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Solving Fractional Partial Differential Equations by Triple g-Transformation

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ABSTRACT

In this article, we use triple g-transformation for solving fractional partial differential equations. There are many studies about finding solutions of fractional partial differential equation by using many transformations like Laplace transform, Fourier transform and Elzaki transform. In this paper we use triple g-transformation because this transformation can be used to solve most of the fractional partial differential equations by choosing the appropriate functions $P_1(s), P_2(\alpha), P_3(\beta)$.

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1. *Introduction:*

The integral transformations are one of import methods for solving many of problems. Such that we can convert ordinary differential equation to algebraic equation and then we back by inverse of this integral transformation, but the case partial differential equations we use the integral transformation to convert partial differential equations to differential equations of type ordinary [1]. By using double integral transformation we can convert partial differential equations of two independent variables to algebraic equations , thus we need to triple transformation to solve partial differential equations with three independent variables. the triple integral transformation is has import applications [5]. This transformation is distinguished by the generalities of the most known integral transformations and the possibility of finding new integral transformations from it.

In [4], H.Jaferi presented general integral transformation and he called it g-transformation also he studied properties of g-transformation and its applications in differential equations.

2. Fractional differential equations and its solutions

2.1 Definition [2] :

Let $n \in R^+$. Then the oprator J_a^n defind on $L_1[a,b]$ as following that:

$$J_a^n f(x) := \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt$$

For $a \le x \le b$, is called the Riemann-liouville fractional integral operator of order n.

For n=0, we set $J_a^n = I$, the identity operator.

One important property of integer-order integral operators is preserved by our generalization:

2.2 Theorem [3] :

let $m, n \geq 0$ *and* $\phi \in L_1[a,b]$ Then,

$$J_a^m J_a^n \phi = J_a^n J_a^m \phi = J_a^{m+n} \phi \tag{2.1}$$

2.3 Example:

Let $f(x) = (x-a)^\beta$ for some $\beta > -1$ *and* $n > 0$.

Then $J_a^n f(x) = \frac{1}{\Gamma(n)} \int_a^x (t-a)^\beta (x-t)^{n-1} dt$

$$= \frac{1}{\Gamma(n)} (x-a)^{n+\beta} \int_0^1 s^\beta (1-s)^{n-1} ds = \frac{\Gamma(\beta+1)}{(a)_k (n+\beta+1)} (x-a)^{n+\beta}$$

Where $(a)_k = (a+1)(a+2) \ldots (a+k)$

So

$$J_a^n f(x) = \frac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)} (x-a)^{n+\beta}.$$

2.4 Definition (Riemann-Liouville Derivatives) [8]:

Let $n \in R^+$ and $m = \lceil n \rceil$. The Riemann-Liouville fractional differential operator defined as following that:

$$D_a^n f := D^m J_a^{m-n} f \tag{2.2}$$

2.5 Lemma [5,6]:

Let $n \in R^+$ *and let* $m \in \mathbb{N}$ *such that* $m > n$. Then,

$$D_a^n := D^m J_a^{m-n}$$

2.6 Definition (Caputo's Derivative) [7]:

Let $n \geq 0$ *and* $m = \lceil n \rceil$, *then* we define the Caputo's operator by:

$$D_a^C f = J_a^{m-n} D^m f$$

Or $$D_a^C f(x) = \frac{1}{\Gamma(m-n)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{n+1-m}} dt \tag{2.3}$$

2.7 Proposition [6]:

Let f be a continuous function and $n \geq 0$, then

$$D_a^C J^n f = f$$

2.8 Remark:

Let A be a constant number. Then

$$D_0^n A = \frac{A x^{-n}}{\Gamma(1-n)}$$

$$D_0^C A = \frac{1}{\Gamma(m-n)} \int_0^x \frac{A^{(m)}}{(x-t)^{n+1-m}} dt = 0$$

2.9 Example:

Let $f(x) = (x-a)^\beta$, $\beta \geq 0$, *then*

$$D_a^C(x-a)^\beta = \begin{cases} 0 & ; \quad \beta = 0,1,2,\dots,m-1 \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}(x-a)^{\beta-n} & ; \ \beta \in \mathrm{N}, \beta \geq m \ or \ \ \beta \notin \mathrm{N}, \beta > m-1 \end{cases}$$

2.10 Proposition [8]:

Let α , β are a positive real number

1- $T_{2g}\left(u_x^{(\alpha)}\right) = q_1^\alpha \overline{\overline{u}} - p_1 \sum_{k=0}^{n-1} q_1^{\alpha-1-k} g\left(u_x^{(k)}(0,y), p_2, q_2\right)$

