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2-Semi-Bounded Linear Operators

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ABSTRACT

In this Article, we introduced a new definition of 2- semi bounded operator in 2- inner product space. Then, we investigate a new Space of bounded operators and proved it as vector space. After that we show this space as Banach space. Finally, we discussed some properties of this space.

MSC.

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1. Introduction

Gahler is the first one who introduced 2- inner product space in 1963 [1] [2] define as [3]

1- Let W be a vector space with $\dim(X) > 1$ over field K = R. Assume that $(. , . | .) : W \times W \times W \to \mathbb{C}$ the conditions are below satisfying

1) $(x, x| z) \geq 0 \ \forall x, z \in W$ and $(x, x| z) = 0$ iff x, z dependent

2) $(x, x| z) = (z, z| x)$

3) $\overline{(x, y| z)} = (y, x| z) \ \forall x, y, z \in W$

4) $(\varepsilon x, y| z) = \varepsilon (x, y| z)$ where $\varepsilon \in K$

5) $(x_1 + x_2, y| z) \leq (x_1, y| z) + (x_2, y| z)$

2- So, the $(. , . | .)$ is said to be 2- pre-Hilbert on W and $(W, (. , . | .))$ is called 2-inner space. In [4] defined the 2-normed space as below

Assume W vector space over field R with dim W > 1. The map $\|., .\| : W \times W \to R$ satisfy the conditions below:

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1) $\|x\,,y\| \geq 0 \quad \forall x,y \in W \quad and \; \|x\,,y\| = 0 \; iff \; x \; and \; y \; dependent$

2) $\|x\,,y\| = \|y\,,x\|$

3) $\|\varepsilon\, x\,,y\| = |\varepsilon|\,\|x\,,y\| \quad \forall\, x\,,y \in W \; and \; \varepsilon \in R$

4) $\|x_1 + x_2, y\| \leq \|x_1\,,y\| + \|x_2\,,y\|$

So the $\|.,.\|$ *is said to be* $2 - normed$ *on* W

and $(W\,, \|.,.\|)$ *is called* $2 - normed$ *spaces* . Every 2-inner product space (2-pre-Hilbert space) is 2-normed space [5] and we can define the 2-norm as

$\|x,z\| = \sqrt{(x\;,x|\,z)}$

After that 2- pre- Hilbert space structures were developed by several researcher as Frees et al, Gahler, Cho et al, Diminnie et al and Gunawan et al [6] [7] [8] [9] [10] [11] [1] [2] [12] [13]. The definition of 2- linear operator and many properties of 2- linear operators were mention by P. K. Harikrishnan , K. T. Ravindran and Giles [14] [15]. Also, many authors discussed bounded linear operator [16] [17] [18]. Finally, there are many researchers discuss semi-normed space and its properties [19].

The definitions of Cauchy sequences and convergent sequence was given as below [20] :

Let $\{x_n\}$ be a sequence in 2-inner product space, we say

1- $\{x_n\}$ convergent to x if and only if $\lim_{n\to\infty} \|x_n - x, b\| = 0 \qquad \forall b.$

2- $\{x_n\}$ Cauchy sequence if and only if $\lim_{n,m\to\infty} \|x_n - x_m, b\| = 0, \qquad \forall\, b \in X.$

In [16] define of continuous function as Let $T: (X, \|,\|) \to (X, \|,\|)$. We say T is continuous at x_0 if $\forall\, \epsilon > 0 \;\exists\; \delta > 0$ such that if $\|x - x_0, b\| < \delta$, then $\|T(x) - T(x_0), b\| < \epsilon \;\forall\, b$.

Also, in [21] there is another definition of continuous function was defined as let $f: X \to X$, we say f is continuous if and only if for every $\{x_n\} \to x$, then $f(x_n) \to f(x)$.

2. Main Results.

We will give a new definition of 2-semi- bounded operators define on 2-inner product space.

Definition 2.1 Let X be 2-inner product space and $T: X \to X$ be a linear then T is 2-semi-bounded function if $\exists\; c \geq 0$ s.t.

$|< Tx, Tx, b >| \leq c^2 |< x, x, b >| \qquad x, b$ independent ….(*)

Define $\|T\|_{\mathcal{B}} = \inf\{c: where\; c\; satisty\; (*)\}$.

Definition 2.2 Let X be 2-inner product space and define the space

$B_s(X) = \{T: X \to X \;|T is\; linear\; and\; T\; satisfy\; (*)\}$ is called 2-Semi- bounded space.

Before discuss about anything in this space we have to show this space as vector space

Theorem 2.3 $B_s(X)$ is vector space with respect to the usual addition and multiplication operation of functions.

