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Principally g-radical Supplemented Modules

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ABSTRACT

In this article we present a proper generalization of the class of g-radical supplemented modules. This class termed by P-g-radical supplemented. We determined it is structure. Several of these modules' characterizations, properties, and instances are described.

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1. Introduction

In this article, all rings $\boldsymbol{R}$ are associative with unity, and all modules are left unitary. We will go through some of the key definitions that we will require in our work. Let $\boldsymbol{\mu}$ be a module and $\boldsymbol{\beta}$ a submodule of $\boldsymbol{\mu}$, denoted by $\boldsymbol{\beta} \leq \boldsymbol{\mu}$. Also, we refer to the direct summand $\boldsymbol{\beta}$ of $\boldsymbol{\mu}$ by $\boldsymbol{\beta} \leq^{\oplus} \boldsymbol{\mu}$. Principally semisimple module is a module which all its cyclic submodules are direct summand [1]. An essential module $\boldsymbol{\beta}$ in $\boldsymbol{\mu}$ symbolized by $\boldsymbol{\beta} \trianglelefteq \boldsymbol{\mu}$, is a submodule which satisfying $\boldsymbol{\beta} \cap \boldsymbol{\Upsilon} = \boldsymbol{0}$ implies $\boldsymbol{\Upsilon} = \boldsymbol{0}$ for any $\boldsymbol{\Upsilon}$ in $\boldsymbol{\mu}$ [2]. As dual, a submodule $\boldsymbol{\beta}$ of $\boldsymbol{\mu}$ called to be small in $\boldsymbol{\mu}$, denoted by $\boldsymbol{\beta} \ll \boldsymbol{\mu}$ if, whenever $\boldsymbol{\mu} = \boldsymbol{\beta} + \boldsymbol{\Upsilon}$ for $\boldsymbol{\Upsilon} \leq \boldsymbol{\mu}$ implies $\boldsymbol{\Upsilon} = \boldsymbol{\mu}$ [1]. Zhou and Zhang [3] recall that, a submodule $\boldsymbol{\Upsilon}$ of $\boldsymbol{\mu}$ is called generalized small, which gives the symbol $\boldsymbol{\Upsilon} \ll_g \boldsymbol{\mu}$, if for $\boldsymbol{\beta} \trianglelefteq \boldsymbol{\mu}$ with $\boldsymbol{\mu} = \boldsymbol{\Upsilon} + \boldsymbol{\beta}$ implies $\boldsymbol{\beta} = \boldsymbol{\mu}$. Again Zhou and Zhang gave the definition of Jacobson generalized radical of an $\boldsymbol{R}$-module $\boldsymbol{\mu}$ as; $\boldsymbol{Rad_g(\mu)} = \cap\{\boldsymbol{\Upsilon} \trianglelefteq \boldsymbol{\mu} | \boldsymbol{\Upsilon} \textbf{ is maximal in } \boldsymbol{\mu}\} = \sum\{\boldsymbol{\Upsilon} | \boldsymbol{\Upsilon} \ll_g \boldsymbol{\mu}\}$. If $\boldsymbol{\mu} = \boldsymbol{\Upsilon} + \boldsymbol{\beta}$ and $\boldsymbol{\Upsilon} \cap \boldsymbol{\beta} \ll_g \boldsymbol{\beta}$, then $\boldsymbol{\beta}$ is called a g-supplement of $\boldsymbol{\Upsilon}$ in $\boldsymbol{\mu}$ [4]. Let $\boldsymbol{\mu}$ be a module, $\boldsymbol{\Upsilon}, \boldsymbol{\beta} \leq \boldsymbol{\mu}$ and $\boldsymbol{\mu} = \boldsymbol{\Upsilon} + \boldsymbol{\beta}$ such that $\boldsymbol{\Upsilon} \cap \boldsymbol{\beta} \leq \boldsymbol{Rad_g(\beta)}$, then $\boldsymbol{\beta}$ is called a generalized radical supplement, briefly g-radical supplement of $\boldsymbol{\Upsilon}$ in $\boldsymbol{\mu}$ [5]. In this paper, we are going to present the concept of a Principally g-radical supplemented modules and give eight equivalent definitions. next some of the numerous this class's properties of modules and their relationships to other types of modules.

ENCLATURE

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2. Principally g-radical Supplemented Modules

First, we will start in presenting our main definition.

