Journal of Al-Qadisiyah for Computer Science and Mathematics Vol. 15(1) 2023 , pp Math. 92–100

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

A Certain Class of Analytic Functions Associated with Beta Negative Binomial Distribution Defined on Complex Hilbert Space

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ARTICLE INFO

Article history:
Received: 10/02/2023
Rrevised form: 11/03/2023
Accepted : 14/03/2023
Available online: 31/03/2023

Keywords:

Analytic function, Convex set, Hilbert space, Extreme point, Beta negative binomial distribution, Radii of starlikeness and convexity.

ABSTRACT

In this work , we show and research a new class of univalent and analytic functions with negative coefficients linked to the beta negative binomial distribution on a complicated Hilbert space. We find fascinating geometric features such as the convex set, coefficient estimates, distortion and growth theorems, starlikeness and convexity radii, and describe the extreme points for functions in this class.

MSC..

https://doi.org/10.29304/jqcm.2023.15.1.1171

1. Introduction

Let $\mathcal{A}$ represent the class of all functions f which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$of the form

$$f(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa \,, \qquad (1.1)$$

And denote by $S^*(\rho)$ the class of starlike functions of order $\rho, (0 \le \rho < 1)$ $when\ f \in \mathcal{A}$ if it

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \qquad (z \in U)\,,$$

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And denote by $\mathcal{K}(\rho)$ the class of convex functions of order $\rho, (0 \leq \rho < 1)$ $when\ f \in \mathcal{A}$ if it

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \rho \quad (z \in U).$$

Let $W \subseteq \mathcal{A}$, where $f(z) \in W$,such that

$$f(z) = z - \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa \quad (a_\kappa \geq 0). \tag{1.2}$$

The power series below, whose coefficients are beta negative binomial distribution probabilities, was recently proposed by Wanas and Al-Ziadi [9]

$$\wp_{\nu,\rho}^{\sigma}(z) = z - \sum_{\kappa=2}^{\infty} \frac{(\nu)_\sigma\ (\sigma)_{\kappa-1}\ (\rho)_{\kappa-1}}{(\nu+\rho)_\sigma\ (\sigma+\nu+\rho)_{\kappa-1}\ (\kappa-1)!} z^\kappa, \quad (z \in U, \nu, \rho, \sigma > 0).$$

We see using the Ratio Test that the above series' radius of convergence is infinite.

The linear operator $\mathfrak{P}\wp_{\nu,\rho}^{\sigma} : W \longrightarrow W$ is defined as follows (see [9])

$$\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(z) = \wp_{\nu,\rho}^{\sigma}(z) * f(z) = z - \sum_{\kappa=2}^{\infty} \frac{(\nu)_\sigma\ (\sigma)_{\kappa-1}\ (\rho)_{\kappa-1}}{(\nu+\rho)_\sigma\ (\sigma+\nu+\rho)_{\kappa-1}\ (\kappa-1)!} a_\kappa z^\kappa, \quad (a_\kappa \geq 0),$$

where $*$ denotes the convolution of two series' Hadamard products.

H stands for the complex field'sHilbert space, and T represents a linear operator on H. For f on U,The Riesz-Dunford integral defines the operator on H, denoted by $f(T)$ [1].

$$f(T) = \frac{1}{2\pi i}\int_c f(z)(zI-T)^{-1}\, dz ,$$

The identification operator on H is I, The simple closed rectifiable contour c is positively orientated, lies in U, and has the spectrum $\sigma(T)$ of T in its interior domain.[2]. The series $f(T)$ can also define

$$f(T) = \sum_{\kappa=0}^{\infty} \frac{f^{(\kappa)}(0)}{\kappa!} T^\kappa ,$$

in which the topology of the norm converges [3].

If $f(z)$ is given by (1.2), then also have

$$\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T) = T - \sum_{\kappa=2}^{\infty} \frac{(\nu)_\sigma\ (\sigma)_{\kappa-1}\ (\rho)_{\kappa-1}}{(\nu+\rho)_\sigma\ (\sigma+\nu+\rho)_{\kappa-1}\ (\kappa-1)!} a_\kappa T^\kappa.$$

Now, We define the class $E(\delta,\gamma,\nu,\rho,\sigma,T)$ made up of the functions $f \in W$ so that

$$\left\|\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)' + \delta T\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)''\right\| > \gamma\left\|\delta\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)' + (1-\delta)\right\|, \tag{1.3}$$

where $0 < \delta < 1$, $0 \leq \gamma < 1$ and $\forall$ operator T, $\|T\| < 1$, $T \neq \emptyset$ (In H, $\emptyset$ represents the zero operator).