2- $T_{2g}\left(u_y^{(\beta)}\right) = q_2^\beta \overline{\overline{u}} - p_2 \sum_{k=0}^{m-1} q_2^{\beta-1-k} g\left(u_y^{(k)}(x,0), p_1, q_1\right)$

Where $n-1 < \alpha < n$, $m-1 < \beta < m$.

$$\frac{q_2^n}{p_2^{n-\alpha+1}} T_{2g}\left(f(t,x)\right) - \frac{1}{p_2^{n-\alpha}} \sum_{k=0}^{n-1} q_2^{n-k-1} T_x\left(f_t^{(k)}(0,x)\right)$$

3. Solving fractional differential equations by using g-transformation

3.1 Proposition:

$$g\left(J^\alpha f(x)\right) = \frac{1}{\left(q(s)\right)^\alpha} g\left(f(x)\right) \qquad (3.1)$$

Proof:

$$g\left(J^\alpha f(x)\right) = g\left(\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt\right)$$

$$= \frac{1}{\Gamma(\alpha)} g\left((h*f)(x)\right) = \frac{1}{\Gamma(\alpha)p(s)} g\left(h(x)\right). g\left(f(x)\right)$$

$where \ \ h(x) = x^{\alpha-1}$

We note that

$$g(x^{\alpha-1}) = \frac{\Gamma(\alpha)p(s)}{\left(q(s)\right)^\alpha}$$

Thus

$$g\left(J^\alpha f(x)\right) = \frac{1}{\Gamma(\alpha)p(s)} g\left(h(x)\right). g\left(f(x)\right) = \frac{1}{\Gamma(\alpha)p(s)} \frac{\Gamma(\alpha)p(s)}{\left(q(s)\right)^\alpha} g\left(f(x)\right)$$

$$g\left(J^\alpha f(x)\right) = \frac{1}{\left(q(s)\right)^\alpha} g\left(f(x)\right)$$

3.2 Proposition:

Let $\alpha \geq 0$, $m-1 < \alpha < m$, $m \in N$, then:

$$g\left(D^\alpha f(x)\right) = q^\alpha(s) g\left(f(x)\right) - p(s) \sum_{k=0}^{m-1} q^{\alpha-1-k}(s) f^{(k)}(0) \qquad (3.2)$$

Proof:

$$g\left(D^\alpha f(x)\right) = g\left(J^{m-\alpha} f^{(m)}(x)\right)$$

$$g\left(D^\alpha f(x)\right) = \frac{1}{\left(q(s)\right)^{m-\alpha}} g\left(f^{(m)}(x)\right)$$

Since $g\left(f^{(m)}(x)\right) = q^m(s) g\left(f(x)\right) - p(s) \sum_{k=0}^{m-1} q^{m-1-k}(s) f^{(k)}(0)$

Thus

$$g\left(D^\alpha f(x)\right) = \frac{1}{\left(q(s)\right)^{m-\alpha}} \left[q^m(s) g\left(f(x)\right) - p(s) \sum_{k==0}^{m-1} q^{m-1--k}(s) f^{(k)}(0)\right]$$

$$g(D^{\alpha}f(x)) = q^{\alpha}(s)g(f(x)) - p(s)\sum_{k==0}^{m-1} q^{\alpha-1--k}(s)f^{(kk)}(0)$$

3.3 Proposition:

$$T_{2g}\left(f^{(\alpha)}(t)\right) = \frac{p_2}{q_2}\left[q_1^{\alpha}\overline{f(t)}_1 - p_1\sum_{k=0}^{n-1} q_1^{\alpha-1-k} f^{(k)}(0)\right]$$

Where $T_{2g}(f(x,y)) = P_1(s)P_2(s)\int_0^{\infty}\int_0^{\infty} e^{-q_1(s)x-q_2(s)y} f(x,y)dxdy$ [11].

where $f^{(\alpha)}(t) = J_0^{n-\alpha}f^{n}(t)$ $n\text{-}1 < \alpha < n$, n is an integer number.