Definition 2.1. A module $\boldsymbol{\mu}$ is said to be principally g-radical supplemented (briefly, P-g-radical supplemented)if for every cyclic submodule $\boldsymbol{\beta}$ of $\boldsymbol{\mu}$, there exists a submodule $\boldsymbol{\Upsilon}$ of $\boldsymbol{\mu}$ such that $\boldsymbol{\mu} = \boldsymbol{\beta} + \boldsymbol{\Upsilon}$ and $\boldsymbol{\beta} \cap \boldsymbol{\Upsilon} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\Upsilon})$.

Examples 2.2.

(1) Let $\boldsymbol{R} = \mathbb{Z}$ and $\boldsymbol{\mu} = \mathbb{Z}_{24}$. All g-small submodules of $\boldsymbol{\mu}$ are $\boldsymbol{0}$, $\boldsymbol{2}\mathbb{Z}_{24}$, $\boldsymbol{4}\mathbb{Z}_{24}$, $\boldsymbol{6}\mathbb{Z}_{24}$, $\boldsymbol{8}\mathbb{Z}_{24}$ and $\boldsymbol{12}\mathbb{Z}_{24}$, hence $\boldsymbol{Rad_g}(\boldsymbol{\mu}) = \boldsymbol{2}\mathbb{Z}_{24}$. Let $\boldsymbol{\Upsilon}$ be any cyclic submodule of $\boldsymbol{\mu}$. So we have three cases:

- if $\boldsymbol{\Upsilon} = \boldsymbol{\mu}$, then trivially $\boldsymbol{0}$ is a g-radical supplement of $\boldsymbol{\Upsilon}$ in $\boldsymbol{\mu}$.

- if $\boldsymbol{3}\mathbb{Z}_{24} \neq \boldsymbol{\Upsilon} \subset \boldsymbol{\mu}$, then $\boldsymbol{\Upsilon} + \boldsymbol{\mu} = \boldsymbol{\mu}$ and $\boldsymbol{\Upsilon} \cap \boldsymbol{\mu} = \boldsymbol{\Upsilon} \subseteq \boldsymbol{2}\mathbb{Z}_{24} = \boldsymbol{Rad_g}(\boldsymbol{\mu})$.

- if $\boldsymbol{\Upsilon} = \boldsymbol{3}\mathbb{Z}_{24}$. Let $\boldsymbol{\beta} = \boldsymbol{4}\mathbb{Z}_{24}$, then $\boldsymbol{\Upsilon} + \boldsymbol{\beta} = \boldsymbol{\mu}$. It is clear to see that every submodule of $\boldsymbol{\beta}$ is g-small, then $\boldsymbol{Rad_g}(\boldsymbol{\beta}) = \boldsymbol{\beta}$. Hence $\boldsymbol{\Upsilon} \cap \boldsymbol{\beta} = \boldsymbol{12}\mathbb{Z}_{24} \subseteq \boldsymbol{4}\mathbb{Z}_{24} = \boldsymbol{Rad_g}(\boldsymbol{\beta})$.

This means that every submodule of $\boldsymbol{\mu}$ has a g-radical supplement in $\boldsymbol{\mu}$. Hence $\boldsymbol{\mu} = \mathbb{Z}_{24}$ is a P-g-radical supplemented as $\mathbb{Z}$-module.

(2) Consider $\boldsymbol{\mu} = \boldsymbol{R} = \mathbb{Z}$. Let $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}_+$ with $\mathbf{gcd}(\boldsymbol{n}, \boldsymbol{m}) = \boldsymbol{1}$, then $\boldsymbol{n}\mathbb{Z} + \boldsymbol{m}\mathbb{Z} = \mathbb{Z}$. But $\boldsymbol{Rad_g}(\boldsymbol{n}\mathbb{Z}) \subseteq \boldsymbol{Rad_g}(\mathbb{Z}) = \boldsymbol{0}$ for any $\boldsymbol{n} \in \mathbb{Z}_+$, we deduce $\boldsymbol{n}\mathbb{Z} \cap \boldsymbol{m}\mathbb{Z} = (\boldsymbol{nm})\mathbb{Z} \neq \boldsymbol{0}$, that is $\boldsymbol{n}\mathbb{Z} \cap \boldsymbol{m}\mathbb{Z} \nsubseteq \boldsymbol{Rad_g}(\boldsymbol{n}\mathbb{Z})$ and $\boldsymbol{n}\mathbb{Z} \cap \boldsymbol{m}\mathbb{Z} \nsubseteq \boldsymbol{Rad_g}(\boldsymbol{m}\mathbb{Z})$. Hence, $\mathbb{Z}$-module $\mathbb{Z}$ is not P-g-radical supplemented.

