Principally g-radical Supplemented Modules

Rasha Najah Mirza* and Thaar Younis Ghawi#

*Department of Math, faculty of Computer sciences and math, University of Kufa, Al-Najaf, Iraq. Email:rasha.mirzah@uokufa.edu.iq

#Department of Mathematics, College of Education, University of Al-Qadisiyah, Al-Qadisiyah, Iraq. e-mail: thar.younis@qu.edu.iq

 ARTICLE INFO

Article history:
Received: 25 /12/2022
Revised form: 29 /01/2023
Accepted: 01 /02/2023
Available online: 17 /02/2023

Keywords: g-small submodules, g-supplemented module, g-radical supplemented modules, P-g-radical supplemented.

 ABSTRACT

In this article we present a proper generalization of the class of g-radical supplemented modules. This class termed by P-g-radical supplemented. We determined it is structure. Several of these modules’ characterizations, properties, and instances are described.

MSC.

1. Introduction

In this article, all rings $R$ are associative with unity, and all modules are left unitary. We will go through some of the key definitions that we will require in our work. Let $\mu$ be a module and $\beta$ a submodule of $\mu$, denoted by $\beta \leq \mu$. Also, we refer to the direct summand $\beta$ of $\mu$ by $\beta \leq \oplus \mu$. Principally semisimple module is a module which all its cyclic submodules are direct summand [1]. An essential module $\beta$ in $\mu$ symbolized by $\beta \leq \mu$, is a submodule which satisfying $\beta \cap Y = 0$ implies $Y = 0$ for any $Y$ in $\mu$ [2]. As dual, a submodule $\beta$ of $\mu$ called to be small in $\mu$, denoted by $\beta \ll \mu$ if, whenever $\mu = \beta + Y$ for $Y \leq \mu$ implies $Y = \mu$ [1]. Zhou and Zhang [3] recall that, a submodule $Y$ of $\mu$ is called generalized small, which gives the symbol $Y \ll_g \mu$, if for $\beta \leq \mu$ with $\mu = Y + \beta$ implies $\beta = \mu$. Again Zhou and Zhang gave the definition of Jacobson generalized radical of an $R$-module $\mu$ as; $Rad_g(\mu) = \cap(Y \leq \mu | Y$ is maximal in $\mu) = \sum(Y \ll_g \mu)$. If $\mu = Y + \beta$ and $Y \cap \beta \ll_g \beta$, then $\beta$ is called a $g$-supplement of $Y$ in $\mu$ [4]. Let $\mu$ be a module, $Y, \beta \leq \mu$ and $\mu = Y + \beta$ such that $Y \cap \beta \leq Rad_g(\beta)$, then $\beta$ is called a generalized radical supplement, briefly $g$-radical supplement of $Y$ in $\mu$ [5]. In this paper, we are going to present the concept of a Principally $g$-radical supplemented modules and give eight equivalent definitions. next some of the numerous this class's properties of modules and their relationships to other types of modules.

ENCLA TURE

---

*Corresponding author

Email addresses:

Communicated by 'sub etitor'
2. Principally g-radical Supplemented Modules

First, we will start in presenting our main definition.

**Definition 2.1.** A module $\mu$ is said to be principally g-radical supplemented (briefly, P-g-radical supplemented) if for every cyclic submodule $\beta$ of $\mu$, there exists a submodule $Y$ of $\mu$ such that $\mu = \beta + Y$ and $\beta \cap Y \subseteq \text{Rad}_g(Y)$.

