(M) =$ G. Hence either $IL \subseteq (E + soc(M) + J(M)) \cap$ G or $I^2 M \subseteq (E + soc(M) + J(M)) \cap$ G. Therefore by modular law we get either $IL \subseteq (E \cap$ G$) + (soc(M) + J(M))$ or $I^2 \subseteq \big[(E \cap$ G$) + \big(soc(M) + J(M)\big) :_R M\big]$. Hence by Proposition 2.20 G $\cap E$ is EXNPS2AB submodule of $M$.

**Lemma 2.33 [13, Theo. (9.1.4) (a)]** Let $\emptyset: M \longrightarrow$ G be an $R$-homomorphism, then $\emptyset(soc(M))$ is a submodule of $soc$(G) (That is $\emptyset(soc(M)) \subseteq soc$(G)). And $\emptyset(J(M)) \subseteq J$(G)).

**Lemma 2.34 [13, Coro. (9.1.5) (a)]** If $\varphi: M \longrightarrow \bar{M}$ be an $R$-epimorphism and $Ker\varphi$ is small submodule of $M$, then $\varphi\big(J(M)\big) = J(\bar{M})$ and $\varphi^{-1}\big(J(\bar{M})\big) = J(M)$.

**Proposition 2.35** Let $\varphi: M \to \bar{M}$ be an $R$-epimorphism with ker $(\varphi)$ is a small submodule of $M$ and G be EXNPS2AB submodule of $\bar{M}$, then $\varphi^{-1}$(G) is EXNPS2AB submodule of $M$.

**Proof** Since $\varphi$ onto, then $\varphi^{-1}$(G) is a proper submodule of $M$, if not, we have $\varphi^{-1}$(G) $= M$, implies thatG $= \varphi(M) = \bar{M}$ contradiction. Let $\alpha^2 x \in \varphi^{-1}$(G) , for $\alpha \in R$, $x \in M$, implies that $\alpha^2\varphi(x) \in$ G, but G is EXNPS2AB submodule of $\bar{M}$, implies that either $\alpha\varphi(x) \in$ G $+ soc(\bar{M}) + J(\bar{M})$ or $\alpha^2 \bar{M} \subseteq$ G $+ soc(\bar{M}) + J(\bar{M})$, it follows that $\alpha x \in \varphi^{-1}$(G) $+ \varphi^{-1}\big(soc(\bar{M})\big) + \varphi^{-1}\big(J(\bar{M})\big) \subseteq \varphi^{-1}$(G) $+ soc(M) + J(M)$ or $\alpha^2 M \subseteq \varphi^{-1}$(G) $+ \varphi^{-1}\big(soc(\bar{M})\big) + \varphi^{-1}\big(J(\bar{M})\big) \subseteq \varphi^{-1}$(G) $+ soc(M) + J(M)$. Hence either $\alpha x \in \varphi^{-1}$(G) $+ soc(M) + J(M)$ or $\alpha^2 M \subseteq \varphi^{-1}$(G) $+ soc(M) + J(M)$. Therefore $\varphi^{-1}$(G) is EXNPS2AB submodule of $M$.

**Proposition 2.36** Let $\varphi: M \to \bar{M}$ be an $R$-epimorphism, G is a *proper* submodule of $M$ and ker$(\varphi)$ is a small submodule of $M$ with ker$(\varphi) \subseteq$ G. Then G is EXNPS2AB submodule of $M$ if and only if $\varphi$(G) is EXNPS2AB submodule of $\bar{M}$.

**Proof (⇒)** Let $a^2 \bar{x} \in \varphi$(G), for $a \in R$ and $x \in \bar{M}$. Since $\varphi$ is onto, then $\bar{x} = \varphi(m)$, for some $x \in M$, that is $a^2\varphi(x) \in \varphi$(G), implies that $a^2\varphi(x) = \varphi(y)$ for some $y \in$ G, then $\varphi(a^2 x - y) = 0$, it follows that $a^2 x - y \in$ ker$(\varphi) \subseteq$ G , then