Recently, various researchers studied the operator in Hilbert space, can refer, for example, to [4,5,6,7,8,10,11,12,13].

2.Main Results

For functions in the class $E(\delta,\gamma,\nu,\rho,\sigma,T)$, the first theorem provides sharp coefficient estimates..

Theorem 2.1: Let $f \in W$ be given by (1.2). Then $f \in E(\delta,\gamma,\nu,\rho,\sigma,T)$ for all $T \neq \emptyset$ if and only if

$$\sum_{\kappa=2}^{\infty} \frac{\kappa(\delta\gamma + \delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!} a_\kappa \leq 1-\gamma\,, \tag{2.1}$$

where $0 \leq \delta < 1$, $0 \leq \gamma < 1$.

For the given function f, the outcome is clear

$$f(T) = T - \frac{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}} T^\kappa\,,\ \ \kappa \geq 2. \tag{2.2}$$

Proof: Consider inequality (2.1) is true. It is sufficient to show

$$\left\|\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)' + \delta T\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)''\right\| > \gamma\left\|\delta\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)' + (1-\delta)\right\|.$$

We consider

$$\gamma\left\|\delta\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)' + (1-\delta)\right\| - \left\|\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)' + \delta T\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)''\right\|$$

$$= \gamma\left\|1 - \sum_{\kappa=2}^{\infty} \frac{\delta\,\kappa\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!} a_\kappa T^{\kappa-1}\right\|$$

$$-\left\|1 - \sum_{\kappa=2}^{\infty} \frac{\kappa(\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!} a_\kappa T^{\kappa-1}\right\|$$

$$\leq \sum_{\kappa=2}^{\infty} \frac{\delta\,\gamma\,\kappa\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!} a_\kappa \|T\|^{\kappa-1} - (1-\gamma)$$

$$+ \sum_{\kappa=2}^{\infty} \frac{\kappa(\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!} a_\kappa \|T\|^{\kappa-1}$$

$$\leq \sum_{\kappa=2}^{\infty} \frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!} a_\kappa - (1-\gamma) \leq 0.$$

Therefore, $f \in E(\delta,\gamma,\nu,\rho,\sigma,T)$.

To show the converse, let $f \in E(\delta,\gamma,\nu,\rho,\sigma,T)$. Then

$$\left\|\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)' + \delta T\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)''\right\| > \gamma\left\|\delta\left(\mathfrak{P}\wp_{\nu,\rho}^{\sigma}(T)\right)' + (1-\delta)\right\|$$

gives

$$\left\|1-\sum_{\kappa=2}^{\infty}\frac{\kappa(\delta(\kappa-1)+1)\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}a_{\kappa}T^{\kappa-1}\right\|$$

$$>\gamma\left\|1-\sum_{\kappa=2}^{\infty}\frac{\delta\,\kappa\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}a_{\kappa}T^{\kappa-1}\right\|$$

In the inequality above, if $T = rI$ $(0 < r < 1)$, we obtain

$$\frac{\gamma-\sum_{\kappa=2}^{\infty}\frac{\delta\,\gamma\,\kappa\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}a_{\kappa}r^{\kappa-1}}{1-\sum_{\kappa=2}^{\infty}\frac{\kappa(\delta(\kappa-1)+1)\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}a_{\kappa}r^{\kappa-1}} < 1. \tag{2.3}$$

After deleting the denominator in (2.3) and allowing $r \longrightarrow 1$, we get

$$\sum_{\kappa=2}^{\infty}\frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}a_{\kappa} \leq 1-\gamma\,,$$

Corollary 2.1: If $f \in E(\delta,\gamma,\nu,\rho,\sigma,T)$, then

$$a_{\kappa} \leq \frac{(1-\gamma)(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}\;,\quad \kappa \geq 2.$$