**Examples 2.2.**

(1) Let $R = \mathbb{Z}$ and $\mu = \mathbb{Z}_{24}$. All g-small submodules of $\mu$ are $0, 2\mathbb{Z}_{24}, 4\mathbb{Z}_{24}, 6\mathbb{Z}_{24}, 8\mathbb{Z}_{24}$ and $12\mathbb{Z}_{24}$, hence $\text{Rad}_g(\mu) = 2\mathbb{Z}_{24}$. Let $Y$ be any cyclic submodule of $\mu$. So we have three cases:

- if $Y = \mu$, then trivially $0$ is a g-radical supplement of $Y$ in $\mu$.
- if $3\mathbb{Z}_{24} \neq Y \subseteq \mu$, then $Y + \mu = \mu$ and $Y \cap \mu = Y \subseteq 2\mathbb{Z}_{24} = \text{Rad}_g(\mu)$.
- if $Y = 3\mathbb{Z}_{24}$, then $Y + \beta = \mu$. It is clear to see that every submodule of $\beta$ is g-small, then $\text{Rad}_g(\beta) = \beta$. Hence $Y \cap \beta = 12\mathbb{Z}_{24} \subseteq 4\mathbb{Z}_{24} = \text{Rad}_g(\beta)$.

This means that every submodule of $\mu$ has a g-radical supplement in $\mu$. Hence $\mu = \mathbb{Z}_{24}$ is a P-g-radical supplemented as $\mathbb{Z}$-module.

(2) Consider $\mu = R = \mathbb{Z}$. Let $n, m \in \mathbb{Z}$ with $\gcd(n, m) = 1$, then $n\mathbb{Z} + m\mathbb{Z} = \mathbb{Z}$. But $\text{Rad}_g(n\mathbb{Z}) \subseteq \text{Rad}_g(\mathbb{Z}) = 0$ for any $n \in \mathbb{Z}$, we deduce $n\mathbb{Z} \cap m\mathbb{Z} = (nm)\mathbb{Z} \neq 0$, that is $n\mathbb{Z} \cap m\mathbb{Z} \subseteq \text{Rad}_g(n\mathbb{Z})$ and $n\mathbb{Z} \cap m\mathbb{Z} \subseteq \text{Rad}_g(m\mathbb{Z})$. Hence, $\mathbb{Z}$-module $\mathbb{Z}$ is not P-g-radical supplemented.

**Theorem 2.3.** Let $\mu$ be a P-g-radical supplemented duo $R$-module. Then every direct summand of $\mu$ is P-g-radical supplemented.

**Proof.** Let $\mu = \mu_1 \oplus \mu_2$ be a P-g-radical supplemented duo $R$-module and let $a \in \mu_1$. There is $Y \leq \mu$ with $\mu = aR + Y$ and $aR \cap Y \subseteq \text{Rad}_g(Y)$. By the modular law, $\mu_1 = aR + (\mu_1 \cap Y)$. By [6, Lemma 2.1], $Y = (\mu_1 \cap Y) \oplus (\mu_2 \cap Y)$. We prove that $aR \cap (\mu_1 \cap Y) \subseteq \text{Rad}_g(\mu_1 \cap Y)$. If $x \in aR \cap (\mu_1 \cap Y)$ and as $aR \cap (\mu_1 \cap Y) \subseteq aR \cap Y$, then $x \in aR \cap Y$, so that $x \in \text{Rad}_g(Y)$. Thus $xR \ll_g Y$, by [4, Lemma 5]. As $xR \leq \mu_1 \cap Y \leq \text{Rad}_g(\mu_1 \cap Y)$, it follows that $xR \ll_g \mu_1 \cap Y$, by [7, Proposition 3.2], thus $xR \subseteq \text{Rad}_g(\mu_1 \cap Y)$, and hence $x = x.1 \in xR \subseteq \text{Rad}_g(\mu_1 \cap Y)$ that is what we have to prove.

Now, we must prove the next lemma.

**Lemma 2.4.** Suppose $\mu = \mu_1 \oplus \mu_2 = Y + \beta$ is an $R$-module and $Y \leq \mu_1$. If $\mu$ is weakly distributive and $Y \cap \beta \subseteq \text{Rad}_g(\beta)$, then $Y \cap \beta \subseteq \text{Rad}_g(\mu_1 \cap \beta)$.