Proof:

First we prove that:

$$T_{2g}(J_0^{\alpha}f(t)) = gsD\left(\frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t)\right)$$

$$= \frac{1}{\Gamma(\alpha)}\frac{p_2}{p_1q_2}\overline{t^{\alpha-1}}_1.\overline{f(t)}_1$$

$$= \frac{1}{\Gamma(\alpha)}\frac{p_2}{p_1q_2}.\frac{\Gamma(\alpha)p_1}{q_1^{\alpha}}\overline{f(t)}_1$$

$$T_{2g}(J_0^{\alpha}f(t)) = \frac{p_2}{q_2q_1^{\alpha}}.\overline{f(t)}_1$$

Second we get:

$$T_{2g}(f^{(\alpha)}) = T_{2g}(J_0^{n-\alpha}f^{(n)}(t))$$

$$= \frac{p_2}{q_2q_1^{n-\alpha}}.\overline{f^{(n)}(t)}_1$$

$$T_{2g}(f^{(\alpha)}) = \frac{p_2}{q_2q_1^{n-\alpha}}\left[q_1^{n}\overline{f(t)}_1 - p_1\sum_{k=0}^{n-1} q_1^{n-1-k} f^{(k)}(0)\right]$$

$$= \frac{p_2}{q_2}\left[q_1^{\alpha}\overline{f(t)}_1 - p_1\sum_{k=0}^{n-1} q_1^{\alpha-1-k} f^{(k)}(0)\right]$$

3.4 Proposition:

$$1 - T_{3g}\left({}^{c}D_t^{\alpha}f(x,y,t)\right) = q_3^{\alpha}T_{3g}(f(x,y,t)) - \sum_{k=0}^{n-1} q_3^{n-k-1}T_xT_y\left(\frac{\partial^k}{\partial t^k}f(x,y,0)\right)$$

$$2 - T_{3g}\left({}^{c}D_x^{\alpha}f(x,y,t)\right) = q_1^{\alpha}T_{3g}(f(x,y,t)) - \sum_{k=0}^{n-1} q_1^{n-k-1}T_yT_t\left(\frac{\partial^k}{\partial x^k}f(0,y,t)\right)$$

$$3 - T_{3g}\left({}^{c}D_y^{\alpha}f(x,y,t)\right) = q_2^{\alpha}T_{3g}(f(x,y,t)) - \sum_{k=0} q_2^{n-k-1}T_xT_t\left(\frac{\partial^k}{\partial y^k}f(x,0,t)\right)$$

Proof:

$$1\text{-}T_{3g}\left({}^{c}D_t^{\alpha}f(x,y,t)\right) = P_1P_2P_3\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-q_1x-q_2y-q_3t}\ {}^{c}D_t^{\alpha}f(x,y,t)dt\,dx\,dy$$

Since

$${}^{c}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\infty}(t-\tau)^{n-\alpha-1}\frac{\partial n}{\partial t^n}f(t)d\tau = \frac{1}{\Gamma(n-\alpha)}\left(t^{n-\alpha-1} * \frac{\partial n}{\partial t^n}f(t)\right) \quad (3.3)$$

By Eq.(3.3) we get

$$T_{3g}\left({}^{c}D_t^{\alpha}f(x,y,t)\right) = \frac{p_1p_2p_3}{\Gamma(n-\alpha)}\int_0^{\infty}\int_0^{\infty} e^{-q_1x-q_2y}\left[\int_0^{\infty} e^{-q_3t}\left(t^{n-\alpha-1} * \frac{\partial n}{\partial t^n}\right)f(x,y,t)dt\right]dx\,dy$$

$$= \frac{p_1p_2}{\Gamma(n-\alpha)}\int_0^{\infty}\int_0^{\infty} e^{q_1x-q_2y}\left[\frac{1}{p_3}T_{tg}(t^{n-\alpha-1})\cdot T_{tg}\left(\frac{\partial n}{\partial t^n}f(x,y,t)\right)\right]dx\,dy$$

$$= \frac{p_1p_2\Gamma(n-\alpha)}{\Gamma(n-\alpha)p_3\,q_3^{n-\alpha}}\int_0^{\infty}\int_0^{\infty} e^{q_1x-q_2y}\left[q_3^{\alpha}T_{tg}(f(x,y,t)) - p_3\sum_{k=0}^{n-1} q_3^{n-k-1}\frac{\partial k}{\partial t^k}f(x,y,t)\right]dx\,dy$$

$$= q_3^{\alpha}T_{3g}(f(x,y,t)) - \sum_{k=0}^{n-1} q_3^{n-k-1}T_xT_y\frac{\partial k}{\partial t^k}f(x,y,0)$$

$$2 - T_{3g}\left({}^{c}D_x^{\alpha}f(x,y,t)\right) = P_1P_2P_3\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-q_1x-q_2y-q_3t}\ {}^{c}D_x^{\alpha}f(x,y,t)dt\,dx\,dy$$

Since

$${}^{c}D_t^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\infty}(x-\tau)^{n-\alpha-1}\frac{\partial n}{\partial t^n}f(x)d\tau = \frac{1}{\Gamma(n-\alpha)}\left(x^{n-\alpha-1} * \frac{\partial n}{\partial x^n}f(x)\right)$$