$a^2x \in$ G. Since G is EXNPS2AB submodule of $M$, then either $ax \in \mathrm{G} + soc(M) + J(M)$ or $a^2M \subseteq \mathrm{G} + soc(M) + J(M)$. Thus we have either $\varphi(\alpha x) \in \varphi(\mathrm{G}) + \varphi\big(soc(M)\big) + \varphi\big(J(M)\big) \subseteq \varphi(\mathrm{G}) + soc(\overline{M}) + J(\overline{M}$ or $\varphi(a^2M) = a^2\varphi(M) = a^2\overline{M} \subseteq \varphi(\mathrm{G}) + \varphi\big(soc(M)\big) + \varphi\big(J(M)\big) \subseteq \varphi(\mathrm{G}) + soc(\overline{M}) + J(\overline{M})$. Hence $\varphi(\mathrm{G})$ is EXNPS2AB submodule of $\overline{M}$.

**(⇐)** Suppose that $\alpha^2h \in$ G, for $\alpha \in R, h \in M$ so $\varphi(\alpha^2h) \in \varphi(\mathrm{G})$, that is $\alpha^2\varphi(h) \in \varphi(\mathrm{G})$. But $\varphi(\mathrm{G})$ is EXNPS2AB submodule of $\overline{M}$, implies that either $\alpha\varphi(h) \in \varphi(\mathrm{G}) + soc(\overline{M}) + J(\overline{M})$ or $\alpha^2\overline{M} \subseteq \varphi(\mathrm{G}) + soc(\overline{M}) + J(\overline{M})$. If $\alpha\varphi(h) \in \varphi(\mathrm{G}) + soc(\overline{M}) + J(\overline{M})$, then $\varphi(\alpha h) \in \varphi(\mathrm{G}) + soc(\overline{M}) + J(\overline{M})$, since $\ker(\varphi)$ is a small submodule of $M$, then by Lemma 2.33 and Lemma 2.34 $\alpha h \in \varphi^{-1}\big(\varphi(\mathrm{G})\big) + \varphi^{-1}\big(soc(\overline{M})\big) + \varphi^{-1}\big(J(\overline{M})\big) \subseteq \mathrm{G} + soc(M) + J(M)$, that is $\alpha h \in \mathrm{G} + soc(M) + J(M)$. If $\alpha^2\overline{M} \subseteq \varphi(\mathrm{G}) + soc(\overline{M}) + J(\overline{M})$, then $\varphi(\alpha^2M) \subseteq \varphi(\mathrm{G}) + soc(\overline{M}) + J(\overline{M})$, It follows that $\alpha^2M \subseteq \varphi^{-1}\big(\varphi(\mathrm{G})\big) + \varphi^{-1}\big(soc(\overline{M})\big) + \varphi^{-1}\big(J(\overline{M})\big) \subseteq \mathrm{G} + soc(M) + J(M)$, that is $\alpha^2M \subseteq \mathrm{G} + soc(M) + J(M)$. Hence G is EXNPS2AB submodule of $M$.

**Proposition 2.37** Let $M$ be an $R$-module with $soc(M)$ is Semi-2-Absorbing submodule of $M$. If $\mathrm{G} \subset M$ such that $\mathrm{G} \subseteq soc(M)$, then G is EXNPS2AB submodule of $M$.

**Proof** Let $r^2M \subseteq$ G for $r \in R$. Since $\mathrm{G} \subseteq soc\,(M)$, it follows that $r^2M \subseteq soc\,(M)$. But $soc(M)$ is Semi-2-Absorbing subomdule of $M$, then either $rM \subseteq soc\,(M) \subseteq \mathrm{G} + soc\,(M) + J(M)$ or $r^2M \subseteq soc\,(M) \subseteq \mathrm{G} + soc\,(M) + J(M)$. That is either $rM \subseteq \mathrm{G} + soc(M) + J(M)$ or $r^2 \in [\,\mathrm{G} + soc\,(M) + J(M):_R M]$. Therefore by Corollary 2.25 G is EXNPS2AB submodule of $M$.

**Proposition 2.38** Let $M$ be an $R$-module with $J(M)$ is Semi-2-Absorbing submodule of $M$. If $\mathrm{G} \subset M$ such that $\mathrm{G} \subseteq J(M)$, then G is EXNPS2AB submodule of $M$.