Theorem 2.2: If $f \in E(\delta,\gamma,\nu,\rho,\sigma,T)$ and $\|T\| < 1, T \neq \emptyset$, then

$$\|T\| - \frac{(1-\gamma)(\nu+\rho)_{\sigma}}{(\delta\gamma+\delta+1)\,(\nu)_{\sigma}}\|T\|^2 \leq \|f(T)\| \leq \|T\| + \frac{(1-\gamma)(\nu+\rho)_{\sigma}}{(\delta\gamma+\delta+1)\,(\nu)_{\sigma}}\|T\|^2$$

and

$$1 - \frac{(1-\gamma)(\nu+\rho)_{\sigma}}{(\delta\gamma+\delta+1)\,(\nu)_{\sigma}}\|T\| \leq \|f'(T)\| \leq 1 + \frac{(1-\gamma)(\nu+\rho)_{\sigma}}{(\delta\gamma+\delta+1)\,(\nu)_{\sigma}}\|T\|.$$

Proof: According to the Theorem 2.1, we see that

$$\frac{(\delta\gamma+\delta+1)\,(\nu)_{\sigma}}{(1-\gamma)(\nu+\rho)_{\sigma}}\sum_{\kappa=2}^{\infty}a_{\kappa} \leq \sum_{\kappa=2}^{\infty}\frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(1-\gamma)(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}a_{\kappa} \leq 1,$$

which gives

$$\sum_{\kappa=2}^{\infty}a_{\kappa} \leq \frac{(1-\gamma)(\nu+\rho)_{\sigma}}{(\delta\gamma+\delta+1)\,(\nu)_{\sigma}}.$$

Thus, we have

$$\|f(T)\| \geq \|T\| - \sum_{\kappa=2}^{\infty} a_\kappa \|T\|^\kappa \geq \|T\| - \|T\|^2 \sum_{\kappa=2}^{\infty} a_\kappa \geq \|T\| - \frac{(1-\gamma)(\nu+\rho)_\sigma}{(\delta\gamma+\delta+1)\,(\nu)_\sigma} \|T\|^2.$$

Also,

$$\|f(T)\| \leq \|T\| + \sum_{\kappa=2}^{\infty} a_\kappa \|T\|^\kappa \leq \|T\| + \frac{(1-\gamma)(\nu+\rho)_\sigma}{(\delta\gamma+\delta+1)\,(\nu)_\sigma} \|T\|^2.$$

In the light of Theorem 2.1, and by noting the relation

$$\frac{\kappa\,(\delta\gamma+\delta+1)\,(\nu)_\sigma}{(1-\gamma)(\nu+\rho)_\sigma} \leq \frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}, \qquad (\kappa \geq 2),$$

we deduce that

$$\sum_{\kappa=2}^{\infty} \frac{\kappa\,(\delta\gamma+\delta+1)\,(\nu)_\sigma}{(1-\gamma)(\nu+\rho)_\sigma} a_\kappa \leq \sum_{\kappa=2}^{\infty} \frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!} a_\kappa \leq 1.$$

That is

$$\sum_{\kappa=2}^{\infty} \kappa a_\kappa \leq \frac{(1-\gamma)(\nu+\rho)_\sigma}{(\delta\gamma+\delta+1)\,(\nu)_\sigma}.$$

Hence

$$\|f'(T)\| \geq 1 - \sum_{\kappa=2}^{\infty} \kappa a_\kappa \|T\|^{\kappa-1} \geq 1 - \|T\| \sum_{\kappa=2}^{\infty} \kappa a_\kappa \geq 1 - \frac{(1-\gamma)(\nu+\rho)_\sigma}{(\delta\gamma+\delta+1)\,(\nu)_\sigma} \|T\|$$

and

$$\|f'(T)\| \leq 1 + \|T\| \sum_{\kappa=2}^{\infty} \kappa a_\kappa \leq 1 + \frac{(1-\gamma)(\nu+\rho)_\sigma}{(\delta\gamma+\delta+1)\,(\nu)_\sigma} \|T\|.$$

Theorem 2.3: The class $E(\delta,\gamma,\nu,\rho,\sigma,T)$ is a convex set.

