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Some Generalizations of g-lifting Modules

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ABSTRACT

In this work we will attempt to define and investigate new classes of modules named ⊕-g-supplemented and ⊕-g-radical supplemented as a proper generalization of class of g-lifting modules and identify several distinct characterizations of these modules. Additionally, we'll attempt to explain the concepts of projective g-covers and g-semiperfect modules. It is shown that the two buildings of g-semiperfect and ⊕-g-supplemented modules are the same for the class of projective modules.

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1. Introduction

In this article, all rings are associative with unit, and all modules are unital right. To demonstrate that D is a submodule and direct smmand of a module M, respectively, we use $T \hookrightarrow M$ and (T is a d.s. of M or, $T \hookrightarrow^{\oplus} M$). $Rad(M)$ represents the radical of a module M. Mod-$\mathcal{R}$ denotes the set of all right modules over a ring $\mathcal{R}$.

We will go through some of the fundamental notions we use often in our work. For $M \in$ Mod-$\mathcal{R}$, $T \hookrightarrow M$ is named small in M, we write $T \hookrightarrow^{s} M$ if, $T \neq M$ and $\forall$ proper $B \hookrightarrow M$, we have $T + B \neq M$; and T named δ-small in M, we write $T \hookrightarrow^{\delta s} M$ if, $\forall$ $B \hookrightarrow M$ with $M = T + B$ and M/B singular, then $M = B$, see [1]. For $M \in$ Mod-$\mathcal{R}$, we will insert $\delta(M) = \sum\{T \,|\, T \hookrightarrow^{\delta s} M\}$. $M(\neq 0) \in$ Mod-$\mathcal{R}$ is named hollow if, $\forall$ proper $T \hookrightarrow M$, then $T \hookrightarrow^{s} M$. If the sum of the proper submodules of $M(\neq 0) \in$ Mod-$\mathcal{R}$ is also proper in M, then $M \in$ Mod-$\mathcal{R}$ is named local.

For $T, H \hookrightarrow M \in$ Mod-$\mathcal{R}$. A submodule H is named to be a supplement (δ-supplement) of T in M, respectively, if

$T + H = M$ and $T \cap H \hookrightarrow^{s} H$ ($T \cap H \hookrightarrow^{\delta s} H$). $M \in$ Mod-$\mathcal{R}$ is named supplemented (δ-supplemented), respectively, if $\forall$ submodule of M have a supplement (δ-supplement) in M, according to ([2] and [3], resp.). $H \hookrightarrow M \in$ Mod-$\mathcal{R}$ is named to be a generalized (supplement) δ-supplement of T, respectively, if $T + H = M$ and ($T \cap H \hookrightarrow Rad(H)$) $T \cap H \hookrightarrow \delta(H)$. $M \in$Mod-$\mathcal{R}$ is named to be a (GS-module) δ-GS-module according to ([4] and [5]), respectively, if $\forall$ submodule of $M \in$ Mod-$\mathcal{R}$ have a generalized (supplement) δ-supplement in M.

However, most authors have referred to GS-modules as Rad-supplemented modules in a number of works, thus we will use this title in our study. $(0 \neq) T \hookrightarrow M \in$ Mod-$\mathcal{R}$ is named large in M, we write $T \hookrightarrow^{e} M$ if $T \cap H \neq 0$, $\forall$ $(0 \neq) H \hookrightarrow M$. If $H = M$, $\forall$ $H \hookrightarrow^{e} M$ with $T + H = M$, then T is named a g-small submodule of M, we write $T \hookrightarrow^{gs} M$ (in [6], it is named an e-small submodule of M and denoted as $T \ll_{e} M$). If $M = T + K = T + H \in$ Mod-$\mathcal{R}$ with $H \hookrightarrow^{e} K$ implies that $H = K$, or

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equivalently, $M = T + H$ and $T \cap H \hookrightarrow^{gs} H$, then recall [7] that H is a g-supplement of T in M. $M \in$ Mod-$\mathcal{R}$ is named g-supplemented if, ∀ submodule of M have a g-supplement in M. These notions were also discussed in [8]. According to Zhou and Zhang [6], the generalized radical of $M \in$ Mod-$\mathcal{R}$ as follows:

$$Rad_g(M) = \cap\{T \hookrightarrow^{e} M \mid T \text{ is maximal in } M\} = \sum\{T \mid T \hookrightarrow^{gs} M\}.$$

A submodule H is named to be a g-radical supplement of U in $M \in$ Mod-$\mathcal{R}$ if, whenever $U + H = M$ and $U \cap H \hookrightarrow Rad_g(H)$. $M \in$Mod-$\mathcal{R}$ is named generalized radical supplemented, we write g-radical supplemented if, ∀ submodule of $M \in$ Mod-$\mathcal{R}$ have a g-radical supplement in M, as shown in [9].

The concept of ⊕-supplemented modules was first suggested by Mohamed and Müller [10]. Harmanci, Keskin and Smith [11] carried done further research on ⊕-supplemented modules and arrived to some significant conclusions. However, $M \in$ Mod-$\mathcal{R}$ is named a ⊕-supplemented module if, ∀ $H \hookrightarrow M$ have a supplement, say U, such that $U \hookrightarrow^{\oplus} M$. $M \in$ Mod-$\mathcal{R}$ is named ⊕-δ-supplemented if, ∀ $H \hookrightarrow M$ have a δ-supplement, say U, such that $U \hookrightarrow^{\oplus} M$. see [12]. The authors recently expanded the class of ⊕-supplemented modules to Rad-⊕-supplemented in [13]. $M \in$ Mod-$\mathcal{R}$ is named Rad-⊕-supplemented if, ∀ $T \hookrightarrow M$, ∃ $H \hookrightarrow^{\oplus} M$ such that $M = T + H$ and $T \cap H \hookrightarrow Rad(H)$. If $Rad(H)$ is changed out for $\delta(H)$ in accordance with the prior notion of Rad-⊕-supplemented, then $M \in$ Mod-$\mathcal{R}$ is referred to as generalized ⊕-δ-supplemented module, (see [14]).

In our study, we generalized the idea of g-lifting modules in the same way that the idea of lifting has been generalized to several ideas, such as G-Rad-lifting modules, see [15], FI-J-supplemented and FI-⊕-J-supplemented modules, see [16]. The mentioned ideas provided as the motivation for the introduction of two structures, named, ⊕-g-supplemented module and ⊕-g-radical supplemented module. $M \in$ Mod-$\mathcal{R}$ is named to be ⊕-g-supplemented (⊕-g-radical supplemented), respectively, if ∀ $H \hookrightarrow M$ have a g-supplement (g-radical supplement), say U, such that $U \hookrightarrow^{\oplus} M$.