Theorem 2.3. Let $\boldsymbol{\mu}$ be a P-g-radical supplemented duo $\boldsymbol{R}$-module. Then every direct summand of $\boldsymbol{\mu}$ is P-g-radical supplemented.

Proof. Let $\boldsymbol{\mu} = \boldsymbol{\mu_1} \oplus \boldsymbol{\mu_2}$ be a P-g-radical supplemented duo $\boldsymbol{R}$-module and let $\boldsymbol{a} \in \boldsymbol{\mu_1}$. There is $\boldsymbol{\Upsilon} \leq \boldsymbol{\mu}$ with $\boldsymbol{\mu} = \boldsymbol{aR} + \boldsymbol{\Upsilon}$ and $\boldsymbol{aR} \cap \boldsymbol{\Upsilon} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\Upsilon})$. By the modular law, $\boldsymbol{\mu_1} = \boldsymbol{aR} + (\boldsymbol{\mu_1} \cap \boldsymbol{\Upsilon})$. By [6, Lemma 2.1], $\boldsymbol{\Upsilon} = (\boldsymbol{\mu_1} \cap \boldsymbol{\Upsilon}) \oplus (\boldsymbol{\mu_2} \cap \boldsymbol{\Upsilon})$. We prove that $\boldsymbol{aR} \cap (\boldsymbol{\mu_1} \cap \boldsymbol{\Upsilon}) \subseteq \boldsymbol{Rad_g}(\boldsymbol{\mu_1} \cap \boldsymbol{\Upsilon})$. If $\boldsymbol{x} \in \boldsymbol{aR} \cap (\boldsymbol{\mu_1} \cap \boldsymbol{\Upsilon})$ and as $\boldsymbol{aR} \cap (\boldsymbol{\mu_1} \cap \boldsymbol{\Upsilon}) \subseteq \boldsymbol{aR} \cap \boldsymbol{\Upsilon}$, then $\boldsymbol{x} \in \boldsymbol{aR} \cap \boldsymbol{\Upsilon}$, so that $\boldsymbol{x} \in \boldsymbol{Rad_g}(\boldsymbol{\Upsilon})$. Thus $\boldsymbol{xR} \ll_g \boldsymbol{\Upsilon}$, by [4, Lemma 5]. As $\boldsymbol{xR} \leq \boldsymbol{\mu_1} \cap \boldsymbol{\Upsilon} \leq^{\oplus} \boldsymbol{\Upsilon}$, it follows that $\boldsymbol{xR} \ll_g \boldsymbol{\mu_1} \cap \boldsymbol{\Upsilon}$, by [7, Proposition 3.2], thus $\boldsymbol{xR} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\mu_1} \cap \boldsymbol{\Upsilon})$, and hence $\boldsymbol{x} = \boldsymbol{x}.\boldsymbol{1} \in \boldsymbol{xR} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\mu_1} \cap \boldsymbol{\Upsilon})$ that is what we have to prove. ■

Now, we must to prove the next lemma.

Lemma 2.4. Suppose $\boldsymbol{\mu} = \boldsymbol{\mu_1} \oplus \boldsymbol{\mu_2} = \boldsymbol{\Upsilon} + \boldsymbol{\beta}$ is an $\boldsymbol{R}$-module and $\boldsymbol{\Upsilon} \leq \boldsymbol{\mu_1}$. If $\boldsymbol{\mu}$ is weakly distributive and $\boldsymbol{\Upsilon} \cap \boldsymbol{\beta} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\beta})$, then $\boldsymbol{\Upsilon} \cap \boldsymbol{\beta} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\mu_1} \cap \boldsymbol{\beta})$.

