Journal of Al-Qadisiyah for Computer Science and Mathematics Vol. 15(1) 2023 , pp Math. 52–60

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Q_{P}_ Continuous Multifunctions and $Q_{P}-$ Closed Multifunctions

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ARTICLE INFO

Article history:
Received: 05/01/2023
Rrevised form: 18/02/2023
Accepted : 20/02/2023
Available online: 31/03/2023

Keywords:

Q_{P}_open set , $Q_P-closed$, $Q_P-closure$, $Q_P-interior$ upper $Q_P-continuous$, lower $Q_P-continuous$, $Q_P-O-closed multifunctions$, contra $Q_P-open\ multifunctions$.

ABSTRACT

In this paper, by means of Q_{P}_open and $Q_P-closed$ sets, we introduce we have provided some basic definitions that we need in the research in addition to the definition of, Q_P-continuous multifunctions and investigate certain ramifications of Q_P-continuous multifunctions, along with their several properties, characterizations and mutual relationships. Further we introduce new types of multifunctions, called $Q_P-O-closed$ multifunctions via Q_P-open sets. The relationship between these multifunctions and Q_{P}_ continuous multifunction are studied .

MSC..

https://doi.org/10.29304/jqcm.2023.15.1.1167

1.Introduction

Continuity is an important concept to study and investigation of classical point set topology theory, generalization of this concept can be made by using weaker forms of open groups . In 1965, O. Njaste[**11**] *introduced the concept and definition of α-open and* $\boldsymbol{\alpha-closed}$*sets in topology . In 1970 N.* **Levine**[**10**] *introduced and studied the notion of Generalized closed sets* ($\boldsymbol{g-closed}$) ***in*** *topology . In 1982*[**2**]*, gives the concept of term* $\boldsymbol{pre-open}$ ***and*** $\boldsymbol{pre-closed}$ *sets and studies its properties . In* **1993** [**5**] *introduced the notion of Generalized α-closed* sets in topology ($\mathbf{g\alpha-closed\ set}$) *. In the year* **1994** [**6**], *introduced and investigated the notion of* $\boldsymbol{\alpha g-closed}$ *sets and* $\boldsymbol{\alpha g-open}$ *sets by involving* $\boldsymbol{\alpha-closed}$ *and* $\boldsymbol{g-closed}$ *sets. In* **1996** [**7**] *, obtained a new class of sets in a topological space, known as* $\boldsymbol{gp-closed}$ *sets and* $\boldsymbol{pg-clsed}$.

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In 2022 [**8**] *, introduced and study new class of sets is called* $\boldsymbol{Q_P - closed}$ *set and* $\boldsymbol{Q_P - open}$ *sets in a topological spaces. The theory of multivalued functions (multifunction) was first codified by* ***ber***. [**13**] .

The aim of this paper is to a new form of continuous multivalued functions called $\boldsymbol{Q_P}$ *−continuous multifunctions , and a new type of the multivalued functions called* $\boldsymbol{Q_P - O - closed}$ *multifunctions are introduced and studied by using* $\boldsymbol{Q_P - open}$ *and* $\boldsymbol{Q_P - closed}$ *sets.*

2-Preliminaries and Definitions

Throughout the present paper , X, Z *and* Y *are always topological spaces (short :* TO.S*) . Let* ω *be a subset of a* TO.S X*, We denote the interior and the closure of a set* ω *by (*$mt(\omega)$ *and* $Cl(\omega)$*) , respectively .*$\mathcal{O}(X)$ *is the of all open sets in* TO.S X*. .*$\mathrm{C}(X)$ *is the of all closed sets in* TO.S X *.the complement of a subset* ω *in* X *denoted by* ω^{C} *. By a topological multifunction* $\Gamma: X \rightarrow Y$ *, it is meant a multifunction from a* TO.S X *to another* TO.S *. We will use the symbol ♦ to indicate end of the proof .*

***Definition* 2. 1 :** *Let* X *and* Y *be a* TO.S *:*

1) *The corresponding* $\Gamma: X \rightarrow Y$ *is called a multivalued function if given any* $X \ni x$ *, then* $\Gamma(x)$ *will be a nonempty subset of* Y [1].

2) *Let* $\Gamma: X \rightarrow Y$ *be a multivalued and* $X \supseteq \omega, Y \supseteq \upsilon$ *then* $\Gamma(\omega) = \cup \{\Gamma(x) : \omega \ni x\}$ *is called image of the set* ω *,* $\Gamma^{+1}(\upsilon) = \{X \ni x : \upsilon \supseteq \Gamma(x)\}$ *is called the upper inverse of the set* υ*,* $\Gamma^{-1}(\upsilon) = \{X \ni x : \Gamma(x) \cap \upsilon \neq \emptyset\}$ *is called the lower inverse of the set* υ [13] ,

***Remark 2.2*[12] :** *Let* $\Gamma: X \rightarrow Y$ *be a multivalued function and* $Y \supseteq \upsilon$ *,then* $\Gamma^{+1}(\upsilon) = (\Gamma^{-1}(\upsilon))^{C}$ *and* $\Gamma^{-1}(\upsilon) = \left(\Gamma^{+1}(\upsilon)\right)^{C}$.

***Definition* 2. 3 :** *A subset* ω *of a* TO.S X *is called*

1) *Preclosed [preopen] set if* $\omega \supseteq Cl(mt(\omega))[\omega \subseteq mt(Cl(\omega))]$ [2].