We define the concept of ⊕-g-supplemented modules, in section 2. This section displays a several different characteristics of this class of modules as well as how it relates to many other types of modules. We present some interesting counterexamples to differentiate between various other classes of modules and the ⊕-g-supplemented characteristic of modules. Section 3 discusses direct summands and decompositions of the category of ⊕-g-supplemented modules. It is illustrated that whenever a ⊕-g-supplemented module has the (D_3) property, its direct summands inherit it. Section 4 defines the notions of projective g-covers and g-semiperfect modules. Section 5 is dedicated to the study and investigation of some of the characteristics and relations of ⊕-g-radical supplemented modules. The study and examination of some of the features and connections of ⊕-g-radical supplemented modules are the focus of Section 5. The idea of ⊕-g-radical supplemented modules was developed as a result of numerous ⊕-g-supplemented module results. Summands and decompositions of modules with ⊕-g-radical supplements are covered in Section 6. You can find the ideas that are not covered here in [17, 2].

Lemma 1.1. ([6]) Let $M \in$ Mod-$\mathcal{R}$, we deduce:

(*i*) For $U \hookrightarrow M$, the next are identical.

(a) $U \hookrightarrow^{gs} M$.

(b) If $H + U = M$, then $H \hookrightarrow^{\oplus} M$ with a module M/H semisimple.

(*ii*) Assume U, T and H are submodules of M with $T \hookrightarrow U$.

(a) If $U \hookrightarrow^{gs} M$, then $T \hookrightarrow^{gs} M$ and $U/T \hookrightarrow^{gs} M/T$.

(b) $U + H \hookrightarrow^{gs} M$ if and only if $U \hookrightarrow^{gs} M$ and $H \hookrightarrow^{gs} M$.

(*iii*) If $H \hookrightarrow^{gs} M$ and $f: M \longrightarrow M'$ is any homomorphism, then $f(H) \hookrightarrow^{gs} M'$. In particuler, if $H \hookrightarrow^{gs} M \hookrightarrow M'$, then $H \hookrightarrow^{gs} M'$.

(*iv*) Assume $H_1 \hookrightarrow M_1 \hookrightarrow M$, $H_2 \hookrightarrow M_2 \hookrightarrow M$ and $M = M_1 \oplus M_2$, then $H_1 \oplus H_2 \hookrightarrow^{gs} M_1 \oplus M_2$ if and only if $H_1 \hookrightarrow^{gs} M_1$ and $H_2 \hookrightarrow^{gs} M_2$.

Lemma 1.2. ([18]) $Rad_g(M) = \oplus_{t\in\tau} Rad_g(H_t)$, for $M = \oplus_{t\in\tau} H_t \in$ Mod-$\mathcal{R}$.

2. ⊕-g-supplemented modules

We describe the notion of ⊕-g-supplemented modules in this section and look at some of its unique features.

Definition 2.1. $M \in$ Mod-$\mathcal{R}$ is named to be ⊕-g-supplemented if, ∀ $H \hookrightarrow M$ have a g-supplement, say U, such that $U \hookrightarrow^{\oplus} M$. If a ring $\mathcal{R}$ is ⊕-g-supplemented as $\mathcal{R}$-module, it is referred to as a ⊕-g-supplemented ring. To represent the category of all ⊕-g-supplemented modules, we shall use the symbol DGS.

Evidently, all modules that are ⊕-supplemented or ⊕-δ-supplemented modules, including ⊕-g-supplemented which implies g-supplemented modules.

If there exists submodules T and B of M such that $M = T \oplus B$ with $T \hookrightarrow U$ and $U \cap B \hookrightarrow^{gs} M$ (so in B) $\forall\, U \hookrightarrow M$, then $M \in$Mod-$\mathcal{R}$ is named to be e-lifting (in our work, g-lifting) [8]. $M \in$ Mod-$\mathcal{R}$ is named to be generalized hollow if, $\forall$ proper $T \hookrightarrow M$, then $T \hookrightarrow^{gs} M$ [7]. The category of hollow modules, as you can see, is generalized hollow in general. If $Rad_g(M)$ is maximal and g-small in M, then $M \in$ Mod-$\mathcal{R}$ is referred to as g-local (in [18], e-local). Moreover, as seen in the next example, it indicates no link between the classes (g-local and generalized hollow; local and g-local) modules.

Example 2.2. We have $\mathbb{Z}$-module $\mathbb{Z}_{24}$ is g-local, because that $Rad_g(M) = 2\mathbb{Z}_{24} \hookrightarrow^{gs} \mathbb{Z}_{24}$ and maximal in $\mathbb{Z}_{24}$. As $3\mathbb{Z}_{24}$ is not g-small in $\mathbb{Z}_{24}$, then $\mathbb{Z}_{24}$ does not generalized hollow, also it is easily to see that $\mathbb{Z}_{24}$ not local $\mathbb{Z}$-module. Since all of its submodules are g-small, each semisimple $M \in$ Mod-$\mathcal{R}$ is generalized hollow, however, it is not g-local; in fact, $Rad_g(M) = M$. The class of simple modules is local but not g-local.

However, we will show that the g-local module is an element of the DGS. The following lemma, which can be found in [18, Proposition 2.14], must be used first.

Lemma 2.3. Assume $M \in$ Mod-$\mathcal{R}$ and $U \hookrightarrow M$. If M is g-local, so either $U \hookrightarrow^{gs} M$ or there is semisimple $T \hookrightarrow M$ such that $M = U \oplus T$.

The next outcome is implied from Lemma 2.3.

Proposition 2.4. $M \in DGS$, whenever M is g-local.

Each semisimple module is belong to DGS but not g-local. According to this illustration, the inverse of the previously stated statement was never true, in general.

Remark 2.5. Any generalized hollow module is g-lifting, and hence $\oplus$-g-supplemented.

In general, a $\oplus$-g-supplemented module does not have to be generalized hollow; for example, in Example 2.2, the $\mathbb{Z}$-module $\mathbb{Z}_{24}$ is not generalized hollow, while it is $\oplus$-g-supplemented, in fact $\mathbb{Z}_{24}$ as $\mathbb{Z}$-module g-local. For another illustration of class of $\oplus$-g-supplemented modules that is not g-lifting, see Example 2.18.