Proof *clearly* $B_s(X) \neq \emptyset$ *because* $I \in B_s(X)$ *, Now, Let* $T_1, T_2 \in B_s(X)$ *. So there exist* $c_1, c_2 \geq 0$ *satisfy* $|< T_1x, T_1x, b >| \leq c_1^2 |< x, x, b >| \;\forall\, x\,, b$ *independent and* $|< T_2x, T_2x, b >| \leq c_2^2 |< x, x, b >| \quad \forall\, x, b$ *independent*

$\Rightarrow\ T_1, T_2$ *linear and bounded*

$\Rightarrow\ T_1 + T_2$ *linear*

To Complete the proof of this theorem we have to prove $T_1 + T_2$ *bounded*

$|< T_1 + T_2(x), T_1 + T_2(x), b >|$

$= |< T_1x, T_1x, b > + < T_1x, T_2x, b > + < T_2x, T_1x, b > + < T_2x, T_2x, b >|$

$\leq |< T_1x, T_1x, b >| + 2|< T_1x, T_2x, b >| + |< T_2x, T_2x, b >|$

$\leq c_1^2 < x, x, b > + 2\sqrt{< T_1x, T_1x, b >}\sqrt{< T_2x, T_2x, b >} + c_2^2 < x, x, b > \quad \forall\, x, b$ *independent*

$\leq c_1^2 < x, x, b > + 2c_1c_2 < x, x, b > + c_2^2 < x, x, b > \quad \forall\, x, b$ *independent*

Hence $|< T_1 + T_2(x), T_1 + T_2(x), b >| \leq (c_1 + c_2)^2 < x, x, b > \quad \forall\, x, b$ *independent*

Thus $T_1 + T_2$ *bounded*

So $T_1 + T_2 \in B_s(X)$

To prove $\propto T \in B_s(X)$ *where* $\propto \in R$.

Let $T \in B_s(X)$ *where* $\alpha \in R$,

So, there is $c \geq 0 \quad s.t.\ |< Tx, Tx, b >| \leq c^2 |< x, x, b >| \quad x, b$ *independent*

Since T *is linear* $\Rightarrow\ \propto T$ *is linear*

Only we have to prove more $\propto T$ *is bounded*

$|<\propto Tx, \propto Tx, b >| = |\propto|^2 |< Tx, Tx, b >|$

$\leq |\propto|^2 c^2 |< x, x, b >| \quad x, b$ *independent*

$\Rightarrow\ \propto T$ *is bounded*

$\Rightarrow\ \propto T \in B_s(X)$

The other conditions of vector space are easily satisfied some omit the proves

$\Rightarrow\ B_s(X)$ *is vector space*

Proposition 2.4 If $T \in \mathcal{B}(X)$, $\ell = \sup_{x,b}\left\{\sqrt{\frac{<T(x),T(x),b>}{<x,x,b>}} :\ x, b \text{ independent}\right\}$, then $\ell = \|T\|_{\mathcal{B}_s}$

Proof Let $\ell = \sup_{x,b}\left\{\sqrt{\frac{<T(x),T(x),b>}{<x,x,b>}} :\ x, b\ \textit{independent}\right\}$

So, c is upper bounded for the set $\left\{\sqrt{\frac{<T(x),T(x),b>}{<x,x,b>}} :\ x, b\ \textit{independent}\right\}$

$\Rightarrow\ \ell \leq c \quad \forall\, c$ *satisfy (*)*

$$\Rightarrow\ \boldsymbol{\ell} \leq \|\boldsymbol{T}\|_{\mathcal{B}_s} \qquad (1)$$

By definition of ℓ.

$\Rightarrow\ \frac{< T(x), T(x), b >}{< x, x, b >} \leq \ell^2 \quad x, b$ *independent*

$\Rightarrow\ \ell \in \{c : c\ satisfy\ (*)\}$

$\Rightarrow \|T\|_{\mathcal{B}_s} = \inf\{c\} \le \ell \Rightarrow \ell = \|T\|_{\mathcal{B}_s}$

Corollary 2.5 Let $T \in B_s(X)$ then $|< Tx, Tx, b >| \le \|T\|_{B_s}^2 |< x, x, b >|$ for any x, b independent

Proof if x, b independent then we have $\sqrt{\frac{<Tx,Tx,b>}{<x,x,b>}} \le \|T\|_{B_s}^2$

$\Rightarrow \left|\sqrt{\frac{<Tx,Tx,b>}{<x,x,b>}}\right| \le \|T\|_{B_s}^2$

Then $|< Tx, Tx, b >| \le \|T\|_{B_s}^2 |< x, x, b >|$

After we proved $B_s(X)$ as a vector space and we define on this space a function. We will show that this space is Semi-normed space.