**Proof** Let $r^2L \subseteq$ G for $r \in R$ and $L$ is a submodule of $M$. Since $\mathrm{G} \subseteq J(M)$, it follows that $r^2L \subseteq J(M)$. But $J(M)$ is Semi-2-Absorbing subomdule of $M$, then either $rL \subseteq J(M) \subseteq \mathrm{G} + soc\,(M) + J(M)$ or $r^2M \subseteq J(M) \subseteq \mathrm{G} + soc\,(M) + J(M)$. That is either $rL \subseteq \mathrm{G} + soc(M) + J(M)$ or $r^2 \in [\mathrm{G} + soc(M) + J(M):_R M]$. Hence by Corollary 2.24 G is EXNPS2AB submodule of $M$.

## 3. The Relationship between the Extend Nearly Pseudo Semi-2-Absorbing Submodules and Other Concepts.

In this part of this search we introduced the relationships between the concept of Extend Nearly Pseudo Semi-2-Absorbing submodules and (2-Absorbing, Semi-2-Absorbing, Nearly-2-Absorbing, Nearly Semi-2-Absorbing, Pseudo-2-Absorbing and Pseudo Semi-2-Absorbing) submodules with all these concepts being equal.

**Lemma 3.1 [15, Theo. (2.2)]** If $R$ is a Boolean ring, then $R$ is a regular ring.

It is well known if $G$ is regular then $J(G) = 0$ **[13]**.

**Proposition 3.2** Let $M$ be a an $R$-module over a Boolean ring $R$ and $\mathrm{G} \subset M$ with $soc(M) \subseteq$ G. Then G is 2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proof (⇒)** By Proposition 2.2.

**(⇐)** Let $\alpha bx \in$ G for $\alpha, b \in R, x \in M$, since $R$ is a Boolean ring, then $\alpha bx = (ab)^2x \in$ G with $(ab)^2 \notin [\mathrm{G} + soc(M) + J(M):_R M]$ and $bx \notin \mathrm{G} + soc(M) + J(M)$. But G is EXNPS2AB submodule of $M$ and $(ab)^2 \notin [\mathrm{G} + soc(M) + J(M):_R M]$, then $abx \in \mathrm{G} + soc(M) + J(M)$. Now, since $R$ is a Boolean ring, then by Lemma 3.1 $R$ is a regular ring, that is $a^2b = a$, hence $abx = a^2bx = ax \in \mathrm{G} + soc(M) + J(M)$. Since $R$ is a regular ring, then $J(G) = 0$. Also $soc(M) \subseteq$ G, that is $soc(M) + \mathrm{G} = \mathrm{G}$, then $ax \in$ G. Since $ab = (ab)^2 \notin [\mathrm{G} + soc(M) + J(M):_R M]$, then $ab \notin [\mathrm{G}:_R M]$ and $bx \notin \mathrm{G} + soc(M) + J(M)$, then $bx \notin$ G. Hence G is 2-Absorbing of $M$.

**Proposition 3.3** Let $M$ be a an $R$-module over a Boolean ring $R$ and G is an essential submodule of $M$. Then G is 2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proof (⇒)** Clear.

**(⇐)** Let $\alpha bx \in$ G for $\alpha, b \in R, x \in M$, since $R$ is a Boolean ring, then $\alpha bx = (ab)^2x \in$ G with $(ab)^2 \notin [\mathrm{G} + soc(M) + J(M):_R M]$ and $bx \notin \mathrm{G} + soc(M) + J(M)$. But G is EXNPS2AB submodule of $M$ and $(ab)^2 \notin [\mathrm{G} + soc(M) + J(M):_R M]$, then $abx \in \mathrm{G} + soc(M) + J(M)$. Now, since $R$ is a Boolean ring, then by Lemma 3.1 $R$ is a regular ring, that is $a^2b = a$, hence $abx = a^2bx = ax \in \mathrm{G} + soc(M) + J(M)$. Since $R$ is a regular ring, then $J(G) = 0$. Also G is an *essential* submodule of $M$, then $soc(M) \subseteq \mathrm{G}$, that is $soc(M) + \mathrm{G} = \mathrm{G}$, hence $ax \in \mathrm{G}$. Since $ab = (ab)^2 \notin [\mathrm{G} + soc(M) + J(M):_R M]$, then $ab \notin [\mathrm{G}:_R M]$ and $bx \notin \mathrm{G} + soc(M) + J(M)$, then $bx \notin \mathrm{G}$. Hence G is 2-Absorbing of $M$.