Lemma 2.6. For $M(\neq 0) \in DGS$, the generalized radical of M is nonzero.
Proof. If $Rad_g(M) = 0$. Let $T \hookrightarrow M$, then $\exists\, U \hookrightarrow^{\oplus} M$ such that $M = T + U$ and $T \cap U \hookrightarrow^{gs} U$. So $T \cap U \hookrightarrow Rad_g(M) = 0$, hence $T \hookrightarrow^{\oplus} M$, and M is semisimple. So $Rad_g(M) = M = 0$, a contradiction. □

Proposition 2.7. Let $M(\neq 0) \in$Mod-$\mathcal{R}$ be indecomposable with $Rad_g(M) \neq M$. Then M is local and g-local, whenever $M \in DGS$.
Proof. Because that $Rad_g(M) \neq M$ and according to [7, Theorem 4], the observation that an $\oplus$-g-supplemented indecomposable module imply generalized hollow proves that M is local. According to Lemma 2.6, we deduce $Rad_g(M) \neq 0$ that implies M is not simple, and is therefore a g-local module according to [18, Proposition 2.7]. □

Lemma 2.8. Assume $A, B \hookrightarrow M \in$ Mod-$\mathcal{R}$ such that T is a g-supplement of $A + B$ in M and G a g-supplement of $A \cap (T + B)$ in A, then $T + G$ is a g-supplement of B in M.
Proof. Look at [8, Lemma 6]. □

Theorem 2.9. A finite direct sums of class DGS is closed.
Proof. Suppose $M_t \in DGS$ for $1 \le t \le n$ where $n \in \mathbb{Z}^+$. To show that $M = \bigoplus_{t \in \tau} M_t \in DGS$. Consider the situation where the index set $\tau = \{1,2\}$. If $L \hookrightarrow M$, then $M = M_1 + M_2 + L$, trivially has 0 as a g-supplement inside M. Suppose H is a g-supplement of $M_2 \cap (M_1 + L)$ inside M_2 such that $H \hookrightarrow^{\oplus} M_2$, Lemma 2.8 implies H is a g-supplement of $M_1 + L$ inside M. Let G be a g-supplement of $M_1 \cap (L + H)$ inside M_1 such that $G \hookrightarrow^{\oplus} M_1$. Again, by Lemma 2.8, $H + G$ is a g-supplement of L inside M. Also, we deduce $H + G = H \oplus G \hookrightarrow^{\oplus} M_1 \oplus M_2 = M$, and then $M = M_1 \oplus M_2 \in DGS$. □

Corollary 2.10. The finite direct sums of g-lifting, generalized hollow or g-local modules belong to DGS.

In Proposition 2.13 we shall examine a sufficient case for a ⊕-g-supplemented module to be inherited by its submodules. The previous is established in [19, Lemma 2.3].

Lemma 2.11. Assume $M \in$ Mod-$\mathcal{R}$ and $T \hookrightarrow M$ with M/T projective. If $B \hookrightarrow^{\oplus} M = B + T$, then $B \cap T \hookrightarrow^{\oplus} M$.

Lemma 2.12. Let $M \in$ Mod-$\mathcal{R}$, $L \hookrightarrow V \hookrightarrow M$ and $U \hookrightarrow M$. If V is a g-supplement of U. Then,
(*i*) $L \hookrightarrow^{gs} V$ if and only if $L \hookrightarrow^{gs} M$.
(*ii*) $Rad_g(V) = V \cap Rad_g(M)$.
Proof. **(*i*)** $\Longrightarrow$) From Lemma 1.1(iii).
$\Longleftarrow$) Let $T \hookrightarrow^{e} V$ such that $L + T = V$. As $M = U + V$, then $M = U + L + T$. Since $L \hookrightarrow^{gs} M$, $M = U + T$ because $U + T \hookrightarrow^{e} M$. Thus $T = V$, since V is a g-supplement of U in M. Hence $L \hookrightarrow^{gs} V$.
(*ii*) In the fact $Rad_g(V) \subseteq V \cap Rad_g(M)$ always holds. Suppose $m \in V \cap Rad_g(M)$ then $m \in Rad_g(M)$, so by [7, Lemma 5] $mR \hookrightarrow^{gs} M$. As $mR \hookrightarrow V$ and V is a g-supplement in M, so (i) implies $mR \hookrightarrow^{gs} V$ and hence $m \in Rad_g(V)$. Therefore $V \cap Rad_g(M) \subseteq Rad_g(V)$ and the necessary equality is achieved. □

In instance, if V is a d.s. of a module M in the prior lemma, then (i) and (ii) are holds.

Proposition 2.13. Let $M \in DGS$ and $T \hookrightarrow M$. If M/T is projective, then $T \in DGS$.
Proof. Suppose that $K \hookrightarrow T$. Then $\exists\, L \hookrightarrow^{\oplus} M$ such that $M = K + L$ and $K \cap L \hookrightarrow^{gs} L$, as $M \in DGS$. It follows that $M = T + L$ and so $T \cap L \hookrightarrow^{\oplus} M$, also for T from Lemma 2.11. Moreover, $T = K + (T \cap L)$ and $K \cap (T \cap L) = K \cap L \hookrightarrow^{gs} M$, Lemma 2.12($i$) implies that $K \cap (T \cap L) \hookrightarrow^{gs} T \cap L$. Hence $T \in DGS$. □

For $M \in$ Mod-$\mathcal{R}$, $T \hookrightarrow M$ is named fully invariant if $f(T) \subseteq T$ for each $f \in End_R(M)$. M is named duo (weak duo), respectively, if all its submodules (d.s(s)) are fully invariant [20]. The class of duo modules are weak duo. Also $A \hookrightarrow M \in$ Mod-$\mathcal{R}$ is named distributive if, $A \cap (B + C) = (A \cap B) + (A \cap C)$ or $A + (B \cap C) = (A + B) \cap (A + C)$ for each $B, C \hookrightarrow M$. $M \in$ Mod-$\mathcal{R}$ is named distributive if all its submodules are distributive [21].

Here, we will show that in some cases, the class DGS is closed under the quotient.

Theorem 2.14. Let $M \in DGS$, and $T \hookrightarrow M$. Then,
(*i*) If for each $B \hookrightarrow^{\oplus} M$, we have $(T + B)/T \hookrightarrow^{\oplus} M/T$, then $M/T \in DGS$.
(*ii*) If for each decomposition $M = M_1 \oplus M_2$, $T = (T \cap M_1) \oplus (T \cap M_2)$, then $M/T \in DGS$.
(*iii*) $M/T \in DGS$, if whenever T is fully invariant. In specifically, each duo module in DGS has also quotient module in DGS.
(*iv*) $M/T \in DGS$, if whenever T is distributive. In specifically, each distributive module in DGS has also quotient module in DGS.
Proof. **(*i*)** Consider $T \hookrightarrow X \hookrightarrow M$. As $M \in DGS$, it follows that $M = X + B$ and $X \cap B \hookrightarrow^{gs} B$ for some $B \hookrightarrow^{\oplus} M$. Thus, $M/T = X/T + (T + B)/T$. Let $\pi : B \longrightarrow (T + B)/T$ be a natural map. As $X \cap B \hookrightarrow^{gs} B$, Lemma 1.1($iii$) implies $\pi(X \cap B) = (T + (X \cap B))/T = (X/T) \cap (T + B)/T \hookrightarrow^{gs} (T + B)/T$. By hypothesis, $(T + B)/T \hookrightarrow^{\oplus} M/T$ and so $M/T \in DGS$.
(*ii*) Let $T \hookrightarrow M$ and let $B \hookrightarrow^{\oplus} M$. Therefore, $M = B \oplus \grave{B}$ for some $\grave{B} \hookrightarrow M$. To prove that $(T + L)/T \hookrightarrow^{\oplus} M/T$. By assumption, $T = (T \cap B) \oplus (T \cap \grave{B})$. Thus $(B + T) \cap (\grave{B} + T) \hookrightarrow (B + T + \grave{B}) \cap T + (B + T + T) \cap \grave{B}$. So $(B + T) \cap (\grave{B} + T) \hookrightarrow T + (B + T \cap B + T \cap \grave{B}) \cap \grave{B}$ this implies $(B + T) \cap (\grave{B} + T) \hookrightarrow T$, thus $M/T = (B + T)/T \oplus (\grave{B} + T)/T$. This means that $(B + T)/T \hookrightarrow^{\oplus} M/T$. By ($i$), the result has been as follows.
The implications **(*iii*)** and **(*iv*)** followed directly from (ii). □