Proof. Let $\boldsymbol{a} \in \boldsymbol{\Upsilon} \cap \boldsymbol{\beta}$, then $\boldsymbol{a} \in \boldsymbol{Rad_g}(\boldsymbol{\beta})$ and so $\boldsymbol{aR} \ll_g \boldsymbol{\beta}$, by [4, Lemma 5]. As $\boldsymbol{\mu}$ is weakly distributive, $\boldsymbol{\beta} = (\boldsymbol{\mu_1} \cap \boldsymbol{\beta}) \oplus (\boldsymbol{\mu_2} \cap \boldsymbol{\beta})$. As $\boldsymbol{aR} \leq \boldsymbol{\Upsilon} \leq \boldsymbol{\mu_1}$ and $\boldsymbol{aR} \leq \boldsymbol{\beta}$ then $\boldsymbol{aR} \leq \boldsymbol{\mu_1} \cap \boldsymbol{\beta} \leq^{\oplus} \boldsymbol{\beta}$, [7, Proposition 3.2] implies that $\boldsymbol{aR} \ll_g \boldsymbol{\mu_1} \cap \boldsymbol{\beta}$, therefore $\boldsymbol{aR} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\mu_1} \cap \boldsymbol{\beta})$, hence $\boldsymbol{a} = \boldsymbol{a}.\boldsymbol{1} \in \boldsymbol{aR} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\mu_1} \cap \boldsymbol{\beta})$. ■

Theorem 2.5. Let $\boldsymbol{\mu}$ be a weakly distributive $\boldsymbol{R}$-module and $\boldsymbol{\mu_1} \leq^{\oplus} \boldsymbol{\mu}$. If $\boldsymbol{\mu}$ is a P-g-radical supplemented $\boldsymbol{R}$-module, then $\boldsymbol{\mu_1}$ is so P-g-radical supplemented.

Proof. Let $\boldsymbol{\mu} = \boldsymbol{\mu_1} \oplus \boldsymbol{\mu_2}$ be a distributive P-g-radical supplemented and $\boldsymbol{a} \in \boldsymbol{\mu_1}$, for some $\boldsymbol{\mu_2} \leq \boldsymbol{\mu}$. There is a submodule $\boldsymbol{\beta}$ of $\boldsymbol{\mu}$ such that $\boldsymbol{\mu} = \boldsymbol{aR} + \boldsymbol{\beta}$ and $\boldsymbol{aR} \cap \boldsymbol{\beta} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\beta})$, as $\boldsymbol{\mu}$ is P-g-radical supplemented. By the modular law, we have $\boldsymbol{\mu_1} = \boldsymbol{\mu_1} \cap \boldsymbol{\mu} = \boldsymbol{\mu_1} \cap (\boldsymbol{aR} + \boldsymbol{\beta}) = \boldsymbol{aR} + (\boldsymbol{\mu_1} \cap \boldsymbol{\beta})$. As $\boldsymbol{\mu} = \boldsymbol{\mu_1} \oplus \boldsymbol{\mu_2} = \boldsymbol{aR} + \boldsymbol{\beta}$, $\boldsymbol{aR} \leq \boldsymbol{\mu_1}$ and $\boldsymbol{aR} \cap \boldsymbol{\beta} \subseteq$

$\boldsymbol{Rad_g(\beta)}$, we deduce $\boldsymbol{aR \cap \beta = aR \cap (\mu_1 \cap \beta) \subseteq Rad_g(\mu_1 \cap \beta)}$, by Lemma 2.4. This mean that $\boldsymbol{\mu_1 \cap \beta}$ is a P-g-radical supplement of $\boldsymbol{aR}$ in $\boldsymbol{\mu_1}$, and so $\boldsymbol{\mu_1}$ is P-g-radical supplemented. ■

Proposition 2.6. Let $\boldsymbol{\mu = \mu_1 \oplus \mu_2 \oplus}$ ... such that $\boldsymbol{\mu_i}$ are P-g-radical supplemented modules $\{\boldsymbol{\mu_i} | \boldsymbol{i \in I}\}$. If $\boldsymbol{aR \le \mu}$, is fully invariant for some $\boldsymbol{a \in \mu}$, then $\boldsymbol{\mu}$ is P-g-radical supplemented.