Proof:

Let f_1 and f_2 be the unspecified components of $E(\delta,\gamma,\nu,\rho,\sigma,T)$ Then we demonstrate that $\forall\, t$, $(0 \leq t \leq 1)$, we demonstrate $(1-t)f_1 + tf_2 \in E(\delta,\gamma,\nu,\rho,\sigma,T)$. Thus, we obtain

$$(1-t)f_1 + tf_2 = z - \sum_{\kappa=2}^{\infty} \big((1-t)a_\kappa + tb_\kappa\big) z^\kappa .$$

Hence

$$\sum_{\kappa=2}^{\infty} \frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}\big((1-t)a_{\kappa}+tb_{\kappa}\big)$$

$$=(1-t)\sum_{\kappa=2}^{\infty} \frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}a_{\kappa}$$

$$+t\sum_{\kappa=2}^{\infty} \frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}b_{\kappa}$$

$$\le (1-t)(1-\gamma)+t(1-\gamma)=1-\gamma\,.$$

Theorem 2.4: If $f\in E(\delta,\gamma,\nu,\rho,\sigma,T)$, then $f\in S^{*}(\rho)$ “of order θ $(0\le\theta<1)$ in $|z|<r_1$",

$$r_1=\inf_{\kappa}\left\{\frac{\kappa(1-\theta)(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_{\sigma}\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\kappa-\theta)(1-\gamma)(\nu+\rho)_{\sigma}\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}\right\}^{\frac{1}{\kappa-1}},(\kappa\ge 2).$$

The result is clear for f given by (2.2).

Proof: It is sufficient to show

$$\left\|\frac{Tf'(T)}{f(T)}-1\right\|\le 1-\theta \quad . \tag{2.4}$$

We get

$$\left\|\frac{Tf'(T)}{f(T)}-1\right\|\le \frac{\sum_{k=2}^{\infty}(\kappa-1)a_{\kappa}\|T\|^{\kappa-1}}{1-\sum_{k=2}^{\infty}a_{\kappa}\|T\|^{\kappa-1}}.$$

Hence (2.4) will be satisfied if

$$\sum_{\kappa=2}^{\infty}\left(\frac{\kappa-\theta}{1-\theta}\right)a_{\kappa}\|T\|^{\kappa-1}\le 1. \tag{2.5}$$

In the light of Theorem 2.1, if $f\in E(\delta,\gamma,\nu,\rho,\sigma,T)$, then

$$\sum_{\kappa=2}^{\infty} \frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}\,a_\kappa \leq 1. \tag{2.6}$$

Using (2.6), we can show that (2.5) is true if

$$\frac{\kappa-\theta}{1-\theta}\|T\|^{\kappa-1} \leq \frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!},$$

or equivalently

$$\|T\| \leq \left\{\frac{\kappa(1-\theta)(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\kappa-\theta)(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}\right\}^{\frac{1}{\kappa-1}},$$

Theorem 2.5: If $f \in E(\delta,\gamma,\nu,\rho,\sigma,T)$, then $f \in \mathcal{K}(\rho)$ of order θ $(0 \leq \theta < 1)$ in $|z| < r_2$,

$$r_2 = \inf_{\kappa}\left\{\frac{(1-\theta)(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(\kappa-\theta)(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}\right\}^{\frac{1}{\kappa-1}}, (\kappa \geq 2).$$

The result is obvious for f provided by (2.2).

Proof: It is sufficient to show

$$\left\|\frac{Tf''(T)}{f'(T)}\right\| \leq 1-\theta\,.$$

By employing arguments like to those used in the proof of Theorem 2.3, the conclusion is achieved.

Theorem 2.6: If $f_1(z) = z$ &

$$f_\kappa(z) = z - \frac{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}z^\kappa\,, \quad \kappa \geq 2.$$

Then $f \in E(\delta,\gamma,\nu,\rho,\sigma,T)$ iff it may be shown in the form:

$$f(z) = \sum_{\kappa=1}^{\infty} \lambda_\kappa\, f_\kappa(z)\,, \tag{2.7}$$

where $\lambda_\kappa \geq 0$ and $\sum_{\kappa=1}^{\infty} \lambda_\kappa = 1$.

Proof: Suppose f may be represented as follows: (2.7). Next, we see that

$$f(z) = \sum_{\kappa=1}^{\infty} \lambda_\kappa\, f_\kappa(z) = \lambda_1\, f_1(z) + \sum_{\kappa=2}^{\infty} \lambda_\kappa\, f_\kappa(z)$$

$$= z - \sum_{\kappa=2}^{\infty} \frac{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}\lambda_\kappa z^\kappa.$$

Thus

$$\sum_{\kappa=2}^{\infty} \frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!} \times \frac{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}\lambda_\kappa$$

$$= \sum_{\kappa=2}^{\infty} \lambda_\kappa = 1 - \lambda_1 \leq 1\,,$$

and so $f \in E(\delta,\gamma,\nu,\rho,\sigma,T)$.