Proposition 2.15. For an arbitrary nonsingular $M \in$ Mod-$\mathcal{R}$, M is ⊕-δ-supplemented if and only if $M \in DGS$.
Proof. The requirement is evident. Assume that $M \in DGS$. If $T \hookrightarrow M$, then $\exists\, V \hookrightarrow^{\oplus} M$ such that $M = T + V$ and $T \cap V \hookrightarrow^{gs} V$. Suppose $V = (T \cap V) + K$ whenever V/K is singular. As M is nonsingular, then V is also nonsingular, and so $K \hookrightarrow^{e} V$ that implies $K = V$. Thus $T \cap V \hookrightarrow^{\delta s} V$ and hence M is ⊕-δ-supplemented. □

Moreover, $M \in$ Mod-$\mathcal{R}$ is named to be refinable if for each U,$V \hookrightarrow M$ with $M = U + V$, $\exists\, \grave{U} \hookrightarrow^{\oplus} M$ such that $\grave{U} \hookrightarrow U$ and $M = \grave{U} + V$.

Proposition 2.16. If $M \in$Mod-$\mathcal{R}$ is refinable, then $M \in DGS$ if and only if it is g-supplemented.
Proof. The requirement is evident. Suppose $M \in$ Mod-$\mathcal{R}$ is g-supplemented, and $T \hookrightarrow M$. Thus $M = T + V$ and

$T \cap V \hookrightarrow^{gs} V$ for some $V \hookrightarrow M$. Since M is refinable, $M = T + \grave{V}$ for a $\grave{V} \hookrightarrow^{\oplus} M$ with $\grave{V} \hookrightarrow V$. Evidently, $T \cap \grave{V} \hookrightarrow^{gs} M$, Lemma 2.12($i$) implies that $T \cap \grave{V} \hookrightarrow^{gs} \grave{V}$. Hence T has a g-supplement $\grave{V} \hookrightarrow^{\oplus} M$. Therefore $M \in DGS$. □

Each g-lifting module, by definition, belongs to DGS. The following example demonstrates the existence of a ⊕-g-supplemented module that does not seem to be g-lifting.

Example 2.17. According to [8], assume that $\mathcal{R} = \mathbb{Z}_8$ therefore $2\mathcal{R}_{\mathcal{R}}/4\mathcal{R}_{\mathcal{R}}$ and $\mathcal{R}_{\mathcal{R}}$ are both g-lifting, while $(2\mathcal{R}_{\mathcal{R}}/4\mathcal{R}_{\mathcal{R}}) \oplus \mathcal{R}_{\mathcal{R}}$ does not be g-lifting. It follows that $(2\mathcal{R}_{\mathcal{R}}/4\mathcal{R}_{\mathcal{R}}) \oplus \mathcal{R}_{\mathcal{R}}$ is a ⊕-g-supplemented module, from Corollary 2.10.

If $\forall\, U, V \hookrightarrow M \in$ Mod-$\mathcal{R}$ with $M = U + V$, $\exists$ an $f \in End_R(M)$ such that $Imf \hookrightarrow U$ and $Im(1-f) \hookrightarrow V$, then M is named π-projective. If $\forall\, U, V \hookrightarrow^{\oplus} M \in$ Mod-$\mathcal{R}$, $U \cap V \hookrightarrow^{\oplus} M$ then M named a module with SIP.

The theorem below shows that the two categories g-lifting modules and ⊕-g-supplemented modules coincide in some cases.

Theorem 2.18. Assume $M \in DGS$ and any of the next claims must be satisfied:

(*i*) M is duo.

(*ii*) M is distributive.

(*iii*) M is π-projective.

(*iv*) M is refinable and have the SIP.

Then $M \in$ Mod-$\mathcal{R}$ is g-lifting.

Proof. **(*i*)** Let $T \hookrightarrow M$. As $M \in DGS$, then $\exists\, B \hookrightarrow^{\oplus} M$ such that $M = T + B$ and $T \cap B \hookrightarrow^{gs} B$. Then $\exists\, D \hookrightarrow M$ with $M = B \oplus D$. Since T is fully invariant, $T = (T \cap B) \oplus (T \cap D)$, and hence $M = (T \cap D) \oplus B$, where $T \cap D \hookrightarrow X$ and $T \cap B \hookrightarrow^{gs} B$.

(*ii*) Comparable to proof (i).

(*iii*) Let $T \hookrightarrow M$, then $\exists\, B \hookrightarrow^{\oplus} M$ such that $M = T + B$ and $T \cap B \hookrightarrow^{gs} B$, since $M \in DGS$. From π-projectivity for M, $\exists\, K \hookrightarrow T$ with $M = K \oplus L$, see [2, 41.14]. Hence M is g-lifting.

(*iv*) Since $M \in DGS$ and $T \hookrightarrow M$, then $\exists\, B \hookrightarrow^{\oplus} M$ such that $M = T + B$ and $T \cap B \hookrightarrow^{gs} B$. Since M is refinable,$\exists\, K \hookrightarrow^{\oplus} M$ with $K \hookrightarrow T$ such that $M = K + B$. So $B \cap K \hookrightarrow^{\oplus} M$, as M have the SIP. Then $\exists\, U \hookrightarrow M$ such that $M = (B \cap K) \oplus U$. Thus, $B = (B \cap K) \oplus (B \cap U)$, and so $M = K + B = K \oplus (B \cap U)$. Evidently, $T \cap (B \cap U) \hookrightarrow^{gs} M$. □

Theorem 2.19. For $M \in$ Mod-$\mathcal{R}$, consider the following:

(*i*) $M \in DGS$.