Thus

$$T_{3g}\left({}^{c}D_x^{\alpha} f(x,y,t)\right) = \frac{p_1p_2p_3}{\Gamma(n-\alpha)}\int_0^{\infty}\int_0^{\infty} e^{-q_2y-q_3t}\left[\int_0^{\infty} e^{-q_1x}\left(x^{n-\alpha-1} * \frac{\partial n}{\partial x^n}\right)f(x,y,t)dx\right]dt\,dy$$
$$= \frac{p_2p_3}{\Gamma(n-\alpha)}\int_0^{\infty}\int_0^{\infty} e^{q_2y-q_3t}\left[\frac{1}{p_1}T_{tg}(x^{n-\alpha-1})\cdot T_{xg}\left(\frac{\partial n}{\partial x^n}f(x,y,t)\right)\right]dt\,dy$$
$$= \frac{p_1p_2}{p_1\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha)p_1}{q_1^{n-\alpha}}\frac{1}{p_1}\int_0^{\infty}\int_0^{\infty} e^{q_2y-q_3t}\left[q_1^{\alpha}T_{xg}(f(x,y,t)) - p_1\sum_{k=0}^{n-1} q_1^{n-k-1}\frac{\partial k}{\partial x^k}f(0,y,t)\right]dx\,dy$$
$$= q_1^{\alpha}T_{3g}(f(x,y,t)) - \sum_{k=0}^{n-1} q_1^{n-k-1}T_yT_t\frac{\partial k}{\partial x^k}f(0,y,t)$$

$$3 - T_{3g}\left({}^{c}D_y^{\alpha} f(x,y,t)\right) = P_1P_2P_3\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-q_1x-q_2y-q_3t}\ {}^{c}D_y^{\alpha} f(x,y,t)dy\,dx\,dt$$

Since

$${}^{c}D_y^{\alpha} f(y) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\infty}(y-\tau)^{n-\alpha-1}\frac{\partial n}{\partial y^n}f(y)d\tau = \frac{1}{\Gamma(n-\alpha)}\left(y^{n-\alpha-1} * \frac{\partial n}{\partial y^n}f(y)\right)$$

Thus

$$T_{3g}\left({}^{c}D_y^{\alpha} f(x,y,t)\right) = \frac{p_1p_2p_3}{\Gamma(n-\alpha)}\int_0^{\infty}\int_0^{\infty} e^{-q_1x-q_3t}\left[\int_0^{\infty} e^{-q_2y}\left(y^{n-\alpha-1} * \frac{\partial n}{\partial y^n}\right)f(x,y,t)dy\right]dt\,dx$$
$$= \frac{p_1p_3}{\Gamma(n-\alpha)}\int_0^{\infty}\int_0^{\infty} e^{q_1x-q_3t}\left[\frac{1}{p_2}T_{yg}(y^{n-\alpha-1})\cdot T_{yg}\left(\frac{\partial n}{\partial y^n}f(x,y,t)\right)\right]dt\,dx$$
$$= \frac{p_1p_2}{p_1\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha)p_1}{q_1^{n-\alpha}}\frac{1}{p_2}\int_0^{\infty}\int_0^{\infty} e^{q_2y-q_3t}\left[q_1^{\alpha}T_{yg}(f(x,y,t)) - p_2\sum_{k=0}^{n-1} q_2^{n-k-1}\frac{\partial k}{\partial y^k}f(x,0,t)\right]dx\,dy$$
$$= q_2^{\alpha}T_{3g}(f(x,y,t)) - \sum_{k=0}^{n-1} q_2^{n-k-1}T_xT_t\frac{\partial k}{\partial y^k}f(x,0,t)$$

3.5 Solution of generalized fractional Heat equation By using T_{3g} – Transformation:

Consider the equation

$${}_{t}^{c}D_{o}^{\alpha}u(x,y,t) = \frac{1}{\pi^2}\left(D_x^2 + D_y^2\right)u(x,y,t) \qquad (3.5.1)$$

With the initial and boundary conditions :

$$\left.\begin{array}{c} u(0,0,t) = u(0,y,t) = u(x,0,t) = 0 \quad D_x(0,y,t) = \pi E_{\alpha}(y^{\alpha}) \\ y > 0, t > 0 \quad D_y(x,0,t) = \pi E_{\alpha}(x^{\alpha}),\ u(x,y,0) = \sin\pi x \sin\pi y,\ x,y,t > 0, 0 < \alpha < 1 \end{array}\right\} \qquad (3.5.2)$$

Eq. (3.5.1 , 3.5.2) are called generalized fractional Heat equation.