2) $\alpha - closed$ *[*$\alpha - open$*] set if* $\omega \supseteq Cl(mt(Cl(\omega)))$ $[\omega \supseteq mt(Cl(mt(\omega)))]$ [11] .

3) $g - closed$ *set if* $\upsilon \supset Cl(\omega)$ *whenever* $\upsilon \supset \omega$ *and* $\mathcal{O}(X) \ni \upsilon$ *and* (ω^{C}) *is* $g - open$ [10].

4) $gp - closed$ *set if* $\omega \supseteq Cl_P(\omega)$ *whenever* $\upsilon \supset \omega$ *and* $\mathcal{O}(X) \ni \upsilon$*, where* $Cl_P(\omega)$ *perclosuer of* ω *(the intersection of all perclosed sets containing* ω *is called perclosure of* ω *) and* (ω^{C}) *is* $gp - open$ [7].

***Definition 2.4*[8]:** *A subset* ω *of a TO.S X is called* $Q_P - closed$ *[*$Q_P - open$*] set if* $\omega \supseteq Cl_{gp}(ınt(\omega))$*, where* $Cl_{gp}(ınt(\omega))$ *is gp −closure of* $ınt(\omega)$ $[\omega \subseteq ınt_{gp}(Cl(\omega))$ *,where* $Cl_{gp}(ınt(\omega))$ *gp interior of* $Cl(\omega)]$ $Q_P C(X)[Q_P O(X)]$ *is the set of all* $Q_P - closed$ $[Q_P - open$ *sets in a* TO.S X.

***Definition* 2. 5:**

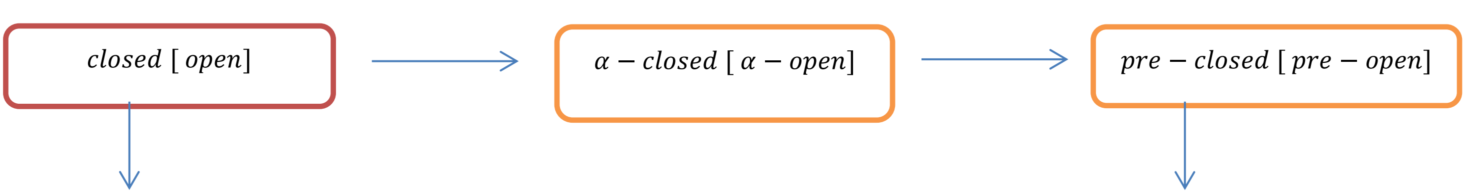
1) $Cl_{QP}(\omega)$ *is called* Q_P *− closure of* ω *in a* TO.S X *if* $Cl_{QP}(\omega) = \cap \{\upsilon : \upsilon \supseteq \omega$ *and* $Q_P C(X) \ni \upsilon\}$[8] .

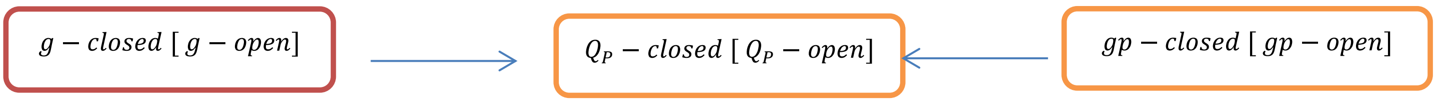
2) $mt_{QP}(\omega)$ *is called* Q_P *− interior of* ω *in a* TO.S X *if* $mt_{QP}(\omega) = \cup \{\upsilon : \upsilon \subseteq \omega$ *and* $Q_P O(X) \ni \upsilon\}$[8] .

3) *We say that a set* υ *of a* TO.S X *is a* Q_P *− neighborhood of a point x if* υ *contains a* Q_P *− open set to which x belongs .*

***Remark* 2. 6 [8]**

The diagram below represents the relationship between the Q_P *− closed type sets and the other closed sets :*





Remark 2.7 [8]

1) In a TO.S X , arbitrary intersection of $Q_P-closed$ sets is $Q_P-closed$ set.

2) In a TO.S X, arbitrary union of Q_P $-$ open sets is Q_P $-$ open set .

3) In a TO.S X, union of two $Q_P-closed$ sets need not $Q_P-closed$ set.

4) In a TO.S X, intersection of two Q_P $-$ open sets need not Q_P $-$ open set.

5) If $Cl_{QP}\ (\omega)=\omega$, then $Q_PC(X)\ni\omega$.

6) If $mt_{QP}\ (\omega)=\omega$, then $Q_PO(X)\ni\omega$

Definition 2.6

1) Let $(D,\geq)$ be a directed set . A function $\chi: D\rightarrow X$ is called net in X , denoted by $(\chi_\alpha : D\ni\alpha)$[1] .

2) The net χ_α is eventually in ω if there exists $\alpha_0\in I$ such that $\omega\ni\chi_\alpha$, for each $\alpha\geq\alpha_0$ [1] .

3) The net χ_α in a TO.S X is called converges to a point x , if χ_α is eventually in every neighborhood of x [1] .

4) The net χ_α in a TO.S X is called $Q_P-converges$ to a point x , if χ_α is eventually in every $Q_P-neighborhood$ of x .

Definition 2.7

1) A TO.S X is called Q_P-H space if for each two points $x_1\neq x_2$ in X ,there exist $Q_PO(X)\ni\upsilon_1,\upsilon_2$ such that $\upsilon_1\ni x_1$, $\upsilon_2\ni x_2$ and $\upsilon_2\cap\upsilon_2=\emptyset$.

2) A subset ω of a TO.S X is said to be $Q_P-compact$ relative to TO.S X if every cover of ω by Q_P-open sets of X has a finite sub cover.