(*ii*) $M/Rad_g(M)$ is semisimple.

Then $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$, whenever M is distributive, and $(ii) \Longrightarrow (i)$ whenever M is refinable with $Rad_g(M) \hookrightarrow^{gs} M$.

Proof. $(i) \Longrightarrow (ii)$ Let $T \hookrightarrow M$, then $\exists\, B \hookrightarrow^{\oplus} M$ such that $M = T + B$ and $T \cap B \hookrightarrow^{gs} B$, and so $T \cap B \hookrightarrow^{gs} M$. Since $M \in$ Mod-$\mathcal{R}$ is distributive and $T \cap B \hookrightarrow Rad_g(M)$, then we deduce $Rad_g(M) = (T \cap B) + Rad_g(M) = (T + Rad_g(M)) \cap (B + Rad_g(M))$. Hence $\frac{M}{Rad_g(M)} = \frac{T+Rad_g(M)}{Rad_g(M)} \oplus \frac{B+Rad_g(M)}{Rad_g(M)}$, as required.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Assume that $T \hookrightarrow M$. From (ii), $\exists\, B \hookrightarrow M$ with $\frac{M}{Rad_g(M)} = \frac{T+Rad_g(M)}{Rad_g(M)} \oplus \frac{B}{Rad_g(M)}$. Therefore $M = T + B$ and $Rad_g(M) = (T + Rad_g(M)) \cap B = (T \cap B) + Rad_g(M)$, it follows that $T \cap B \hookrightarrow Rad_g(M)$ implies $T \cap B \hookrightarrow^{gs} M$. Since $M = T + B \in$ Mod-$\mathcal{R}$ is refinable, then $\exists\, U \hookrightarrow^{\oplus} M$ with $M = T + U$ where $U \hookrightarrow B$. From Lemma 2.12(i), $T \cap U \hookrightarrow^{gs} U$, and this end the proof. □

3. Main results

This part examines the cases under which direct summands of ⊕-g-supplemented modules can be ⊕-g-supplemented.

Let $n \in \mathbb{Z}^+$ and $\tau = \{1,2,\dots,n\}$. If all of the modules in the collection $\{M_i \mid i \in \tau\}$ are M_j-projective for all $(i \neq j) \in \tau$, then the collection is referred to as relatively projective. It is unknown wether direct summands inherit the property ⊕-g-supplemented.

Theorem 3.1. Let $\{M_t \mid t \in \{1,2,\dots,n\}\}$ be a family of relatively projective modules. Then, for each $t \in \{1,2,\dots,n\}$, $M_t \in DGS$ if and only if $M = \bigoplus_{t=1}^{n} M_t \in DGS$.
Proof. Assume $M = \bigoplus_{t=1}^{n} M_t \in DGS$. We will prove $M_1 \in DGS$. If $T \hookrightarrow M_1$, then $\exists\ B \hookrightarrow^{\oplus} M$ such that $M = T + B$ and $T \cap B \hookrightarrow^{gs} B$. We have $M = T + B = M_1 + B$, then $M = M_1 \oplus B_1$ for some $B_1 \hookrightarrow B$, see [10, Lemma 4.47]. Therefore, $B = B_1 \oplus (M_1 \cap B)$. It is easily to see that $M_1 = T + (M_1 \cap B)$ and $M_1 \cap B \hookrightarrow^{\oplus} M_1$. Because $T \cap (M_1 \cap B) = T \cap B \hookrightarrow^{gs} B$ and $M_1 \cap B \hookrightarrow^{\oplus} B$ implies $T \cap (M_1 \cap B) \hookrightarrow^{gs} M_1 \cap B$, from Lemma 2.12($i$). Hence M_1 is $\oplus$-g-supplemented. In Theorem 2.9, the reverse is demonstrated. □

For $M \in$ Mod-$\mathcal{R}$, the next requirement will be considered:

(D_3) If $A, B \hookrightarrow^{\oplus} M = A + B$, then $A \cap B \hookrightarrow^{\oplus} M$.

Proposition 3.2. If $M \in DGS$ has (D_3) and $T \hookrightarrow^{\oplus} M$, then $T \in DGS$.
Proof. Assume that $T \hookrightarrow^{\oplus} M$ and $U \hookrightarrow T$. Since $M \in DGS$, $M = U + B$ and $U \cap B \hookrightarrow^{gs} B$ for some $B \hookrightarrow^{\oplus} M$. It follows that $T = U + (T \cap B)$. Since $B, T \hookrightarrow^{\oplus} M$ with $M = T + B$ implies that $T \cap B \hookrightarrow^{\oplus} M$, as M has (D_3). Also $U \cap (T \cap B) = U \cap B \hookrightarrow^{gs} T \cap B$, from Lemma 2.12($i$). Therefore $T \in DGS$. □

Corollary 3.3. Let $M \in$ Mod-$\mathcal{R}$ has the SIP. Then, for each $T \hookrightarrow^{\oplus} M$, $T \in DGS$ if and only if $M \in DGS$.

Let $M \in$ Mod-$\mathcal{R}$ and $T \hookrightarrow M$. T is referred to as closed if it has no proper essential extensions inside M. However, M is named to be extending if, each closed $U \hookrightarrow M$, we have $U \hookrightarrow^{\oplus} M$. Recall from [22] that if, all partial endomorphisms of M have closed kernels, then $M \in$ Mod-$\mathcal{R}$ is named polyform.

Corollary 3.4. Let $M \in$ Mod-$\mathcal{R}$ be extending and polyform. Then, for each $T \hookrightarrow^{\oplus} M$, $T \in DGS$ if and only if $M \in DGS$.
Proof. Evident by [23, Lemma 11] and Corollary 3.3. □

Corollary 3.5. If $M \in$ Mod-$\mathcal{R}$ is quasi-projective, then
(*i*) $M \in DGS$ if and only if if $T \in DGS$, for each $T \hookrightarrow^{\oplus} M$.
(*ii*) M is $\oplus$-δ-supplemented if and only if T is $\oplus$-δ-supplemented, for each $T \hookrightarrow^{\oplus} M$.
Proof. It follows M has (D_3) property, see [10, Lemma 4.6 and Proposition 4.38]. Therefore, Proposition 3.2 and [12, Theorem 2.5] directly follow (i) and (ii), respectively. □

Corollary 3.6. Let $M \in$ Mod-$\mathcal{R}$ be projective. The next assertions are then identical.
(*i*) $M \in DGS$.
(*ii*) M is $\oplus$-δ-supplemented.
(*iii*) $T \in DGS$, for each $T \hookrightarrow^{\oplus} M$.
(*iv*) T is $\oplus$-δ-supplemented, for each $T \hookrightarrow^{\oplus} M$.
Proof. Corollary 3.5 deduce that (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv). According to [6], the two subclasses δ-small and g-small submodules are identical in terms of projectivity for M, which yields (i) $\Leftrightarrow$ (ii). □