The following Corollaries are direct consequence of Proposition 3.2 and Proposition 3.3.

**Corollary 3.4** Let $M$ be an $R$-module over a Boolean ring $R$ and $\mathrm{G} \subset M$ with $soc(M) + J(M) \subseteq \mathrm{G}$. Then G is 2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Corollary 3.5** Let $M$ be a an $R$-module over a Boolean ring $R$ and $\mathrm{G} \subset M$ with $soc(M) + J(M) = 0$. Then G is 2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Corollary 3.6** Let $M$ be a an $R$-module over a Boolean ring $R$ and $soc(M) = \mathrm{G}$. Then G is 2-Absorbing if and only if G is EXNPS2AB submodule of $M$.

**Corollary 3.7** Let $M$ be a an $R$-module over a Boolean ring $R$ and $soc(M) = 0$. Then G is 2-Absorbing if and only if G is EXNPS2AB submodule of $M$.

**Proposition 3.8** Let $M$ be $R$-module and $\mathrm{G} \subset M$ with $J(M/\mathrm{G}) = (0)$ and $soc(M) \subseteq \mathrm{G}$. Then G is Semi-2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proof (⇒)** By Proposition 2.5.

**(⇐)** Since $J(M/\mathrm{G}) = (0)$, then by [5, Theo. (9.1.4)(b)] we get $J(M) \subseteq \mathrm{G}$. Let $r^2m \in \mathrm{G}$ for $r \in R, m \in M$. Since G is EXNPS2AB, then either $rm \in \mathrm{G} + soc(M) + J(M)$ or $r^2 \in [\mathrm{G} + soc(M) + J(M):_R M]$. But $soc(M) \subseteq \mathrm{G}$ and $J(M) \subseteq \mathrm{G}$, hence $\mathrm{G} + soc(M) = \mathrm{G}$ and $\mathrm{G} + soc(M) + J(M) = \mathrm{G} + J(M) = \mathrm{G}$. Thus either $rm \in \mathrm{G}$ or $r^2 \in [\mathrm{G}:_R M]$. Therefore G is Semi-2-Absorbing submodule of $M$.

**Proposition 3.9** Let $M$ be $R$-module and G is an essential submodule of $M$ with $J(M) \subseteq \mathrm{G}$. Then G is Semi-2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proof (⇒)** Clear.

**(⇐)** Let $s^2m \in \mathrm{G}$ for $s \in R$, $m \in M$. Since G is EXNPS2AB, then either $sm \in \mathrm{G} + soc(M) + J(M)$ or $s^2 \in [\mathrm{G} + soc(M) + J(M):_R M]$. Since G is essential submodule of $M$, then $soc(M) \subseteq \mathrm{G}$ and by hypotheses $J(M) \subseteq \mathrm{G}$, we get $\mathrm{G} + soc(M) = \mathrm{G}$ and $\mathrm{G} + J(M) = \mathrm{G}$, thus $\mathrm{G} + soc(M) + J(M) = \mathrm{G}$. Hence either $sm \in \mathrm{G}$ or $s^2 \in [\mathrm{G}:_R M]$. Therefore G is Semi-2-Absorbing submodule of $M$.

The following Corollaries are direct consequence of Proposition 3.8 and Proposition 3.9.

**Corollary 3.10** Let $M$ be $R$-module and $\mathrm{G} \subset M$ with $soc(M) + J(M) \subseteq \mathrm{G}$. Then G is Semi-2-Absorbing if and only if G is EXNPS2AB submodule of $M$.

**Corollary 3.11** Let $M$ be $R$-*module* and G is maximal submodule of $M$ with $soc(M) \subseteq \mathrm{G}$. Then G is Semi-2-Absorbing if and only if G is EXNPS2AB submodule of $M$.

**Corollary 3.12** Let $M$ be a semi-simple $R$-module and $\mathrm{G} \subset M$ with $soc(M) \subseteq \mathrm{G}$. Then G is Semi-2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Corollary 3.13** Let $M$ be a regular $R$-module and $\mathrm{G} \subset M$ with $soc(M) \subseteq \mathrm{G}$. Then G is Semi-2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proposition 3.14** Let G be a proper submodule of an $R$-module $M$ with $soc(M) = (0)$ and $J(M) = (0)$. Then G is Semi-2-Absorbing if and only if G is EXNPS2AB submodule of $M$.