Proof. We will prove when $\boldsymbol{I = \{1, 2\}}$. Suppose that $\boldsymbol{\mu = \mu_1 \oplus \mu_2}$ is an $\boldsymbol{R}$-module and $\boldsymbol{a \in \mu}$. By hypothesis, $\boldsymbol{aR}$ is fully invariant, then $\boldsymbol{aR = (aR \cap \mu_1) \oplus (aR \cap \mu_2)}$, by [6, Lemma 2.1]. Since $\boldsymbol{aR \cap \mu_1}$ and $\boldsymbol{aR \cap \mu_2}$ are cyclic submodules of $\boldsymbol{\mu_1}$ and $\boldsymbol{\mu_2}$ respectively, then there exist a submodule $\boldsymbol{\Upsilon}$ of $\boldsymbol{\mu_1}$ such that $\boldsymbol{\mu_1 = (aR \cap \mu_1) + \Upsilon}$ and $\boldsymbol{(aR \cap \mu_1) \cap \Upsilon = aR \cap \Upsilon \subseteq Rad_g(\Upsilon)}$, and $\boldsymbol{\beta \le \mu_2}$ such that $\boldsymbol{\mu_2 = (aR \cap \mu_2) + \beta}$ and $\boldsymbol{(aR \cap \mu_2) \cap \beta = aR \cap \beta \subseteq Rad_g(\beta)}$. Hence $\boldsymbol{\mu = \mu_1 + \mu_2 = aR + (\Upsilon + \beta)}$. Now we have $\boldsymbol{aR \cap (\Upsilon + \beta) = (aR \cap \Upsilon) + (aR \cap \beta)}$. Therefore $\boldsymbol{aR \cap (\Upsilon + \beta) \subseteq Rad_g(\Upsilon) + Rad_g(\beta) \subseteq Rad_g(\Upsilon + \beta)}$. Hence $\boldsymbol{\mu}$ is P-g-radical supplemented. ■

Recall [6] If all submodules of a module μ are fully invariant,then $\mathcal{M}$ is called a duo module. A submodule A of a module μ is called distributive if $A \cap (B + C) = (A \cap B) + (A \cap C)$ or $A + (B \cap C) = (A + B) \cap (A + C)$ for all submodules B, C of μ. Recall [8] A module μ is said to be distributive if all submodules of μ are distributive. A submodule A of a module μ is weak distributive if $A = (A \cap X) + (A \cap Y)$ for all submodules X, Y of μ with $X + Y = \mu$. A module μ is said to be weakly distributive if every submodule of μ is a weak distributive submodule of μ.

Proposition 2.7. Let $\mu = \mu_1 \oplus \mu_2$ be a direct sum of P-g-radical supplemented R-modules μ_1 and μ_2. If every cyclic submodule of μ is weak distributive, then μ is P-g-radical supplemented.

Proof. Analogous proof of Proposition 2.6. ■

Corollary 2.8. Let μ be a module,

(1) If μ is duo and $\mu = \mu_1 \oplus \mu_2 \oplus$ .... Then μ_i are P-g-radical supplemented modules $\{\mu_i | i \in I\}$ if and only if μ is P-g-radical supplemented.

(2) if μ is weakly distributive and $\mu = \mu_1 \oplus \mu_2$. Then μ_1 and μ_2 are P-g-radical supplemented modules if and only if μ is P-g-radical supplemented.

Proof. (1) Clear by Theorem 2.3 and Proposition 2.6.

(2) Clear by Theorem 2.5 and Proposition 2.7. ■

Lemma 2.9. If $f: \mu \to \beta$ is a homomorphism and β is a g-radical supplement in μ such that $Kerf \le \beta$, then $f(\beta)$ is a g-radical supplement in $f(\mu)$.

Proof. If β is a g-radical supplement in μ, then there is $\Upsilon \le \mu$ with $\Upsilon + \beta = \mu$ and $\Upsilon \cap \beta \subseteq Rad_g(\beta)$. Thus $f(\Upsilon) + f(\beta) = f(\mu)$. Since $Kerf \le \beta$, then $f(\Upsilon) \cap f(\beta) = f(\Upsilon \cap \beta) \subseteq f(Rad_g(\beta)) \subseteq Rad_g(f(\beta))$. This means $f(\beta)$ is a g-radical supplement of $f(\Upsilon)$ in $f(\mu)$. ■

In following, we will investigate the factors of P-g-radical supplemented modules under some cases.

Proposition 2.10. Let μ be a P-g-radical supplemented R-module and $\beta \le \mu$. If any cyclic submodule of μ has a g-radical supplement contains β, then μ/β is P-g-radical supplemented.

Proof. Suppose that $\bar{a}R$ is any cyclic submodule of μ/β, then $\bar{a}R = (aR + \beta)/\beta$ for some $a \in \mu$. By hypothesis, there exists $\Upsilon \le \mu$ such that $\beta \le \Upsilon$, $aR + \Upsilon = \mu$ and $aR \cap \Upsilon \subseteq Rad_g(\Upsilon)$. Consider a natural map $\pi: \mu \to \mu/\beta$. Since $Ker\pi = \beta \le \Upsilon$, so by Lemma 2.9, $\pi(\Upsilon)$ is a g-radical supplement of $\pi(aR) = (aR + \beta)/\beta = \bar{a}R$ in μ/β, and this completes the proof. ■

The next instance declare the converse of Proposition 2.10 is not correct, in general.