Proposition 3.7. Let $M \in DGS$ whose any g-supplement is a d.s. in M. Then $T \in DGS$, for each $T \hookrightarrow^{\oplus} M$
Proof. Let $T \hookrightarrow^{\oplus} M$, so $\exists\ K \hookrightarrow M$ such that $M = T \oplus K$. Since $M \in DGS$, so it is g-supplemented and thus M/K is g-supplemented, from [7, Theorem 2], that deduce T is g-supplemented. Let $B \hookrightarrow T$, then B has a g-supplement C in T. To show $H \hookrightarrow^{\oplus} T$. Note $M = T \oplus K = (B + K) + C$, and $(B + K) \cap C \hookrightarrow (B + C) \cap K + (C + K) \cap B = (C + K) \cap B \hookrightarrow B$. Thus $(B + K) \cap C \hookrightarrow B \cap C \hookrightarrow^{gs} C$. Hence $B + K$ has a g-supplement C in M. By the assumption, $M = C \oplus D$ for some $D \hookrightarrow M$. Therefore, $T = C \oplus (L \cap D)$. □

Proposition 3.8. Let $M \in$ Mod-$\mathcal{R}$ be π-projective. Then $M \in DGS$ if and only if $T \in DGS$, for each $T \hookrightarrow^{\oplus} M$.
Proof. $\Longrightarrow$) Directly from Theorem 2.18 and [8, Lemma 3].
$\Longleftarrow$) Evident. □

Proposition 3.9. Let $M \in$ Mod-$\mathcal{R}$, and $T \hookrightarrow M$ is (fully invariant or distributive) d.s. of M. Then $M \in DGS$ if and only if $T \in DGS$ and $M/T \in DGS$.
Proof.$\Longrightarrow$) Assume $T \hookrightarrow M$ is fully invariant, then $M/T \in DGS$, from Theorem 2.14(iii). Let $B \hookrightarrow T$, then $\exists\ K \hookrightarrow^{\oplus} M$ such that $M = B + K$ and $B \cap K \hookrightarrow^{gs} K$, since $M \in DGS$. Then $\exists\ \grave{K} \hookrightarrow M$ such that $M = K \oplus \grave{K}$. So $T = T \cap (B + K) = B + (T \cap K)$. By [20, Lemma 2.1], $T = (T \cap K) \oplus (T \cap \grave{K})$, hence $T \cap K \hookrightarrow^{\oplus} T$. On the other side, $B \cap (T \cap K) = B \cap$

$K \hookrightarrow^{gs} M$, and as $T \cap K \hookrightarrow^{\oplus} M$, imply $B \cap (T \cap K) \hookrightarrow^{gs} T \cap K$, according to Lemma 2.12(i). Thus, $T \in DGS$. Similarly, when T is distributive.

$\Leftarrow$) By Theorem 2.9. □

Corollary 3.10. Let $M \in$ Mod-$\mathcal{R}$ be (weak duo or distributive), and $T \hookrightarrow^{\oplus} M$. Then $M \in DGS$ if and only if $T \in DGS$ and $M/T \in DGS$.

Corollary 3.11. Let $M \in$ Mod-$\mathcal{R}$ be (weak duo or distributive). Then $M \in DGS$ if and only if $T \in DGS$, for each $T \hookrightarrow^{\oplus} M$.

According to Theorem 2.9 and by some cases, if $M = \oplus_{t=1}^{n} M_t \in$ Mod-$\mathcal{R}$ is weak duo or, distributive. Then, for each $t \in \{1,2,\ldots,n\}$, $M_t \in DGS$ if and only if $M \in DGS$.

Corollary 3.12. Assume $M \in DGS$, then $M/Rad_g(M) \in DGS$. Also, if $Rad_g(M) \hookrightarrow^{\oplus} M$, then $Rad_g(M) \in DGS$.

Proof. According to [6, Corollary 2.11] $Rad_g(M) \hookrightarrow M$ is fully invariant. The consequence is followed directly by Theorem 2.14(iii) and Proposition 3.9. □

The following lemma demonstrates a situation under which the subclass g-small is coincide to small submodule.

Lemma 3.13. Let $M \in$ Mod-$\mathcal{R}$ be indecomposable and let $T \hookrightarrow M$ is proper. Then $T \hookrightarrow^{gs} M$ if and only if $T \hookrightarrow^{s} M$.

Proof. It is evidently. □

So, we deduce:

Lemma 3.14. Assume $M \in$ Mod-$\mathcal{R}$ is indecomposable. Then $M \in DGS$ if and only if M is $\oplus$-supplemented.

Proposition 3.15. For an indecomposable $M \in$ Mod-$\mathcal{R}$, the next are coincide.

(*i*) M is hollow.

(*ii*) M is generalized hollow.

(*iii*) M is $\oplus$-supplemented.

(*iv*) M is $\oplus$-g-supplemented.

(*v*) T is $\oplus$-supplemented, for each $T \hookrightarrow^{\oplus} M$.

(*vi*) T is $\oplus$-g-supplemented, for each $T \hookrightarrow^{\oplus} M$.

Proof. $(\boldsymbol{i}) \Leftrightarrow (\boldsymbol{ii})$ Follows directly from Lemma 3.13.

$(\boldsymbol{iii}) \Leftrightarrow (\boldsymbol{iv})$ and $(\boldsymbol{v}) \Leftrightarrow (\boldsymbol{vi})$ Follows directly from Lemma 3.14.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{vi})$ Evident.

$(\boldsymbol{vi}) \Longrightarrow (\boldsymbol{ii})$ Assume $T \hookrightarrow M \in$ Mod-$\mathcal{R}$ is proper, then $\exists\, B \hookrightarrow^{\oplus} M$ such that $M = T + B$ and $T \cap B \hookrightarrow^{gs} B$, as $M \in DGS$. Because M is indecomposable and $B \neq 0$, it follows that $B = M$ and hence $T \hookrightarrow^{gs} M$. Therefore, (ii) holds.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{iv})$ Evident. $(\boldsymbol{iv}) \Longrightarrow (\boldsymbol{ii})$ The same as proof $(vi) \Longrightarrow (ii)$. □

Proposition 3.15 and Remark 2.5 focus attention to the fact that, in addition to Theorem 2.18, an indecomposable notion is also seen as a requirement that defines $\oplus$-g-supplemented as g-lifting.