**Proof** Direct.

**Proposition 3.15** Let $M$ be an $R$-module over a Boolean ring $R$ with G is a proper submodule of $M$ and $soc(M) \subseteq J(M)$. Then G is Nearly-2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proof (⇒)** By Proposition 2.14.

**(⇐)** Let $\alpha bx \in \text{G}$ for $\alpha, b \in R, x \in M$, since $R$ is a Boolean ring, then $\alpha bx = (ab)^2x \in \text{G}$ with $(ab)^2 \notin [\text{G} + soc(M) + J(M):_R M]$ and $bx \notin \text{G} + soc(M) + J(M)$. But G is EXNPS2AB submodule of $M$ and $(ab)^2 \notin [\text{G} + soc(M) + J(M):_R M]$, then $abx \in \text{G} + soc(M) + J(M)$. Now, since $R$ is a Boolean ring, then by Lemma 3.1 $R$ is a regular ring, that is $a^2b = a$, hence $abx = a^2bx = ax \in \text{G} + soc(M) + J(M)$. Since $soc(M) \subseteq J(M)$, then $soc(M) + J(M) = J(M)$, then $ax \in \text{G} + J(M)$. Since $ab = (ab)^2 \notin [\text{G} + soc(M) + J(M):_R M]$, then $ab \notin [\text{G} + J(M):_R M]$ and $bx \notin \text{G} + soc(M) + J(M)$, then $bx \notin \text{G} + J(M)$. Thus is G is Nearly-2-Absorbing of $M$.

**Proposition 3.16** Let $M$ be an $R$-module over a Boolean ring $R$ with $\text{G} \subset M$ and $soc(M) \subseteq \text{G}$. Then G is Nearly-2-Absorbing if and only if G is EXNPS2AB submodule of $M$.

**Proof (⇒)** Clear.

**(⇐)** Let $\alpha bx \in \text{G}$ for $\alpha, b \in R, x \in M$, since $R$ is a Boolean ring, then $\alpha bx = (ab)^2x \in \text{G}$ with $(ab)^2 \notin [\text{G} + soc(M) + J(M):_R M]$ and $bx \notin \text{G} + soc(M) + J(M)$. But G is EXNPS2AB submodule of $M$ and $(ab)^2 \notin [\text{G} + soc(M) + J(M):_R M]$, then $abx \in \text{G} + soc(M) + J(M)$. Now, since $R$ is a Boolean ring, then by Lemma 3.1 $R$ is a regular ring, that is $a^2b = a$, hence $abx = a^2bx = ax \in \text{G} + soc(M) + J(M)$. Since $soc(M) \subseteq \text{G}$, then $soc(M) + \text{G} = \text{G}$, hence $ax \in \text{G} + J(M)$. Since $ab = (ab)^2 \notin [\text{G} + soc(M) + J(M):_R M]$, then $ab \notin [\text{G} + J(M):_R M]$ and $bx \notin \text{G} + soc(M) + J(M)$, then $bx \notin \text{G} + J(M)$. Thus G is Nearly-2-Absorbing of $M$.

The Proof of the following results is direct.

**Proposition 3.17** Let $M$ be an $R$-module over a Boolian ring $R$ with $\text{G} \subset M$ and $soc(M) = (0)$. Then G is Nearly-2-Absorbing if and only if G is EXNPS2AB submodule of $M$.

**Proposition 3.18** Let $M$ be an $R$-module over a Boolian ring $R$ and G is an essential submodule of $M$. Then G is Nearly-2-Absorbing if and only if G is EXNPS2AB submodule of $M$.