Lemma 2.6 the space $B_s(X)$ is normed space where $\|T\|_{\mathcal{B}_s} = \inf\{c: where\ c\ satisfy\ (*)\}$

Proof since $\frac{<Tx,Tx,b>}{<x,x,b>} \ge 0 \quad \forall x, b\ indep.$

$\Rightarrow \|T\|_{B_s} \ge 0 \qquad \forall\, T \in B_s(X)$

2) $\|\propto T\|_{B_s} = \sup\{\sqrt{\frac{<\propto Tx,\propto Tx,b>}{<x,x,b>}}\ where\ x, b\ indep\}$

$= |\propto|\ \sup\{\sqrt{\frac{<Tx,Tx,b>}{<x,x,b>}}\ where\ x, b\ indep\}$

$= |\propto|\ \|T\|_{B_s} \qquad \forall\, T \in B_s(X)\quad and\ \forall \alpha \in R$

3) $\|T_1 + T_2\|_{B_s} = \sup\{\sqrt{\frac{<T_1+T_2x,T_1+T_2x,b>}{<x,x,b>}}\ ,\ x, b\ independent\}$

$= \sup\{\sqrt{\frac{<T_1x,T_1x,b>+<T_1x,T_2x,b>+<T_2x,T_1x,b>+<T_2x,T_2x,b>}{<x,x,b>}}\ ,\ x, b\ independent\}$

$\le \sup\{\sqrt{\frac{\|T_1\|_{B_s}^2<x,x,b>+2\,\|T_1\|_{B_s}\|T_2\|_{B_s}<x,x,b>+\|T_2\|_{B_s}^2<x,x,b>}{<x,x,b>}}\ ,\ x, b\ independent\}$ (By using Cauchy-Schwartz inequality)

$= \sup\{\sqrt{\left(\|T_1\|_{B_s} + \|T_2\|_{B_s}\right)^2}\ , x\ and\ b\ independent\}$

$= \|T_1\|_{B_s} + \|T_2\|_{B_s}$

$\Rightarrow \left(B_s(X), \|.\|_{B_s}\right)$ is semi normed space.

Proposition 2.7 Let $TS \in B_s(X)\ \ \forall\ S, T \in B_s(X)$.

Proof Let *x, b is independent*

Case 1) if S(x) = 0 then this inequality $< T(Sx), T(Sx), b > \le c_T^2 < Sx, Sx, b >$ *is true*

So, TS is bounded

Case 2) if $S(x) \ne 0$

If $S(x), x$ *dependent Then* $S(x), b$ *are independent also* $S(x) = \alpha x$

$< T(Sx), T(Sx), b > \le c_T^2 < Sx, Sx, b > = c_T^2 < \alpha x, \alpha x, b > = c_T^2 \alpha^2 < x, x, b >$

$\Rightarrow$ *TS is bounded*

Case 3) If x, S(x) independent. Then

$< TSx, TSx, b > \le c_T^2 < Sx, Sx, b >$

$\le c_T^2 c_S^2 < x, x, b >$

Thus $TS \in B_s(X)$

Theorem 2.8 Let X *be 2-Hilbert space, then* $B_s(X)$ *be complete semi normed space.*

Proof Let $\{T_n\}$ be Cauchy sequence in $B_s(X)$

(i.e) $\|T_n - T_m\|_{B_s(X)} \to 0$

We need to show that. $\{T_n\} \to T$ and $T \in B_s(X)$

Case 1: if x,b independent

Since $\{T_n\}$ Cauchy sequence

So, $|< T_n(x) - T_m(x), T_n(x) - T_m(x), b >|$

$= |< T_n - T_m(x), T_n - T_m(x), b >|$

$\leq \|T_n - T_m\|_{B_s(X)} < x, x, b > \to 0 \quad n, m \to \infty \qquad \forall b$

$\therefore$ $\{T_n(x)\}$ Cauchy sequence

Thus $\{T_n(x)\}$ convergent in X because it is complete.

Say $\{T_n(x)\} \to T(x)$

T.P. $T \in B_s(X)$

First prove T is linear

$\because$ $\{T_n(x_1 + x_2)\} \to T(x_1 + x_2)$ ……(1)

Also $\{T_n(x_1) + T_n(x_2)\} \to T(x_1) + T(x_2)$ ……(2)

Because $T_n(x_1 + x_2) = T_n(x_1) + T_n(x_2)$

From (1) and (2) and unique of convergent p

We gave $T(x_1 + x_2) = T(x_1) + T(x_2)$

$\{T_n(\propto x)\} \to T(\propto x)$ and $\{\propto T_n(x)\} \to \propto T(x)$

Because $\{T_n(\propto x)\} = \{\propto T_n(x)\}$

$\Rightarrow$ $T(\propto x) = \propto T(x)$

T is linear.

T.P. $T \in B_s(X)$

$< Tx, Tx, b > = < Tx - T_n x + T_n x, Tx - T_n x + T_n x, b >$

$< T_n x - Tx, T_n x - Tx, b > + < T_n x - Tx, T_n x, b > + < T_n x, T_n x - Tx, b > + < T_n x, T_n x, b > \leq 0 + 0 + 0 + c^2 = c^2$

$\Rightarrow$ $T \in B_s(X)$

$\Rightarrow$ $B_s(X)$ is semi complete normed space

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