3) Let X be a TO.S , we say that X is $Q_P-disconnected$ if it is the union of two non-empty Q_P-open sub sets , otherwise is $Q_P-connected$.

Definition 2.8 [12]: Let $(D,\geq)$ be a directed set , $\{\Gamma_\alpha : D\ni\alpha\}$ be a net of multivalued function $\Gamma_\alpha : X\rightarrow Y$ and Γ a multivalued function on X into Y, , $\{\Gamma_\alpha : D\ni\alpha\}$ is said to be

1) Upper pointwise convergent to Γ, if for each $X\ni x$ and each open set $Y\supseteq\omega$ containing $\Gamma(x)$, there exists $D\ni\beta$ such that $\Gamma^{+1}{}_\alpha(\omega)\ni x$,for each $\alpha\geq\beta$.

2) Lower pointwise convergent to Γ, if for each $X\ni x$ and each open set $Y\supseteq\omega$ meeting $\Gamma(x)$, there exists $D\ni\beta$ such that $\Gamma^{-1}{}_\alpha(\omega)\ni x$,for each $\alpha\geq\beta$.

3) Pointwise convergent if it is both upper pointwise convergent and lower pointwise convergent .

Definition 2.9 Let $\Gamma : X\rightarrow Y$ be a topological multifunction , then

1) Γ is called upper , (lower) continuous (short : $U.C$, $L.C$) , at a point $X\ni x$, if for each open set υ in Y with $\upsilon\supseteq\Gamma(x)$ $(\Gamma(x)\cap\upsilon\neq\phi)$ there exists open set ω containing x such that $\upsilon\supseteq\Gamma(\omega)(\Gamma(z)\cap\upsilon\neq\phi$ for all $z\in\omega)$, Γ is said to be $U..C(L.C$, if Γ is $U.C(L.C)$ at each point $X\ni x$. Γ is called continuous, if it is $U.C$ and $L.C$ at each point $X\ni x$. [12]

2) Γ is called $\big(upper-per\ (lower-per)\ continuous\ (short: U.P.C\ , L.P.C)\big)$, at a point $X \ni x$, if for each open set υ in Y with $\upsilon \supseteq \Gamma(x)$ $(\Gamma(x) \cap \upsilon \neq \phi)$ there exists $per-open$ set ω containing x such that $\upsilon \supseteq \Gamma(\omega)(\Gamma(z) \cap \upsilon \neq \phi$ for all $z \in \omega)$, Γ is said to be $U.P.C(L.P.C)$, if Γ is $U.P.C(L.P.C)$ at each point $X \ni x$. Γ is called $per-continuous$, if it is $U.P.C$ and $L.P.C$ at each point $X \ni x$. [13]

3) Γ is called $\big(upper-\alpha\ (lower-\alpha)continuous\ (short: U.\alpha.C\ , L.\alpha.C)\big)$, at a point $X \ni x$, if for each open set υ in Y with $\upsilon \supseteq \Gamma(x)$ $(\Gamma(x) \cap \upsilon \neq \phi)$ there exists $\alpha-open$ set ω containing x such that $\upsilon \supseteq \Gamma(\omega)(\Gamma(z) \cap \upsilon \neq \phi$ for all $z \in \omega)$, Γ is said to be $U.\alpha.C(L.\alpha.C)$, if Γ is $U.\alpha.C(L.\alpha.C)$ at each point $X \ni x$. Γ is called $\alpha-continuous$, if it is $U.\alpha.C$ and $L.\alpha.C$ at each point $X \ni x$. [14]

4) Γ is called surjection if for each $Y \ni y$ there exists an element $X \ni x$ such that $\Gamma(x) \ni y$. [9]

5) Γ is called point closed (connected , compact) if for each $X \ni x$, $\Gamma(x)$ is closed (connected , compact) in Y. [3]

3. Q_P –continuous multifunctions

In the following section , the new concept of –continuous multifunctions , introduced and studied and several characterization and properties of these forms are proved .

Definition 3.1 Let $\Gamma: X \to Y$ be a topological multifunction ; then

1) Upper Q_P – continuous $(U.Q_P.C)$ at a point $X \ni x$, if for each open set υ in Y with $\Gamma(x) \supseteq \upsilon$ there exists $Q_P O(X) \ni \omega$ containing x such that $\upsilon \supseteq \Gamma(\omega)$. Γ is said to be $U.Q_P.C$, if Γ is $U.Q_P.C$ at each point $X \ni x$.

2) Lower Q_P – continuous $(L.Q_P.C)$ at a point $X \ni x$, if for each open set υ in Y with $\Gamma(x) \cap \upsilon \neq \phi$ there exists $Q_P O(X) \ni \omega$ containing x such that $\Gamma(z) \cap \upsilon \neq \phi$ for all $z \in \omega$. Γ is said to be $L.Q_P.C$, if Γ is $L.Q_P.C$ at each point $X \ni x$.

3) Q_P –continuous $(Q_P.C)$ at a point $X \ni x$, if it is both $(U.Q_P.C)$ and $(L.Q_P.C)$ at x . Γ is called $Q_P.C$, if it is $Q_P.C$ at each point $X \ni x$.

Example 3.2 : Let $X = \{e_1, e_2\}$, $T_X = \{X, \phi, \{e_1\}, \{e_2\}\}$ be a TO.S on X . $Y = \{h_1, h_2, h_3\}$, $T_Y = \{Y, \phi, \{h_1\}, \{h_2, h_3\}\}$ be a TO.S on Y. A multifunction $\Gamma: X \to Y$ defines by Γ by $\Gamma(e_1) = h_1$, and $\Gamma(e_2) = \{h_2, h_3\}\}$. Γ is $U.Q_P.C$ and $L.Q_P.C$.