**Proposition 3.19** Let G be a proper submodule of an $R$-module $M$ and $soc(M) \subseteq J(M)$. Then G is Nearly Semi-2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proof (⇒)** Let G be a Nearly Semi-2-Absorbing submodule of an $R$-module $M$ and $\alpha^2 x \in \text{G}$, for $\alpha \in R, x \in M$. Since G is Nearly Semi-2-Absorbing submodule of $M$, then either $\alpha x \in \text{G} + J(M) \subseteq \text{G} + soc(M) + J(M)$ or $\alpha^2 M \subseteq \text{G} + J(M) \subseteq \text{G} + soc(M) + J(M)$ .That is either $\alpha x \in \text{G} + soc(M) + J(M)$ or $\alpha^2 M \subseteq \text{G} + soc(M) + J(M)$. Hence G is EXNPS2AB submodule of $M$.

**(⇐)** Let $s^2m \in \text{G}$ for $s \in R, m \in M$. Since G is EXNPS2AB submodule of $M$, then either $sm \in \text{G} + soc(M) + J(M)$ or $s^2 \in [\text{G} + soc(M) + J(M):_R M]$. Since $soc(M) \subseteq J(M)$, then $soc(M) + J(M) = J(M)$, thus either $sm \in \text{G} + J(M)$ or $s^2 \in [\text{G} + J(M):_R M]$. Hence G is Nearly Semi-2-Absorbing submodule of $M$.

**Proposition 3.20** Let G be a proper submodule of an $R$-module $M$ and $soc(M) \subseteq \text{G}$. Then G is Nearly Semi-2-Absorbing submodule of $M$ if and only if G is EXNPS-2-Absorbing submodule of $M$.

**Proof (⇒)** Clear.

**(⇐)** Let $s^2m \in \text{G}$ for $s \in R$ , $m \in M$ . Since G is EXNPS2AB, then either $sm \in \text{G} + soc(M) + J(M)$ or $s^2 \in [\text{G} + soc(M) + J(M):_R M]$. Since $soc(M) \subseteq \text{G}$, then $\text{G} + soc(M) = \text{G}$, so $\text{G} + soc(M) + J(M) = \text{G} + J(M)$, thus either $sm \in \text{G} + J(M)$ or $s^2 \in [\text{G} + J(M):_R M]$. Hence G is Nearly Semi-2-Absorbing of $M$.

The Proof of the following results is direct.

**Proposition 3.21** Let $M$ be $R$-module with G is proper of $M$ and $soc(M) = (0)$. Then G is Nearly Semi-2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proposition 3.22** Let $M$ be $R$-module and G is an essential submodule of $M$. Then G is Nearly Semi-2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proposition 3.23** Let $M$ be an $R$-module over a Boolean ring $R$ with $\text{G} \subset M$. Then G is Pseudo-2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proof (⇒)** By Proposition 2.8.

**(⇐)** Let $\alpha bx \in G$ for $\alpha, b \in R, x \in M$, since $R$ is a Boolean ring, then $\alpha bx = (ab)^2x \in G$ with $(ab)^2 \notin [G + soc(M) + J(M):_R M]$ and $bx \notin G + soc(M) + J(M)$. But G is EXNPS2AB submodule of $M$ and $(ab)^2 \notin [G + soc(M) + J(M):_R M]$, then $abx \in G + soc(M) + J(M)$. Now, since $R$ is a Boolean ring, then by Lemma 3.1 $R$ is a regular ring, that is $a^2b = a$, hence $abx = a^2bx = ax \in G + soc(M) + J(M)$. Since $R$ is a regular ring, then $J(M) = (0)$, then $ax \in G + soc(M)$. Since $ab = (ab)^2 \notin [G + soc(M) + J(M):_R M]$, then $ab \notin [G+soc(M):_R M]$ and $bx \notin G + soc(M) + J(M)$, hence $bx \notin G + soc(M)$. Hence G is Pseudo-2-Absorbing of $M$.

The Proof of the following results is direct.

**Proposition 3.24** Let $M$ be an $R$-module over a Boolean ring $R$ and G is an maximal submodule of $M$. Then G is Pseudo-2-Absorbing if and only if G is EXNPS2AB submodule of $M$.

**Proposition 3.25** Let $M$ be $R$-module with $J(M) \subseteq soc(M)$ and $G \subset M$. Then G is Pseudo Semi-2-Absorbing submodule of $M$ if and only if G is EXNPS2AB submodule of $M$.