**Proof.** Let $a \in Y \cap \beta$, then $a \in \text{Rad}_g(\beta)$ and so $aR \ll_g \beta$, by [4, Lemma 5]. As $\mu$ is weakly distributive, $\beta = (\mu_1 \cap \beta) \oplus (\mu_2 \cap \beta)$. As $aR \leq \mu_1$ and $aR \leq \beta$ then $aR \leq \mu_1 \cap \beta \leq \text{Rad}_g(\mu_1 \cap \beta)$, hence $a = a.1 \in aR \subseteq \text{Rad}_g(\mu_1 \cap \beta)$.

**Theorem 2.5.** Let $\mu$ be a weakly distributive $R$-module and $\mu_1 \leq \text{Rad}_g\mu$. If $\mu$ is a P-g-radical supplemented $R$-module, then $\mu_1$ is so P-g-radical supplemented.

**Proof.** Let $\mu = \mu_1 \oplus \mu_2$ be a distributive P-g-radical supplemented and $a \in \mu_1$, for some $\mu_2 \leq \mu$. There is a submodule $\beta$ of $\mu$ such that $\mu = aR + \beta$ and $aR \cap \beta \subseteq \text{Rad}_g(\beta)$, as $\mu$ is P-g-radical supplemented. By the modular law, we have $\mu_1 = \mu_1 \cap \mu = \mu_1 \cap (aR + \beta) = aR + (\mu_1 \cap \beta)$. As $\mu = \mu_1 \oplus \mu_2 = aR + \beta$, $aR \leq \mu_1$ and $aR \cap \beta \subseteq \text{Rad}_g(\beta)$.
We deduce $aR \cap \beta = aR \cap (\mu_1 \cap \beta) \subseteq \text{Rad}_g(\mu_1 \cap \beta)$, by Lemma 2.4. This means that $\mu_1 \cap \beta$ is a $g$-radical supplement of $aR$ in $\mu_1$, hence $\mu_1$ is a $P$-g-radical supplemented module.

**Proposition 2.6.** Let $\mu = \mu_1 \oplus \mu_2 \oplus \ldots$ such that $\mu_i$ are $P$-g-radical supplemented modules $\{\mu_i | i \in I\}$. If $aR \leq \mu$, then $\mu$ is $P$-g-radical supplemented.

**Proof.** We will prove when $I = \{1, 2\}$. Suppose that $\mu = \mu_1 \oplus \mu_2$ is an $R$-module and $a \in \mu$. By hypothesis, $aR$ is fully invariant, then $aR = (aR \cap \mu_1) \oplus (aR \cap \mu_2)$, by [6, Lemma 2.1]. Since $aR \cap \mu_1$ and $aR \cap \mu_2$ are cyclic submodules of $\mu_1$ and $\mu_2$ respectively, there exist a submodule $Y$ of $\mu_1$ such that $\mu_1 = (aR \cap \mu_1) + Y$ and $(aR \cap \mu_1) \cap Y = aR \cap Y \subseteq \text{Rad}_g(Y)$, and $\beta \leq \mu_2$ such that $\mu_2 = (aR \cap \mu_2) + \beta$ and $(aR \cap \mu_2) \cap \beta = aR \cap \beta \subseteq \text{Rad}_g(\beta)$. Hence $\mu = \mu_1 + \mu_2 = aR + (Y + \beta)$. Now we have $aR \cap (Y + \beta) = (aR \cap Y) + (aR \cap \beta)$. Therefore $aR \cap (Y + \beta) \subseteq \text{Rad}_g(Y) + \text{Rad}_g(\beta) \subseteq \text{Rad}_g(Y + \beta)$. Hence $\mu$ is $P$-g-radical supplemented.