Example 2.11. Let $R = \mathbb{Z}$ and $\mu = \mathbb{Z}/24\mathbb{Z} = \mathbb{Z}_{24}$. We illustrate by Example 2.2(1) that $\mathbb{Z}/24\mathbb{Z}$ is a P-g-radical supplemented as $\mathbb{Z}$-module, while the $\mathbb{Z}$-module $\mathbb{Z}$ does not be P-g-radical supplemented, see Example 2.2(2).

Proposition 2.12. Let μ be a distributive P-g-radical supplemented R-module. Then $\mu/Rad_g(\mu)$ is principally semisimple.

Proof. Suppose μ is a P-g-radical supplemented R-module and let $\bar{a}R$ be a cyclic submodule of $\mu/Rad_g(\mu)$, then $\bar{m}R = \left(aR + Rad_g(\mu)\right)/Rad_g(\mu)$ for some $a \in \mu$. Then there is a submodule β of μ such that $aR + \beta = \mu$ and $aR \cap \beta \subseteq Rad_g(\beta)$. Therefore we get that , $\mu/Rad_g(\mu) = \left(aR + Rad_g(\mu)\right)/Rad_g(\mu) + (\beta + Rad_g(\mu))/Rad_g(\mu) = \bar{m}R + (\beta + Rad_g(\mu))/Rad_g(\mu)$. Also, since μ is distributive, we have $\bar{m}R \cap ((\beta + Rad_g(\mu))/Rad_g(\mu)) = [(aR + Rad_g(\mu)) \cap (\beta + Rad_g(\mu))]/Rad_g(\mu) = ((aR \cap \beta) + Rad_g(\mu))/Rad_g(\mu) \subseteq (Rad_g(\beta) + Rad_g(\mu))]/Rad_g(\mu) = Rad_g(\mu)$. Therefore, we have $\mu/Rad_g(\mu) = \bar{m}R \oplus (\beta + Rad_g(\mu))/Rad_g(\mu)$, and hence $\mu/Rad_g(\mu)$ is principally semisimple. ■

Corollary 2.13. Let μ be a distributive P-g-radical supplemented R-module. Then $\mu/Rad_g(\mu)$ is P-g-radical supplemented.

Proposition 2.14. Let $\alpha: \mu_1 \to \mu_2$ be an R-epimorphism, where μ_1 is a P-g-radical supplemented R-module such that any cyclic submodule of μ contains $Ker\alpha$. Then μ_2 is P-g-radical supplemented.

Proof. Let $\grave{a}R$ be any cyclic submodule of μ_2, then $\grave{a} = \alpha(a)$ for some $a \in \mu_1$. Since μ_1 is a P-g-radical supplemented R-module and $aR \le \mu_1$, then there is $\Upsilon \le \mu_1$ such that $aR + \Upsilon = \mu_1$ and $aR \cap \Upsilon \subseteq Rad_g(\Upsilon)$. It follows that $\grave{m}R + \alpha(\Upsilon) = \mu_2$. By assumption $Ker\alpha \le aR$, thus $\alpha(aR) \cap \alpha(\Upsilon) = \alpha(aR \cap \Upsilon) \subseteq \alpha(Rad_g(\Upsilon)) \subseteq Rad_g(\alpha(\Upsilon))$ that is $\grave{a}R \cap \alpha(\Upsilon) \subseteq Rad_g(\alpha(\Upsilon))$. So $\alpha(\Upsilon)$ is a g-radical supplement of $\grave{a}R$ in μ_2, and then μ_2 is P-g-radical supplemented. ■

Corollary 2.15. Let μ be a P-g-radical supplemented R-module and $\Upsilon \le \mu$. If any cyclic submodule of μ contains Υ, then μ/Υ is P-g-radical supplemented.

Proof. Consider the natural epimorphism map $\pi: \mu \to \mu/\Upsilon$. As $Ker\pi = \Upsilon$, then by assumption, every cyclic submodule of μ contains $Ker\pi$, so that μ/Υ is P-g-radical supplemented, by Proposition 2.14. ■

Proposition 2.16. If an R-module $\mu = \mu_1 \oplus \mu_2$. Then μ_2 is P-g-radical supplemented if and only if for any cyclic submodule Υ/μ_1 of μ/μ_1, there is a submodule β of μ_2 such that $\mu = \beta + \Upsilon$ and $\beta \cap \Upsilon \subseteq Rad_g(\beta)$.