**Proof (⇒)** By Proposition 2.11.

**(⇐)** Since $J(M) \subseteq soc(M)$, then $J(M) + soc(M) = soc(M)$, so $G + J(M) + soc(M) = G + soc(M)$. Let $r^2h \in G$ for $r \in R$, $h \in M$. Since G is EXNPS2AB, then either $rh \in G + soc(M) + J(M) = G + soc(M)$ or $r^2 \in [G + soc(M) + J(M):_R M] = [G + soc(M):_R M]$. Thus either $rh \in G + soc(M)$ or $r^2 \in [G + soc(M):_R M]$. Hence G is Pseudo Semi-2-Absorbing of $M$.

**Proposition 3.26** Let $M$ be $R$-module and G is a maximal submodule of $M$. Then G is Pseudo Semi-2-Absorbing if and only if G is EXNPS2AB submodule of $M$.

**Proof** Direct.

**Remark 3.27**

1. Every 2-Absorbing submodule is a Semi-2-Absorbing submodule. [**3**, **Rem and Exa**. (**1.2**)(**2**)].

2. Every Pseudo-2-Absorbing submodule is a Pseudo Semi-2-Absorbing submodule. [**18**, **Rem and Exa**. (**2.2.2**)].

Finally, we will present a Proposition that all concepts are equivalent.

**Proposition 3.28** Let $M$ be a an $R$-module over a Boolean ring $R$ and $G \subset M$ with $soc(M) \subseteq G$. Then the following are equivalent:

**1.** G is 2-Absorbing submodule of $M$.

**2.** G is Semi-2-Absorbing submodule of $M$.

**3.** G is Nearly Semi-2-Absorbing submodule of $M$.

**4.** G is Nearly-2-Absorbing submodule of $M$.

**5.** G is EXNPS2AB submodule of $M$.

**6.** G is Pseudo-2-Absorbing submodule of $M$.

**7.** G is Pseudo Semi-2-Absorbing submodule of $M$.

**Proof**

**(1⇒2)** By Remark 3.23.

**(2⇒3)** Clear.

**(3⇒4)** Let $abx \in$ G for $a, b \in R$, $x \in M$, since $R$ is a Boolean ring, then $abx = (ab)^2x \in$ G with $(ab)^2 \notin [\mathrm{G} + J(M):_R M]$ and $bx \notin \mathrm{G} + J(M)$. But G is Nearly Semi-2-Absorbing submodule of $M$ and $(ab)^2 \notin [\mathrm{G} + J(M):_R M]$, then $(ab)x \in \mathrm{G} + J(M)$. Now, since $R$ is a Boolean ring, then by Lemma 3.1 $R$ is a regular ring, that is $a^2b = a$, hence $abx = a^2bx = ax \in \mathrm{G} + J(M)$. Also $ab = (ab)^2 \notin [\mathrm{G}+J(M):_R M]$, then $ab \notin [\mathrm{G} + J(M):_R M]$ and $bx \notin \mathrm{G} + J(M)$. Hence G is Nearly-2-Absorbing submodule of $M$.

**(4⇔5)** By Proposition 3.16.

**(5⇔6)** By Proposition 3.23.

**(6⇒7)** By Remark 3.27.

**(7⇒1)** Let $abx \in$ G for $a, b \in R$, $x \in M$, since $R$ is a Boolean ring, then $abx = (ab)^2x \in$ G with $(ab)^2 \notin [\mathrm{G} + soc(M):_R M]$ and $bx \notin \mathrm{G} + soc(M)$. But G is Pseudo Semi-2-Absorbing submodule of $M$ and $(ab)^2 \notin [\mathrm{G} + soc(M):_R M]$, then $abx \in \mathrm{G} + soc(M)$. Now, since $R$ is a Boolean ring, then by Lemma 3.1 $R$ is a regular ring, that is $a^2b = a$, hence $abx = a^2bx = ax \in \mathrm{G} + soc(M)$. Since $soc(M) \subseteq$ G, then $soc(M) + \mathrm{G} = \mathrm{G}$ and $J(M) = 0$ hence $ax \in$ G. Also, $ab = (ab)^2 \notin [\mathrm{G} + soc(M):_R M] = [\mathrm{G}:_R M]$, then $ab \notin [\mathrm{G}:_R M]$ and $bx \notin$ G. Hence G is 2-Absorbing of $M$.

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