Theorem 3.3 Let $\Gamma: X \to Y$ be a topological multifunction ; then the following statements are equivalent :

1) Γ is $L.Q_P.C$.

2) $Q_P O(X) \ni \Gamma^{-1}(\upsilon)$, for each open set υ in a TO.S Y .

3) $Q_P C(X) \ni \Gamma^{+1}(\kappa)$, for each closed set κ in a TO.S Y.

4) $Cl(\Gamma(\omega)) \supseteq \Gamma\left(Cl_{QP}(\omega)\right)$, for each set ω in a TO.S X

5) $\Gamma^{+1}(Cl(\beta)) \supseteq Cl_{QP}\left(\Gamma^{+1}(\beta)\right)$, for each set β in a TO.S Y .

6) $mt_{QP}\left(\Gamma^{-1}(\beta)\right) \supseteq \Gamma^{-1}(mt(\beta))$, for each set β in a TO.S Y .

7) For each $X \ni x$ and for each open set υ in a TO.S Y such that $\Gamma(x) \cap \upsilon \neq \phi$, there exists a $Q_P - neighborhood$ ω of x such that if $\omega \ni u$; then $\Gamma(u) \cap \upsilon \neq \phi$.

8) For each $X \ni x$ and for each net χ_α which $Q_P - converges$ to x in X and for each open set υ in Y, such that $\Gamma^{-1}(\upsilon) \ni x$; then the net χ_α is eventually in $\Gamma^{-1}(\upsilon)$.

Proof : 1) ⇒ 2) Let $\Gamma^{-1}(\upsilon) \ni x$ in a TO.S X , υ any open subset of of a TO.S Y , from 1) we get there exists $Q_P O(X) \ni \omega$ such that $\Gamma^{-1}(\upsilon) \supseteq \omega \ni x$; therefore , $Q_P O(X) \ni \Gamma^{-1}(\upsilon)$.

2)⇒3) *Let* κ *be a closed set in a* TO.S Y *; then* κ^C *is open set in* Y *from* 2) *we obtain* $Q_P O(X) \ni \Gamma^{-1}(\kappa^C)$ *, since* $Q_P O(X) \ni \Gamma^{-1}(\kappa^C) = (\Gamma^{+1}(\kappa))^C$ *; then* $Q_P C(X) \ni \Gamma^{+1}(\kappa)$.

3) ⇒4) *Let* $X \supseteq \omega$ *; then* $\Gamma(\omega)$ *is subset of a* TO.S Y *and* $\mathrm{Cl}(\Gamma(\omega))$ *is closed set in* Y *. since* Γ *is* $L.Q_P.C$ *; then* $Q_P C(X) \ni \Gamma^{+1}(\mathrm{Cl}(\Gamma(\omega)))$ *, in view fact that* $\mathrm{Cl}(\Gamma(\omega)) \supseteq \Gamma(\omega)$ *implies that to* $\Gamma^{+1}(\mathrm{Cl}(\Gamma(\omega))) \supseteq \omega$ *, It follows that* $\mathrm{Cl}_{QP}\left(\Gamma^{+1}(\mathrm{Cl}(\Gamma(\omega)))\right) \supseteq \mathrm{Cl}_{QP}(\omega)$ *; therefore ,* $\mathrm{Cl}(\Gamma(\omega)) \supseteq \Gamma\left(\mathrm{Cl}_{QP}\left(\Gamma^{+1}(\mathrm{Cl}(\Gamma(\omega)))\right)\right) \supseteq \Gamma(\mathrm{Cl}_{QP}(\omega))$.

4) ⇒5) *Let* $Y \supseteq \beta$*,then* $Q_P C(X) \ni \mathrm{Cl}_{QP}\left(\Gamma^{+1}(\beta)\right)$ *, from this we get* $\Gamma^{+1}(\beta) \supseteq \mathrm{Cl}_{QP}\left(\Gamma^{+1}(\beta)\right)$*, then* $\left(\Gamma\left(\Gamma^{+1}(\beta)\right)\right) \supseteq \Gamma\left(\mathrm{Cl}_{QP}\left(\Gamma^{+1}(\beta)\right)\right)$ *and* $\mathrm{Cl}(\beta) \supseteq \mathrm{Cl}\left(\Gamma\left(\Gamma^{+1}(\beta)\right)\right)$*, this is implies* $\Gamma^{+1}(\mathrm{Cl}(\beta)) \supseteq \Gamma^{+1}\left(\mathrm{Cl}\left(\Gamma\left(\Gamma^{+1}(\beta)\right)\right)\right) \supseteq \mathrm{Cl}_{QP}\left(\Gamma^{+1}(\beta)\right)$.

5)⇒6) *let* β *any set of a* TO.S Y*, then* $Q_P O(X) \ni mt_{QP}(\Gamma^{-1}(\beta)$ *and* $Q_P C(X) \ni (mt_{QP}(\Gamma^{-1}(\beta)))^C = \mathrm{Cl}_{QP}(\Gamma^{-1}(\beta))^C = \mathrm{Cl}_{QP}\left(\Gamma^{+1}(\beta^C)\right)$

Then$\left(\Gamma^{+1}\mathrm{Cl}(\beta^C)\right) \supseteq \mathrm{Cl}_{QP}\left(\Gamma^{+1}(\beta^C)\right)$ *and* $\left(\Gamma^{+1}\mathrm{Cl}(\beta^C)\right) = \Gamma^{+1}((mt\beta))^C = (\Gamma^{-1}(mt\beta)^C$ *from this we get* $mt_{QP}\left(\Gamma^{-1}(\beta)\right) \supseteq \Gamma^{-1}(mt(\beta))$.