Proof. Suppose that μ_2 is P-g-radical supplemented. Let β/μ_1 be a cyclic submodule of μ/μ_1. Then $\beta/\mu_1 = (xR + \mu_1)/\mu_1$ and $x = \alpha_1 + \alpha_2$ where $\alpha_1 \in \mu_1, \alpha_2 \in \mu_2$. Thus $\beta/\mu_1 = (\alpha_2 R + \mu_1)/\mu_1$. Hence there is $\Upsilon \le \mu_2$ with $\mu_2 = \alpha_2 R + \Upsilon$ with $\alpha_2 R \cap \Upsilon \le Rad_g(\Upsilon)$. It follows that $\beta = \alpha_2 R + \mu_1$ and $\mu = \mu_1 + (\alpha_2 R + \Upsilon) = \beta + \Upsilon$. Now, we have $\beta \cap \Upsilon = (\alpha_2 R + \mu_1) \cap \Upsilon \le [\alpha_2 R \cap (\mu_1 + \Upsilon)] + [\mu_1 \cap (\alpha_2 R + \Upsilon)]$. As $\mu_1 \cap (\alpha_2 R + \Upsilon) \subseteq \mu_1 \cap \mu_2 = 0$, we deduce that $\beta \cap \Upsilon \le \alpha_2 R \cap (\mu_1 + \Upsilon) \le \alpha_2 R$, and so $\beta \cap \Upsilon \le \alpha_2 R \cap \Upsilon$. Since $\alpha_2 R \cap \Upsilon \le Rad_g(\Upsilon)$, we get $\beta \cap \Upsilon \le Rad_g(\Upsilon)$. Conversely, assume $\alpha_2 \in \mu_2$. Consider the cyclic submodule $(\alpha_2 R + \mu_1)/\mu_1$ of μ/μ_1. By our assuming, there is $\Upsilon \le \mu_2$ with $\mu = (\alpha_2 R + \mu_1) + \Upsilon$ and $(\alpha_2 R + \mu_1) \cap \Upsilon \le Rad_g(\Upsilon)$. By the modular law, we deduce that $\mu_2 = \mu_2 \cap \mu = \mu_2 \cap \left((\alpha_2 R + \mu_1) + \Upsilon\right) = (\alpha_2 R + \Upsilon) + (\mu_2 \cap \mu_1) = \alpha_2 R + \Upsilon$. It is enough to show $\Upsilon \cap (\mu_1 + \alpha_2 R) = \alpha_2 R \cap (\mu_1 + \Upsilon) = \alpha_2 R \cap \Upsilon$ .Therefore, $\alpha_2 R \cap (\mu_1 + \Upsilon) \le [\mu_1 \cap (\Upsilon + \alpha_2 R)] + [\Upsilon \cap (\alpha_2 R + \mu_1)] = \Upsilon \cap (\alpha_2 R + \mu_1) \le [\alpha_2 R \cap (\mu_1 + \Upsilon)] + [\mu_1 \cap (\Upsilon + \alpha_2 R)] = \alpha_2 R \cap (\mu_1 + \Upsilon)$, since $\mu_1 \cap (\Upsilon + \alpha_2 R) \subseteq \mu_1 \cap \mu_2 = 0$. It follows that $\Upsilon \cap (\mu_1 + \alpha_2 R) = \alpha_2 R \cap (\mu_1 + \Upsilon) = \alpha_2 R \cap \Upsilon$. Hence $\alpha_2 R \cap \Upsilon \le Rad_g(\Upsilon)$. Therefore μ_2 is principally g- radical supplemented. ■

Proposition 2.17. Let μ be a P-g-radical supplemented R-module and $\Upsilon \le \mu$. If $\Upsilon \cap Rad_g(\mu) = 0$, then Υ is principally semisimple. In particular, a P-g-radical supplemented R-module μ with $Rad_g(\mu) = 0$ is principally semisimple.