The idea of Dual Goldie dimension of $M \in$ Mod-$\mathcal{R}$ was established by Varadarajan [24], and denoted by corank(M). Furthermore, whenever $M = 0$, corank$(M) = 0$. Let $M \neq 0$, and an $k \in \mathbb{Z}^{+}$. For $1 \leq t \leq k$, if $\exists$ an epimorphism $\varphi: M \longrightarrow \Pi_{t=1}^{k} L_t$ where $L_t \neq 0$, then we call that corank$(M) \geq$ k. If corank$(M) \geq k$ and corank$(M) \ngeq k+1$, then corank$(M) = k$ is defined. If corank$(M) \geq k$, for any $k \geq 1$, then we put corank$(M) = \infty$. In [24] it was proved that corank$(M) = k < \infty$ if and only if $\exists$ an epimorphism $\varphi: M \longrightarrow \Pi_{t=1}^{k} N_t$ where N_i is hollow, for $1 \leq t \leq k$, and $Ker\varphi \hookrightarrow^{s} M$. Furthermore, $M \in$ Mod-$\mathcal{R}$ is hollow if and only if corank$(M) = 1$.

If the direct decomposition $H = \bigoplus_{t\in\tau} H_t$ of $H \in$ Mod-$\mathcal{R}$ is the direct sum of indecomposable $H_t \hookrightarrow H$, $t \in \tau$, it is named to as being indecomposable, see [17, P.140].

Proposition 3.16. If $H = \bigoplus_{t=1}^{m} H_t \in$ Mod-$\mathcal{R}$ is an indecomposable decomposition, then the next are coincide.

(*i*) $H_1, H_2, \ldots, H_m$ are hollow.

(*ii*) $H_1, H_2, \ldots, H_m$ are generalized hollow.

(*iii*) T is $\oplus$-supplemented, for each $T \hookrightarrow^{\oplus} H$.

(*iv*) T is $\oplus$-g-supplemented, for each $T \hookrightarrow^{\oplus} H$.

Proof. $(\boldsymbol{i}) \Leftrightarrow (\boldsymbol{ii})$ Suppose that $H = \bigoplus_{t=1}^{m} H_t$ is an indecomposable decomposition module, this means $H_t \hookrightarrow H$ is indecomposable, for all $t = 1,2,\dots,m$. Thus, Proposition 3.15 implies the result.

$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{iii})$ Assume $T \hookrightarrow^{\oplus} H$. If $T = H$, [11, Corollary 1.6] implies T is $\oplus$-supplemented. Assume $T \neq H$, so $\exists\, \grave{T} \hookrightarrow H$ such that $H = T \oplus \grave{T}$. Since corank$(T) = 1$, T is hollow, and thus T is $\oplus$-supplemented.

$(\boldsymbol{iii}) \Longrightarrow (\boldsymbol{i})$ We have $H_t \hookrightarrow H$ is indecomposable, for each $t = 1,2,\dots,m$. From (iii), for each $i = 1,2,\dots,m$, H_t is $\oplus$-supplemented, and hence H_t is hollow, see Proposition 3.15.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{iv})$ According to Proposition 3.15, we deduce $H_1, H_2, \dots, H_m$ are hollow, and by the implication $(i) \Longrightarrow (iii)$, we deduce T is $\oplus$-supplemented, and hence it is $\oplus$-g-supplemented, for each $T \hookrightarrow^{\oplus} H$.

$(\boldsymbol{iv}) \Longrightarrow (\boldsymbol{ii})$ Analogous to $(iii) \Longrightarrow (i)$. □

Additionally, similar to the previous outcome, if it is possible to hypothesis that $H \in$ Mod-$\mathcal{R}$ has a finite decomposition $H = \bigoplus_{t=1}^{m} H_t$ such that $End_R(H_t)$ is local for all $t = 1,2,\dots,m$, this implies H_R is an indecomposable decomposition see [17, Theorem 12.6], and we arrive at a similar conclusion.

The decomposition of $\oplus$-g-supplemented modules will be investigated next.

Proposition 3.17. Let $H \in DGS$, then we can write $H = H_1 \oplus H_2$ where $H_1 \in$ Mod-$\mathcal{R}$ with $Rad_g(H_1) \hookrightarrow^{gs} H_1$ and $H_2 \in$ Mod-$\mathcal{R}$ with $Rad_g(H_2) = H_2$.

Proof. Suppose $M \in DGS$. As $Rad_g(H) \hookrightarrow H$, so $\exists\, H_1, H_2 \hookrightarrow H$ such that $H_1 + Rad_g(H) = H$ and $H_1 \cap Rad_g(H) \hookrightarrow^{gs} H_1$, where $H = H_1 \oplus H_2$. $Rad_g(H_1) \hookrightarrow H_1 \cap Rad_g(H)$ follows that $Rad_g(H_1) \hookrightarrow^{gs} H_1$. From Lemma 1.2, we conclude that $H = H_1 \oplus Rad_g(H_2)$, and thus $Rad_g(H_2) = H_2$. □

Proposition 3.18. If $H \in DGS$, then $H = H_1 \oplus H_2$ where H_1 is semisimple and $H_2 \in$ Mod-$\mathcal{R}$ with $Rad_g(H_2) \hookrightarrow^{e} H_2$.

Proof. Look at [8, Proposition 3]. □

Theorem 3.19. For $H \in$ Mod-$\mathcal{R}$ with (D_3), the next are coincide.

(*i*) $T \in DGS$, for each $T \hookrightarrow^{\oplus} H$.

(*ii*) $H \in DGS$.

(*iii*) $H = H_1 \oplus H_2$ where H_1 is semisimple and $H_2 \in DGS$ with $Rad_g(H_2) \hookrightarrow^{e} H_2$.

(*iv*) $H = H_1 \oplus H_2$ where $H_1, H_2 \in DGS$ with $Rad_g(H_1) \hookrightarrow^{gs} H_1$ and $Rad_g(H_2) = H_2$.

Proof. $(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Follows directly from Proposition 3.2.

$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{iii})$ Follows directly from Proposition 3.18 and part (i).

$(\boldsymbol{iii}) \Longrightarrow (\boldsymbol{ii})$ Because that ecah semisimple module implies $\oplus$-g-supplemented, Theorem 2.9 gives the required.

$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{iv})$ Follows directly from Proposition 3.17 and part (i).

$(\boldsymbol{iv}) \Longrightarrow (\boldsymbol{ii})$ Follows directly form Theorem 2.9. □

4. Applications

The classes projective g-covers and g-semiperfect modules will be explored in this part.

Definition 4.1. If $P \in$ Mod-$\mathcal{R}$ and $f: P \longrightarrow M$ is a surjective with $kerf \hookrightarrow^{gs} P$, the pair (P, f) is referred to as a g-cover of $M \in$ Mod-$\mathcal{R}$. If $P \in$ Mod-$\mathcal{R}$ is projective, (P, f) is referred to as a projective g-cover of the module M. $M \in$ Mod-$\mathcal{R}$ is named g-semiperfect if all of its quotient modules have a projective g-cover. However, if R_R is a g-semiperfect module, a ring R is named g-semiperfect.