Conversely, Assume that f as provided by (1.2), belongs in $E(\delta,\gamma,\nu,\rho,\sigma,T)$. Then by Corollary 2.1, we have

$$a_\kappa \leq \frac{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}.$$

Putting

$$\lambda_\kappa = \frac{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}a_\kappa\,, \quad \kappa \geq 2\,,$$

and $\lambda_1 = 1 - \sum_{\kappa=2}^{\infty} \lambda_\kappa$. Then

$$f(z) = z - \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa$$

$$= z - \sum_{\kappa=2}^{\infty} \frac{(1-\gamma)(\nu+\rho)_\sigma\,(\sigma+\nu+\rho)_{\kappa-1}\,(\kappa-1)!}{\kappa(\delta\gamma+\delta(\kappa-1)+1)\,(\nu)_\sigma\,(\sigma)_{\kappa-1}\,(\rho)_{\kappa-1}}\lambda_\kappa z^\kappa$$

$$= z - \sum_{\kappa=2}^{\infty}\big(z - f_\kappa(z)\big)\lambda_\kappa = \left(1 - \sum_{\kappa=2}^{\infty}\lambda_\kappa\right)z + \sum_{\kappa=2}^{\infty}\lambda_\kappa\, f_\kappa(z)$$

$$= \lambda_1 f_1(z) + \sum_{\kappa=2}^{\infty}\lambda_\kappa\, f_\kappa(z) = \sum_{\kappa=1}^{\infty}\lambda_\kappa\, f_\kappa(z)\,,$$

References

[1] N. Dunford and J. T. Schwarz, Linear Operator, Part I, General Theory, New York - London, Inter Science, 1958.

[2] K. Fan, Analytic functions of a proper contraction, Math. Z., 160(1978), 275-290.

[3] K. Fan, Julia,s lemma for operators, Math. Ann., 239(1979), 241-245.

[4] F. Ghanim and M. Darus, On new subclass of analytic p-valent function with negative coefficients for operators on Hilbert space, Int. Math. Forum, 3(2)(2008), 69-77.

[5] S. B. Joshi, On a class of analytic functions with negative coefficients for operators on Hilbert Space, J. Appr. Theory and Appl., (1998), 107–112.

[6] Y. C. Kim and J. H. Choi, Some subclasses of analytic functions with negative coefficients for operators on Hilbert space, International Journal of Applied Mathematics, 32(4)(2019), 549-562.
[7] C. Selvaraj, A. J. Pamela and M. Thirucheran, On a subclass of multivalent analytic functions with negative coefficients for contraction operators on Hilbert space, Int. J. Contemp. Math. Sci., 4(9)(2009), 447-456.
[8] A. K. Wanas, A certain subclass of multivalent analytic functions with negative coefficients for operator on Hilbert space, J. Indones. Math. Soc., 21 (2)(2015), 77-82.
[9] A. K. Wanas and N. A. Al-Ziadi, Applications of Beta negative binomial distribution series on holomorphic functions, Earthline J. Math. Sci., 6(2)(2021), 271-292.
[10] A. K. Wanas, J. Choi and N. E. Cho, Geometric properties for a family of holomorphic functions associated with Wanas operator defined on complex Hilbert space, Asian-European Journal of Mathematics, 14(7)(2021), 1-14.
[11] A. K. Wanas and B. A. Frasin, Applications of fractional calculus for a certain subclass of multivalent analytic functions on complex Hilbert space, General Mathematics, 26(1-2)(2018), 11–23.
[12] A. K. Wanas and G. Murugusundaramoorthy, Higher-order derivatives of multivalent analytic functions defined on complex Hilbert space, J. Adv. Math. Stud., 14(2)(2021), 197-205.
[13] A. K. Wanas and S. R. Swamy, New family of multivalent analytic functions defined on complex Hilbert space, Earthline Journal of Mathematical Sciences, 3(1)(2020), 155-165.