Recall [6] If all submodules of a module $\mu$ are fully invariant, then $\mathcal{M}$ is called a duo module. A submodule $A$ of a module $\mu$ is called distributive if $A \cap (B + C) = (A \cap B) + (A \cap C)$ for all submodules $B, C$ of $\mu$. Recall [8] A module $\mu$ is said to be distributive if all submodules of $\mu$ are distributive. A submodule $A$ of a module $\mu$ is weakly distributive if $A = (A \cap X) + (A \cap Y)$ for all submodules $X, Y$ of $\mu$ with $X + Y = \mu$. A module $\mu$ is said to be weakly distributive if every submodule of $\mu$ is a weak distributive submodule of $\mu$.

**Proposition 2.7.** Let $\mu = \mu_1 \oplus \mu_2$ be a direct sum of $P$-g-radical supplemented $R$-modules $\mu_1$ and $\mu_2$. If every cyclic submodule of $\mu$ is weakly distributive, then $\mu$ is $P$-g-radical supplemented.

**Proof.** Analogous proof of Proposition 2.6.

**Corollary 2.8.** Let $\mu$ be a module,

1. If $\mu$ is duo and $\mu = \mu_1 \oplus \mu_2 \oplus \ldots$ Then $\mu_i$ are $P$-g-radical supplemented modules $\{\mu_i | i \in I\}$ if and only if $\mu$ is $P$-g-radical supplemented.

2. If $\mu$ is weakly distributive and $\mu = \mu_1 \oplus \mu_2$. Then $\mu_1$ and $\mu_2$ are $P$-g-radical supplemented modules if and only if $\mu$ is $P$-g-radical supplemented.

**Proof.** (1) Clear by Theorem 2.3 and Proposition 2.6.

(2) Clear by Theorem 2.5 and Proposition 2.7.

**Lemma 2.9.** If $f: \mu \rightarrow \beta$ is a homomorphism and $\beta$ is a $g$-radical supplement in $\mu$ such that $\text{Ker}f \leq \beta$, then $f(\beta)$ is a $g$-radical supplement in $f(\mu)$.

**Proof.** If $\beta$ is a $g$-radical supplement in $\mu$, then there is $Y \subseteq \mu$ with $Y + \beta = \mu$ and $Y \cap \beta \subseteq \text{Rad}_g(\beta)$. Thus $f(Y) + f(\beta) = f(\mu)$. Since $f(\mu) \leq \beta$, then $f(Y) \cap f(\beta) = f(Y \cap \beta) \subseteq f(\text{Rad}_g(\beta)) \subseteq \text{Rad}_g(f(\beta))$. This means $f(\beta)$ is a $g$-radical supplement of $f(Y)$ in $f(\mu)$.

In following, we will investigate the factors of $P$-g-radical supplemented modules under some cases.

**Proposition 2.10.** Let $\mu$ be a $P$-g-radical supplemented $R$-module and $\beta \leq \mu$. If any cyclic submodule of $\mu$ has a $g$-radical supplement contains $\beta$, then $\mu/\beta$ is $P$-g-radical supplemented.

**Proof.** Suppose that $aR$ is any cyclic submodule of $\mu/\beta$, then $aR = (aR + \beta)/\beta$ for some $a \in \mu$. By hypothesis, there exists $Y \subseteq \mu$ such that $\beta \leq Y$, $aR + Y = \mu$ and $aR \cap Y \subseteq \text{Rad}_g(Y)$. Consider a natural map $\pi: \mu \rightarrow \mu/\beta$. Since $\text{Ker} \pi = \beta \leq Y$, so by Lemma 2.9, $\pi(Y)$ is a $g$-radical supplement of $\pi(aR) = (aR + \beta)/\beta = aR$ in $\mu/\beta$, and this completes the proof.

The next instance declare the converse of Proposition 2.10 is not correct, in general.
Example 2.11. Let \( R = \mathbb{Z} \) and \( \mu = \mathbb{Z}/24\mathbb{Z} = \mathbb{Z}_{24} \). We illustrate by Example 2.2(1) that \( \mathbb{Z}/24\mathbb{Z} \) is a \( p \)-\( g \)-radical supplemented as \( \mathbb{Z} \)-module, while the \( \mathbb{Z} \)-module \( \mathbb{Z} \) does not be \( p \)-\( g \)-radical supplemented, see Example 2.2(2).