6)⇒7) *Let* $X \ni x$ *and* υ *be any open set in a* TO.S Y *, such that* $\Gamma(x) \cap \upsilon \neq \phi$*, then* $mt_{QP}\left(\Gamma^{-1}(\upsilon)\right) \supseteq \Gamma^{-1}(mt(\upsilon))$*, and* $mt_{QP}\left(\Gamma^{-1}(\upsilon)\right) \supseteq \Gamma^{-1}(\upsilon)$ *. It follows that* $Q_P O(X) \ni \Gamma^{-1}(\upsilon)$*, we take* $\omega = \Gamma^{-1}(\upsilon)$ *is* $Q_P -$ *neighborhood, then for each* $\omega \ni u$ *,* $\Gamma(u) \cap \upsilon \neq \phi$*, .*

7)⇒8) *Let* $X \ni x$ *and let a net* χ_α *which* $Q_P -$ *converges to* x *in a* TO.S X*,* υ *any open set in a* TO.S Y*, such that* $\Gamma^{-1}(\upsilon) \ni x$ *from* 7) *there exists* $Q_P O(X) \ni \omega$ *and containing* x*,* $\Gamma(u) \cap \upsilon \neq \phi$ *for each* $\omega \ni u$*, then* $\omega = \Gamma^{-1}(\upsilon)$*, since* χ_α *is* $Q_P -$ *converges to* x *in a* TO.S X*, then* χ_α *is eventually in every* $Q_P -$ *neighborhood of a point x, therefore , the net* χ_α *is eventually in* $\Gamma^{-1}(\upsilon)$.

8)⇒1) *Suppose that* 1) *is not true , there exists a point* $X \ni x$ *and* υ *an open set in a* TO.S Y *with* $\Gamma^{-1}(\upsilon) \ni x$ *such that for set* $Q_P O(X) \ni \omega$ *containing* x *,* $\Gamma^{-1}(\upsilon) \not\ni \omega$ *. Let* $\omega \ni \chi_\omega$ *and* $\Gamma^{-1}(\upsilon) \not\ni \chi_\omega$ *for each* $Q_P O(X) \ni \omega$ *and containing x, therefore , the net* χ_ω $Q_P -$ *converges to a point x , but* χ_ω *is not eventually in* $\Gamma^{-1}(\upsilon)$*. This is contradiction , hence* Γ *is* $L.Q_P.C.$ ♦

<u>Theorem 3.4</u> *Let* $\Gamma : X \rightarrow Y$ *be a topological multifunction ; then the following statements are equivalent :*

1) Γ *is* $U.Q_P.C.$

2) $Q_P O(X) \ni \Gamma^{+1}(\upsilon)$ *, for each open set* υ *in a* TO.S Y .

3) $Q_P C(X) \ni \Gamma^{-1}(\kappa)$ *, for each closed set* κ *in a* TO.S Y.

4) $\Gamma^{-1}(\mathrm{Cl}(\beta)) \supseteq \mathrm{Cl}_{QP}\left(\Gamma^{-1}(\beta)\right)$ *, for each set* β *in a* TO.S Y .

5) $mt_{QP}\left(\Gamma^{+1}(\beta)\right) \supseteq \Gamma^{+1}(mt(\beta))$ *, for each set* β *in a* TO.S Y .

6) *For each* $X \ni x$ *and for each open set* υ *in a* TO.S Y *such that* $\upsilon \supseteq \Gamma(x)$*, there exists a* $Q_P -$ *neighborhood* ω *of* x *such that if* $\omega \ni u$ *; then* $\upsilon \supseteq \Gamma(u)$.

7) *For each* $X \ni x$ *and for each net* χ_α *which* $Q_P -$ *converges to* x *in* X *and for each open set* υ *in* Y*, such that* $\Gamma^{+1}(\upsilon) \ni x$ *; then the net* χ_α *is eventually in* $\Gamma^{+1}(\upsilon)$.

Proof : *The details of the proof are similar to the details in the previous theorem .* ♦

<u>Remark 3.5</u> *The following implication are hold*

1) Γ *is* $L.C \Rightarrow \Gamma$ *is* $L.\alpha.C \Rightarrow \Gamma$ *is* $L.P.C \Rightarrow \Gamma$ *is* $L.Q_P.C$

2) Γ *is* $U.C \Rightarrow \Gamma$ *is* $U.\alpha.C \Rightarrow \Gamma$ *is* $U.P.C \Rightarrow \Gamma$ *is* $U.Q_P.C.$

While the converses are not true in general as the following

Example 3.6 *: Let $\Gamma:(X,T_X)\rightarrow(Y,T_Y)$ be a topological multifunction, where $X=\{e_1,e_2,e_3\}, T_X=\{X,\phi,\{e_1\}\}$ and $Y=\{h_1,h_2,h_3\}, T_Y=\{Y,\phi,\{h_3\},\{h_1,h_2\}\}$. Define Γ by $\Gamma(e_1)=\Gamma(e_2)=h_1$, and $\Gamma(e_3)=h_3$, then $\{h_3\}$ an open set in a TO.S Y and $\Gamma^{+1}(h_3)=\{e_3\}\in Q_P O(X)$ but $\{e_3\}$ is not per − open set in a TO.S X, therefore Γ is $U.Q_P.C$ but Γ is not $U.P.C$*