By Taking T_{3g} –transformation of both sides of Eq (3.3.1) we get :

$$T_{3g}\left({}_{t}^{c}D_{o}^{\alpha}u(x,y,t)\right) = c^2\left[T_{3g}\left(D_x^2u(x,y,t)\right) + T_{3g}\left(D_y^2u(x,y,t)\right)\right] \qquad (3.5.3)$$

But First we note that $T_xT_y(u(x,y,0)) = T_xT_y(\sin\pi x\ \sin\pi y) = T_x(\sin\pi x).T_y(\sin\pi y)$

$$= \frac{\pi p_1}{q_1^2+\pi^2}\cdot\frac{\pi p_2}{q_2^2+\pi^2} = \frac{\pi^2p_1p_2}{(q_1^2+\pi^2)(q_2^2+\pi^2)}$$

$2 - T_tT_t(u(0,0,t)) = T_tT_y(u(0,y,t)) = T_tT_x(u(x,0,t)) = 0$

$3 - T_tT_y(D_x\ u(0,y,t)) = \dfrac{\pi\ p_2p_3q_2^{\alpha-1}}{q_3(q_2^{\alpha}+1)}$

$4 - T_tT_x\left(D_y\ u(x,0,t)\right) = \dfrac{\pi\ p_1p_3q_1^{\alpha-1}}{q_3(1-q_1^{\alpha})}$

Now we use 1,2,3,and 4 in Eq.(3.5.3):

$$q_3^{\alpha}T_{3g}(u(x,y,t)) - \sum_{k=0}^{n=1} q_3^{\alpha-k-1}TxTy\left(\frac{\partial k}{\partial t^k}u(x,y,0)\right) = \frac{1}{\pi^2}\left[q_1^2T_{3g}(u) - T_tT_y(u(0,y,t)) - q_1T_tT_Y(D_xu(0,y,t)) + q_2^2T_{3g}(u) - T_tT_x(u(x,0,t)) - q_2T_tT_x\left(D_yu(x,0,t)\right)\right]$$

$$q_3^{\alpha}T_{3g}(u) - \frac{\pi^2p_1p_2}{(q_1^2+\pi^2)(q_2^2+\pi^2)} = \frac{1}{\pi^2}\left[(q_1^2+q_2^2)T_{3g}(u) - \frac{\pi q_1p_2p_3q_2^{\alpha-1}}{q_3(q_2^{\alpha}+1)} - \frac{\pi q_2p_1p_3q_1^{\alpha-1}}{q_3(1-q_1^{\alpha-1})}\right]\left(q_3^{\alpha} - \frac{q_1^2+q_2^2}{\pi^2}\right)T_{3g}(u)$$
$$= \frac{\pi^2p_1p_2}{(q_1^2+\pi^2)(q_2^2+\pi^2)} - \frac{\pi q_1p_2p_3q_2^{\alpha-1}}{\pi q_3(q_2^{\alpha}+1)} - \frac{\pi q_2p_1p_3q_1^{\alpha-1}}{\pi q_3(1-q_1^{\alpha-1})}\ T_{3g}(u)$$
$$= \frac{\pi^4p_1p_2}{(q_1^2+\pi^2)(q_2^2+\pi^2)(\pi^2q_3^{\alpha}-q_1^2-q_2^2)} - \frac{\pi p_2p_3q_1q_2^{\alpha-1}}{q_3(1+q_2^{\alpha})(\pi^2q_3^{\alpha}-q_1^2-q_2^2)} - \frac{\pi p_1p_2q_1^{\alpha-1}q_2}{q_3(\pi^2q_3^{\alpha}-q_1^2-q_2^2)(1-q_1^{\alpha})}$$

$$u(x,y,t) = T_{3g}^{-1}\left(\frac{\pi^4p_1p_2}{(q_1^2+\pi^2)(q_2^2+\pi^2)(\pi^2q_3^{\alpha}-q_1^2-q_2^2)}\right) - T_{3g}^{-1}\left(\frac{\pi p_2p_3q_1q_2^{\alpha-1}}{q_3(1+q_2^{\alpha})(\pi^2q_3^{\alpha}-q_1^2-q_2^2)}\right) - T_{3g}^{-1}\left(\frac{\pi p_1p_2q_1^{\alpha-1}q_2}{q_3(\pi^2q_3^{\alpha}-q_1^2-q_2^2)(1-q_1^{\alpha})}\right)$$

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