Proposition 2.12. Let \( \mu \) be a distributive \( p \)-\( g \)-radical supplemented \( R \)-module. Then \( \mu / \text{Rad}_g(\mu) \) is principally semisimple.

Proof. Suppose \( \mu \) is a \( p \)-\( g \)-radical supplemented \( R \)-module and let \( \alpha R \) be a cyclic submodule of \( \mu / \text{Rad}_g(\mu) \), then \( \alpha R \cap \beta \subseteq \text{Rad}_g(\beta) \). Therefore we get that \( \mu / \text{Rad}_g(\mu) = (\alpha R + \text{Rad}_g(\mu)) / \text{Rad}_g(\mu) = (\alpha R + \text{Rad}_g(\mu)) / \text{Rad}_g(\mu) = \alpha R + \beta \). Also, since \( \mu \) is distributive, we have \( \alpha R \cap \beta \subseteq \text{Rad}_g(\beta) \). Therefore, we have \( \mu / \text{Rad}_g(\mu) = (\alpha R + \text{Rad}_g(\mu)) / \text{Rad}_g(\mu) \), and hence \( \mu / \text{Rad}_g(\mu) \) is principally semisimple.

Corollary 2.13. Let \( \mu \) be a distributive \( p \)-\( g \)-radical supplemented \( R \)-module. Then \( \mu / \text{Rad}_g(\mu) \) is \( p \)-\( g \)-radical supplemented.

Proposition 2.14. Let \( \alpha: \mu_1 \rightarrow \mu_2 \) be an \( R \)-epimorphism, where \( \mu_1 \) is a \( p \)-\( g \)-radical supplemented \( R \)-module such that any cyclic submodule of \( \mu \) contains \( K \). Then \( \mu_2 \) is \( p \)-\( g \)-radical supplemented.

Proof. Let \( \alpha \) be any cyclic submodule of \( \mu_2 \), then \( \alpha = \alpha(\alpha) \) for some \( \alpha \in \mu_1 \). Since \( \mu_1 \) is a \( p \)-\( g \)-radical supplemented \( R \)-module and \( \alpha \cap \beta \subseteq \mu_1 \), then there is \( \gamma \subseteq \mu_1 \) such that \( \alpha R + \gamma = \mu_2 \) and \( \alpha \cap \gamma \subseteq \text{Rad}_g(\gamma) \). It follows that \( \alpha R \cap \gamma \subseteq \text{Rad}_g(\gamma) \) and \( \gamma \cap \alpha \gamma = \alpha R \cap \gamma \subseteq \text{Rad}_g(\gamma) \).

Corollary 2.15. Let \( \mu \) be a \( p \)-\( g \)-radical supplemented \( R \)-module and \( Y \leq \mu \). If any cyclic submodule of \( \mu \) contains \( Y \), then \( \mu / Y \) is \( p \)-\( g \)-radical supplemented.

Proof. Consider the natural epimorphism map \( \pi: \mu \rightarrow \mu / Y \). As \( \text{Ker} \pi = Y \), then by assumption, every cyclic submodule of \( \mu \) contains \( K \), so that \( \mu / Y \) is \( p \)-\( g \)-radical supplemented, by Proposition 2.14.

Proposition 2.16. If an \( R \)-module \( \mu = \mu_1 \oplus \mu_2 \). Then \( \mu_2 \) is \( p \)-\( g \)-radical supplemented if and only if for any cyclic submodule \( Y / \mu_1 \) of \( \mu / \mu_1 \), there is a submodule \( \beta \) of \( \mu_2 \) such that \( \mu = \beta + Y \) and \( \beta \cap Y \subseteq \text{Rad}_g(\beta) \).