Theorem 3.7 *Let $\Gamma: X\rightarrow Y$ be a topological multifunction, if Γ is $U.Q_P.C$, then $G: X\rightarrow \Gamma(X)$ is also $U.Q_P.C$ where $G(x)=\Gamma(x)$.*

Proof: *Let υ any open set in a TO.S Y, then $\Gamma(X)\bigcap\upsilon$ is open set in a space $\Gamma(X)$, $G^{+1}(\Gamma(X)\bigcap\upsilon)=X\bigcap G^{+1}(\upsilon)=\Gamma^{+1}(\upsilon)$, since Γ is $U.Q_P.C$, then $Q_P O(X)\ni\Gamma^{+1}(\upsilon)$, therefore , G is $U.Q_P.C$.♦*

Theorem 3.8 *Let $\{\Gamma_\ell: X\rightarrow Y, D\ni\ell, D \text{ index}\}$ be a net of $U.Q_P.C(L.Q_P.C)$ topological multifunctions , Γ_ℓ be an upper (lower) pointwise convergent to $\Gamma: X\rightarrow Y$ and Γ is point closed , if Y is normal (regular) and for each open set $\mathcal{M}$ of Y with $\Gamma^{-1}(\mathcal{M})\neq\phi\left(\Gamma^{+1}(\mathcal{M})\neq\phi\right)$ and $D\ni\mathcal{J}$, there exists $\mathcal{J}\leq\mathcal{L}$ such that $\Gamma_\mathcal{L}(x)\bigcap\mathcal{M}\neq\phi\,(\mathcal{M}\supseteq\Gamma_\mathcal{L}(x)$, for all $\Gamma^{-1}(\mathcal{M})\ni x\left(\Gamma^{+1}(\mathcal{M})\ni x\right)$, then Γ is also $U.Q_P.C(L.Q_P.C)$.*

Proof: *Suppose Γ is not $U.Q_P.C$ at a point $\mathcal{E}$ in X ; then there exists an open set υ in Y containing $\Gamma(\mathcal{E})$ such that for Q_P- open set ω in X containing $\mathcal{E}$, there exists $\omega\ni\mathcal{E}_0$ and $\upsilon\not\supseteq\Gamma(\mathcal{E}_0)$, that's mean $\Gamma(\mathcal{E}_0)\bigcap\upsilon^c\neq\phi$, by the assumption Y is normal and $Y\ni\Gamma(\mathcal{E})$, $\Gamma(\mathcal{E})$ closed and $\upsilon\supseteq\Gamma(\mathcal{E})$, there exists an open set e such that $\upsilon\supseteq Cl(\mathrm{e})\supseteq\mathrm{e}\supseteq\Gamma(\mathcal{E})$, let's say $\mathcal{M}=Y-Cl(\mathrm{e})$, then $\mathcal{M}\supseteq\upsilon^c$. Where $\{\Gamma_\ell: X\rightarrow Y, D\ni\ell\}$ is upper pointwise convergent to Γ at $\mathcal{E}$. that's mean there exists $D\ni\mathcal{J}$ whereas $\Gamma_\ell(\mathcal{E})\subseteq\mathrm{e}$, for each $\mathcal{J}\leq\ell$, where $\Gamma(\mathcal{E}_0)\bigcap\upsilon^c\neq\phi$, and then $\Gamma(\mathcal{E}_0)\bigcap\mathcal{M}\neq\phi$ implies that $\emptyset\neq\Gamma^{-1}(\mathcal{M})\ni\mathcal{E}_0$. subsequently there exists $\mathcal{J}\leq\mathcal{L}$ whereas $\emptyset\neq\mathcal{M}\bigcap\Gamma_\mathcal{L}(x)$, for all $\Gamma^{-1}(\mathcal{M})\ni x$, implies that $\emptyset\neq\mathcal{M}\bigcap\Gamma_\mathcal{L}(\mathcal{E}_0)$, therefore $\mathrm{e}\not\supseteq\Gamma(\mathcal{E}_0)$, whereas $\Gamma_\mathcal{L}$ is not $U.Q_P.C$ at $\mathcal{E}$, this contradiction ; therefore , Γ is $U.Q_P.C$.*

With the same details, it can be shown that Γ is $L.Q_P.C$

Theorem 3.9 *Let $\Gamma: X\rightarrow Y$ be an $U.Q_P.C$ topological multifunction and point compact , if Y is $Hausdorff-$ space and $\Gamma(x_1)\cap\Gamma(x_2)=\phi$, for all $x_1\neq x_2$ in a TO.S X, then X is Q_P-H space .*

Proof: *Let $x_\mathcal{J}, x_\ell$ be any points of X such that $x_\mathcal{J}\neq x_\ell$, since $\Gamma(x_\mathcal{J}), \Gamma(x_\ell)$ are compact sets , $\Gamma(x_\mathcal{J})\cap\Gamma(x_\ell)=\phi$ in a TO.S Y and Y is a Hausdorff space , there exist two open sets $\upsilon_\mathcal{J}$ containing $\Gamma(x_\mathcal{J})$ and υ_ℓ containing $\Gamma(x_\ell)$ and $\upsilon_\mathcal{J}\cap\upsilon_\ell=\phi$. Since Γ is $U.Q_P.C$, then $Q_P O(X)\ni\Gamma^{+1}(\upsilon_\ell), \Gamma^{+1}(\upsilon_\mathcal{J})$ and $\Gamma^{+1}(\upsilon_\ell)\ni x_\ell$, $\Gamma^{+1}(\upsilon_\mathcal{J})\ni x_\mathcal{J}$ such that $\Gamma^{+1}(\upsilon_\mathcal{J})\cap\Gamma^{+1}(\upsilon_\ell)=\Gamma^{+1}(\upsilon_\mathcal{J}\cap\upsilon_\ell)==\Gamma^{+1}(\phi)=\phi$. By definition (2.7(1)) X is Q_P-H space .♦*

4. Q_P-O- Closed Multifunctions

In this section , the new concept of $\boldsymbol{Q_P}-O-$closed multifunctions , introduced and studied , and several properties of these new concept are proved .