Proof. Let $\boldsymbol{a} \in \boldsymbol{\Upsilon}$. Since $\boldsymbol{\mu}$ is a P-g-radical supplemented $\boldsymbol{R}$-module, then there is $\boldsymbol{\beta} \leq \boldsymbol{\mu}$ with $\boldsymbol{\mu} = \boldsymbol{aR} + \boldsymbol{\beta}$ and $\boldsymbol{aR} \cap \boldsymbol{\beta} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\beta}) \subseteq \boldsymbol{Rad_g}(\boldsymbol{\mu})$. By the modular law, $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon} \cap \boldsymbol{\mu} = \boldsymbol{\Upsilon} \cap (\boldsymbol{aR} + \boldsymbol{\beta}) = \boldsymbol{aR} + (\boldsymbol{\Upsilon} \cap \boldsymbol{\beta})$. As $\boldsymbol{aR} \cap (\boldsymbol{\Upsilon} \cap \boldsymbol{\beta}) \subseteq \boldsymbol{\Upsilon} \cap \boldsymbol{Rad_g}(\boldsymbol{\mu}) = \boldsymbol{0}$, we get $\boldsymbol{\Upsilon} = \boldsymbol{aR} \oplus (\boldsymbol{\Upsilon} \cap \boldsymbol{\beta})$. Hence $\boldsymbol{aR} \leq^{\oplus} \boldsymbol{\Upsilon}$, and so $\boldsymbol{\Upsilon}$ is principally semisimple. ■

Proposition 2.18. Let $\boldsymbol{\mu}$ be a cyclic and P-g-radical supplemented module over a PID$\boldsymbol{R}$ (Principal ideal domain ring). Then $\boldsymbol{\mu} = \boldsymbol{\Upsilon} \oplus \boldsymbol{\beta}$ for some principally semisimple module $\boldsymbol{\Upsilon}$ and some module $\boldsymbol{\beta}$ with essential generalized radical.

Proof. Since $\boldsymbol{Rad_g}(\boldsymbol{\mu}) \leq \boldsymbol{\mu}$, there is a submodule $\boldsymbol{\Upsilon}$ of $\boldsymbol{\mu}$ such that $\boldsymbol{\Upsilon} \oplus \boldsymbol{Rad_g}(\boldsymbol{\mu}) \trianglelefteq \boldsymbol{\mu}$. Since $\boldsymbol{\Upsilon} \cap \boldsymbol{Rad_g}(\boldsymbol{\mu}) = \boldsymbol{0}$ and $\boldsymbol{\mu}$ a P-g-radical supplemented $\boldsymbol{R}$-module, then by Proposition 2.17, $\boldsymbol{\Upsilon}$ is principally semisimple. Since $\boldsymbol{\mu}$ is a cyclic module over a PID $\boldsymbol{R}$, and $\boldsymbol{\Upsilon} \leq \boldsymbol{\mu}$, then $\boldsymbol{\Upsilon}$ is a cyclic submodule. As $\boldsymbol{\mu}$ is P-g-radical supplemented, there is a submodule $\boldsymbol{\beta}$ of $\boldsymbol{\mu}$ such that $\boldsymbol{\mu} = \boldsymbol{\Upsilon} + \boldsymbol{\beta}$ and $\boldsymbol{\Upsilon} \cap \boldsymbol{\beta} \subseteq \boldsymbol{Rad_g}(\boldsymbol{\beta}) \subseteq \boldsymbol{Rad_g}(\boldsymbol{\mu})$. As $\boldsymbol{\Upsilon} \cap \boldsymbol{Rad_g}(\boldsymbol{\mu}) = \boldsymbol{0}$, then $\boldsymbol{\Upsilon} \cap \boldsymbol{\beta} = \boldsymbol{0}$. Thus $\boldsymbol{\mu} = \boldsymbol{\Upsilon} \oplus \boldsymbol{\beta}$. It follows that $\boldsymbol{Rad_g}(\boldsymbol{\mu}) = \boldsymbol{Rad_g}(\boldsymbol{\Upsilon} \oplus \boldsymbol{\beta}) = \boldsymbol{Rad_g}(\boldsymbol{\Upsilon}) \oplus \boldsymbol{Rad_g}(\boldsymbol{\beta})$, by [9, Corollary 2.3]. Hence $\boldsymbol{\Upsilon} \oplus \boldsymbol{Rad_g}(\boldsymbol{\mu}) = \boldsymbol{\Upsilon} \oplus \boldsymbol{Rad_g}(\boldsymbol{\beta})$. Therefore $\boldsymbol{\Upsilon} \oplus \boldsymbol{Rad_g}(\boldsymbol{\beta}) \trianglelefteq \boldsymbol{\mu} = \boldsymbol{\Upsilon} \oplus \boldsymbol{\beta}$, then by [3, Proposition 5.20] $\boldsymbol{Rad_g}(\boldsymbol{\beta}) \trianglelefteq \boldsymbol{\beta}$. ■

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