Proof. Suppose that \( \mu_2 \) is \( p \)-\( g \)-radical supplemented. Let \( \beta / \mu_1 \) be a cyclic submodule of \( \mu / \mu_1 \). Then \( \beta / \mu_1 = (xR + \mu_1) / \mu_1 \) and \( x = \alpha_1 + \alpha_2 \) where \( \alpha_1 \subseteq \mu_1, \alpha_2 \subseteq \mu_2 \). Thus \( \beta / \mu_1 = (\alpha_2 R + \mu_1) / \mu_1 \). Hence there is \( Y \leq \mu_2 \) with \( \beta \cap Y \subseteq \text{Rad}_g(\beta) \). It follows that \( \beta \cap Y \subseteq \text{Rad}_g(\beta) \) and \( \beta \cap Y \subseteq \text{Rad}_g(\beta) \).

Proposition 2.17. Let \( \mu \) be a \( p \)-\( g \)-radical supplemented \( R \)-module and \( Y \leq \mu \). If \( Y \cap \text{Rad}_g(\mu) = 0 \), then \( Y \) is principally semisimple. In particular, a \( p \)-\( g \)-radical supplemented \( R \)-module \( \mu \) with \( \text{Rad}_g(\mu) = 0 \) is principally semisimple.
Proof. Let $a \in Y$. Since $\mu$ is a $P$-radical supplemented $R$-module, then there is $\beta \leq \mu$ with $\mu = aR + \beta$ and $aR \cap \beta \subseteq \text{Rad}_g(\beta) \subseteq \text{Rad}_g(\mu)$. By the modular law, $Y = Y \cap \mu = Y \cap (aR + \beta) = aR + (Y \cap \beta)$. As $aR \cap (Y \cap \beta) \subseteq Y \cap \text{Rad}_g(\mu) = 0$, we get $Y = aR \oplus (Y \cap \beta)$. Hence $aR \leq \oplus Y$, and so $Y$ is principally semisimple. □

Proposition 2.18. Let $\mu$ be a cyclic and $P$-radical supplemented module over a PID $R$ (Principal ideal domain ring). Then $\mu = Y \oplus \beta$ for some principally semisimple module $Y$ and some module $\beta$ with essential generalized radical.

Proof. Since $\text{Rad}_g(\mu) \leq \mu$, there is a submodule $Y$ of $\mu$ such that $Y \oplus \text{Rad}_g(\mu) \leq \mu$. Since $Y \cap \text{Rad}_g(\mu) = 0$ and $\mu$ a $P$-radical supplemented $R$-module, then by Proposition 2.17, $Y$ is principally semisimple. Since $\mu$ is a cyclic module over a PID $R$, and $Y \leq \mu$, then $Y$ is a cyclic submodule. As $\mu$ is $P$-radical supplemented, there is a submodule $\beta$ of $\mu$ such that $\mu = Y + \beta$ and $Y \cap \beta \subseteq \text{Rad}_g(\beta) \subseteq \text{Rad}_g(\mu)$. As $Y \cap \text{Rad}_g(\mu) = 0$, then $Y \cap \beta = 0$. Thus $\mu = Y \oplus \beta$. It follows that $\text{Rad}_g(\mu) = \text{Rad}_g(Y \oplus \beta) = \text{Rad}_g(Y) \oplus \text{Rad}_g(\beta)$, by [9, Corollary 2.3]. Hence $Y \oplus \text{Rad}_g(\mu) = Y \oplus \text{Rad}_g(\beta)$. Therefore $Y \oplus \text{Rad}_g(\beta) \leq \mu = Y \oplus \beta$, then by [3, Proposition 5.20] $\text{Rad}_g(\beta) \leq \beta$. □

ACKNOWLEDGMENT

The researchers would like to acknowledge the referee(s) for their supportive and important recommendations that enhanced the article.

References