Definition 4.1 *A topological multifunction $\Gamma: X\rightarrow Y$ is said to be $Q_P-O-Closed$ if for each $X\ni x$ and $Y\ni y$, for which $\Gamma(x)\not\ni y$, there exists $Q_P O(X)\ni\omega$ containing x and open set υ in a TO.S Y containing y , such that $\Gamma(x_0)\cap\upsilon=\phi$, for each $\omega\ni x_0$.*

Example 3.6: *Let $X=\{e_1,e_2,e_3\}, T_X=\{X,\phi,\{e_1\},\{e_3\},\{e_1,e_3\},\{e_2,e_3\}\}$ be a TO.S on X . $Y=\{h_1,h_2,h_3\}, T_Y$ is a discrete , TO.S on Y. A multifunction $\Gamma: X\rightarrow Y$ defines by Γ by $\Gamma(e_1)=\{h_1\}$, $\Gamma(e_2)=\{h_1,h_2\}$ and $\Gamma(e_3)=\{h_2\}$. Γ is $Q_P-O-Closed$. but Γ is not $Q_P-Continuous$*

The following theorem give the relationships between the concept Q_P-continuous multifunctions and $Q_P-O-Closed$ multifunction .

Theorem 4.2 *Let $\Gamma: X\rightarrow Y$ be a multifunction from a TO.S X into a Hausdorff TO.S Y , if Γ is $U.Q_P.C$ and point compact, then Γ is $Q_P-O-Closed$.*

Proof: *Suppose $\Gamma: X\rightarrow Y$ is $U.Q_P.C$ on a TO.S X and $\Gamma(x)\not\ni y$, since $\Gamma(x)$ is compact and Y is Hausdorff TO.S , then there exist disjoint two open set υ_1 and υ_2 in a TO.S Y such that $\upsilon_1\ni y$ and $\upsilon_2\supseteq\Gamma(x)$. Since Γ is $U.Q_P.C$, there exists $Q_P O(X)\ni\omega$ such that $\omega\ni x$ implies $\upsilon_2\supseteq\Gamma(\omega)$, then $\Gamma(\omega)\cap\upsilon_1=\phi$. It follows that Γ is $Q_P-O-closed$.♦*

Theorem 4.3 *If $\Gamma: X\rightarrow Y$ is a $Q_P-O-Closed$ multifunction , then $\Gamma(\mathrm{e})$ is closed subset of a TO.S Y , for each e $Q_P-Compact$ relative to a TO.S X .*

Proof: *Assume $\Gamma(\mathrm{e})\not\ni y$, then for each $\mathrm{e}\ni x$, $\Gamma(x)\not\ni y$. By definition (4.1) there exist $Q_P O(X)\ni\omega_x$ containing x and an open set υ_x containing y such that $\Gamma(\omega_x)\cap\upsilon_x=\phi$. The family $\{\omega_x:\mathrm{e}\ni x\}$ is a cover of a set e and $Q_P O(X)\ni\omega_x$ for each x , since e is $Q_P-Compact$*

there exists a finite subset $\mathcal{M}$ *of* e *such that* $\cup \{\omega_x : \mathcal{M} \ni x\} \supseteq \mathrm{e}$. *Take* $\upsilon = \{\upsilon_x : \mathcal{M} \ni x\}$, *then* υ *is an open set containing element* y *and* $\Gamma(\mathrm{e}) \cap \upsilon = \phi$, *this means that* $\Gamma(\mathrm{e})^C \supseteq \upsilon$, *therefore*, $\Gamma(\mathrm{e})$ *is closed in a* TO.S Y. ♦

Definition 4.4 *A topological multifunction* $\Gamma : X \to Y$ *is called contra* $Q_P -$ *open if the image of every* $Q_P -$ *open set in a* TO.S X *is closed set in a* TO.S Y..

Theorem 4.5 *If* $\Gamma : X \to Y$ *is contra* $Q_P -$ *open multifunction such that the inverse image for each point of a* TO.S Y *is* $Q_P - Closed$ *set in a* TO.S X, *then* Γ *is* $Q_P - O - Closed$.

Proof: *Let* $\Gamma(x) \not\ni y$, *then* $\Gamma^{-1}(y) \not\ni x$, *since* $\Gamma^{-1}(y)$ $Q_P - Closed$, *then there exists* υ $Q_P -$ *open set containing* x *such that* $\upsilon \cap \Gamma^{-1}(y) = \phi$. *Since* Γ *is contra* $Q_P -$ *open* , *then* $\Gamma(\upsilon)$ *is closed set in a* TO.S Y , *this is implies that there exists an open set* $\mathcal{M}$ *in a* TO.S Y *such that* $\mathcal{M} \ni y$ *and* $\Gamma(\upsilon) \cap \mathcal{M} = \phi$. *Hence* Γ *is* $Q_P - O - Closed$.♦

Theorem 4.6 *If* $\{\Gamma_\ell : X \to Y, I \ni \ell\}$ *is a family of* $Q_P - O - Closed$ *multifunction from a* TO.S X *into a* TO.S Y, *then* $\Gamma(x) = \cap \; \Gamma_{I \ni \ell}(x)$. *is* $Q_P - O - Closed$, $\Gamma(x) = \cap_{I \ni \ell} \Gamma_\ell(x)$.

Proof: *let* $\Gamma(x_1) \not\ni y_1$, *then there exists* $I \ni \ell_1$ *such that* $\Gamma_{\ell_1}(x_1) \not\ni y_1$, *since* Γ_{ℓ_1} *is* $Q_P - Closed$, *then there exists* $Q_P O(X) \ni \upsilon$ *containing* x_1 *and* ω *open set in* TO.S Y *containing* y_1 *such that* $\Gamma_{\ell_1}(\upsilon) \cap \omega = \phi$, *from this we obtain* $\Gamma(\upsilon) \cap \omega = \phi$, *therefore* , Γ *is* $Q_P - O - Closed$.♦

Theorem 4.7 *If* $\Gamma : X \to Y$ *is* $Q_P - O - Closed$ *topological multifunction* , *then* $\Gamma^{-1}(\upsilon)$ *is* $Q_P - Closed$, *for each* υ *compact set in a* TO.S Y.

Proof: *Let* υ *be arbitrary compact set in a* TO.S Y *and* $\Gamma^{-1}(\upsilon) \not\ni x$, $\Gamma^{-1}(\upsilon) = \{x : \Gamma(x) \cap \upsilon \neq \phi\}$, *then* $\Gamma^{-1}(\ell) \not\ni x$, *for each* $\upsilon \ni \ell$. *Since* Γ *is* $Q_P - Closed$, *there exist* $Q_P O(X) \ni \mathcal{M}_{x\ell}$ *and open set* $\mathcal{M}_\ell$ *such that* $\mathcal{M}_{x\ell} \ni x$ *and* $\mathcal{M}_\ell \ni \ell$ *implies* $\Gamma(\mathcal{M}_{x\ell}) \cap \mathcal{M}_\ell = \phi$. *The family* $\{\mathcal{M}_\ell : \upsilon \ni \ell\}$ *is open cover of* υ, *since* υ *is compact* , *then there exists* $\{\mathcal{M}_{\ell_1}, \mathcal{M}_{\ell_2}, \mathcal{M}_{\ell_3}, \ldots, \mathcal{M}_{\ell_n}\}$ *is finite sub cover such that* $\cup_{i=1}^{n} \mathcal{M}_{\ell_i} \supseteq \upsilon$, *let* $\mathcal{M}_x = \cap_{i=1}^{n} \mathcal{M}_{x\ell_i}$ *such that* $\mathcal{M}_x \ni x$. *If* $\mathcal{M}_x \cap \Gamma^{-1}(\upsilon) = \phi$, *then* $\Gamma^{-1}(\upsilon)$ *is* $Q_P - Closed$. *Now* , *for showing this* , *suppose* $\mathcal{M}_x \cap \Gamma^{-1}(\upsilon) \neq \phi$, *there exists* $\mathcal{M}_x \ni x_0$ *and* $\Gamma^{-1}(\upsilon) \ni x_0$ *then* $\mathcal{M}_{x\ell_i} \ni x_0$, *for each* $i = 1,2, \cdots, n$ *and* $\Gamma(x_0) \cap \upsilon \neq \phi$, *this implies there exists* $\Gamma(x_0) \ni y$ *and* $\upsilon \ni y$, *since* $\cup_{i=1}^{n} \mathcal{M}_{\ell_i} \supseteq \upsilon$, *then* $\mathcal{M}_{\ell_i} \ni y$ *where* $i = j$, *there exists* $\mathcal{M}_{x\ell_j}$ *such that* $\Gamma\left(\mathcal{M}_{\ell x_j}\right) \cap \mathcal{M}_{\ell_j} \neq \phi$, *this is contradiction* , *therefore* $\mathcal{M}_x \cap \Gamma^{-1}(\upsilon) = \phi$ *and* $\Gamma^{-1}(\upsilon)$ *is* $Q_P - Closed$ *set in* X . ♦

Now , *the following theorem is study the converse of theorem* 4.2 .

Corollary 4.8 *If* $\Gamma : X \to Y$ *is* $Q_P - O - Closed$ *topological multifunction and* Y *is a compact a* TO.S , *then* Γ *is* $U.Q_P.C$.

Proof: *Let* $\Gamma : X \to Y$ *be* $Q_P - Closed$ *and* υ *be any closed subset of a* TO.S , *since* Y *is a compact space* , *then* υ *is compact set in* Y , *from theorem* (4.7) *we get* $\Gamma^{-1}(\upsilon)$ *is* $Q_P - Closed$, *this implies* Γ *is* $U.Q_P.C$. ♦

5-Conclusion

Concepts of openness and continuity is the basic concepts of study and survey of topological spaces. In this work, we introduced some new concepts of a kind of continuous multifunctional functions called $Q_P - Continuous\ multifunctions$ *and studied its properties and it turned out to be an expansion of the space of continuous multifunctional and we have obtained some theorems that can be generalized to this type of continuity.This work is extended to* $Q_P -$ *closed multifunctions and the relationships between* $Q_P - Continuous$ *and* $Q_P -$ *closed multifunctions are explored.*

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