The characterization of projective $\oplus$-g-supplemented modules can be found here.

Theorem 4.2. For a projective $M \in$ Mod-$\mathcal{R}$, the next are coincide.

(*i*) $M \in DGS$.

(*ii*) M is g-semiperfect.

Proof. $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$ Suppose $M \in DGS$. Let $H \hookrightarrow M$, so $\exists\, T \hookrightarrow^{\oplus} M$ such that $M = H + T$ and $H \cap T \hookrightarrow^{gs} T$. Therefore T is projective. Define $f: T \longrightarrow M/H$ by $f(t) = t + H$ for all $t \in T$. Then f is a surjective and $kerf = H \cap T \hookrightarrow^{gs} T$. Hence $f: T \longrightarrow M/H$ is a projective g-cover, and then (ii) holds.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Assume $A \hookrightarrow M$, then from (ii), M/A has a projective g-cover $f: F \longrightarrow \frac{M}{A}$. As M is projective, then $\exists$ a homomorphism $h: M \longrightarrow F$ with $fh = \pi$, as $\pi: M \longrightarrow \frac{M}{A}$ is the natural epimorphism. Thus, $F = h(M) + kerf$. As $kerf \hookrightarrow^{gs} F$, from Lemma1.1$(i)$ $\exists$ a semisimple submodule $Y \hookrightarrow F$ such that $F = h(M) \oplus Y$. Hence $h(M)$ is projective. Then $kerh \hookrightarrow^{\oplus} M$, $M = kerh \oplus T$ for some $T \hookrightarrow M$. Since $kerh \hookrightarrow ker\pi = A$, then $M = A + T$. We claim that $kerf \cap h(T) = h(X \cap T)$. Assume $a \in kerf \cap h(T)$, then $f(a) = 0$ and $a = h(y)$ for some $y \in T$. Then $\pi(y) = f(h(y)) = f(a) = 0$, $y \in A$ and then $a = h(y) \in h(A \cap T)$. Suppose $a \in h(A \cap T)$, $a = h(y)$ for some $y \in A \cap T$. So $x = h(y) \in h(T)$. Also, $y \in A = ker\pi$ imply $f\big(h(y)\big) = \pi(y) = 0$, and so $a = h(y) \in kerf$. Therefore $a \in kerf \cap h(T)$. We have $kerf \cap h(T) = h(A \cap T)$. $M = kerh \oplus T$ implies $h(M) = h(T) \hookrightarrow^{\oplus} F$. Since $kerf \hookrightarrow^{gs} F$, $kerf \cap h(T) \hookrightarrow^{gs} F$, so $h(A \cap T) \hookrightarrow^{gs} F$, from Lemma 2.12$(i)$, we have $h(A \cap T) \hookrightarrow^{gs} h(T)$. Because h is an isomorphism between T and $h(T)$, then $h^{-1}(kerf \cap h(T)) \hookrightarrow^{gs} T$, but $A \cap T \hookrightarrow h^{-1}(kerf \cap h(T))$ that deduce $A \cap T \hookrightarrow^{gs} T$. Therefore T is a g-supplement of A in $M \in$ Mod-$\mathcal{R}$. □

The outcome that followed will then arrive.

Corollary 4.3. $R \in DGS$ if and only if R is g-semiperfect, for each ring R.

5. ⊕-g-radical supplemented modules

In this part, we offer a new category of modules named ⊕-g-radical supplemented, which is an extension of ⊕-g-supplemented modules.

Definition 5.1. $M \in$ Mod-$\mathcal{R}$ is named to be ⊕-g-radical supplemented if, $\forall\ H \hookrightarrow M$ have a generalized radical supplement, say U, such that $U \hookrightarrow^{\oplus} M$. To denote the category of all ⊕-g-radical supplemented modules, we will use the symbol $DGRS$.

In between classes of ⊕-g-supplemented module and g-radical supplemented module, it is obvious that the class of ⊕-g-radical supplemented module exists. Yet, we do have the next simple fact.

Proposition 5.2. Let $M \in$ Mod-$\mathcal{R}$. If $Rad_g(M) = M$, then $M \in DGRS$.
Proof. It is easy. □

Proposition 5.3. Suppose $M \in$ Mod-$\mathcal{R}$ with $Rad_g(M) \hookrightarrow^{gs} M$. If $M \in DGRS$ then $M \in DGS$.
Proof. Evident by the definition and Lemma 2.12(i). □

Lemma 5.4. If $M \in$ Mod-$\mathcal{R}$ is finitely generated, then $Rad_g(M) \hookrightarrow^{gs} M$.
Proof. Let $M \in$ Mod-$\mathcal{R}$ is finitely generated and let $T \hookrightarrow^{e} M$ with $Rad_g(M) + T = M$. As M is finitely generated and $Rad_g(M)$ the sum of all g-small submodules of M, then $\exists$ a finite set of g-small submodules $B_1, B_2, \dots, B_n$ of M with $\sum_{t=1}^{n} B_t + T = M$. According to Lemma 1.1$(ii)$, we deduce that $\sum_{t=1}^{n} B_t \hookrightarrow^{gs} M$, and then $T = M$. Thus $Rad_g(M) \hookrightarrow^{gs} M$. □

The following follows from Lemma 5.4 and Proposition 5.3, and is immediate.

Corollary 5.5. If $M \in$ Mod-$\mathcal{R}$ is finitely generated, then $M \in DGS$ if and only if $M \in DGRS$.

Proposition 5.6. Suppose $M(\neq 0) \in$ Mod-$\mathcal{R}$ is indecomposable with $Rad_g(M) \neq M$. Then the next are coincide.
$(\boldsymbol{i})$ $M \in DGS$.
$(\boldsymbol{ii})$ $M \in DGRS$.
$(\boldsymbol{iii})$ M is g-local.
Proof. $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$ Evident. $(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Proposition 5.3 can be used to establish that $Rad_g(M) \hookrightarrow^{gs} M$ as follows; let $T \hookrightarrow^{e} M$ with $Rad_g(M) + T = M$. Since $M \in DGRS$, then $\exists\ B \hookrightarrow^{\oplus} M$ such that $T + B = M$ and $T \cap B \hookrightarrow Rad_g(B)$. Since M is indecompsable, either $B = M$ or $B = 0$. Assume that $B = M$, then $T \hookrightarrow Rad_g(M)$ and so $Rad_g(M) = M$, which is a contradiction. Hence $B = 0$, and thus $T = M$, as required.
$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{iii})$ and $(\boldsymbol{iii}) \Longrightarrow (\boldsymbol{i})$ Directly from Propositions 2.7 and 2.4, respectively. □

Proposition 5.7. Let $H_1, H_2 \in DGRS$. If $H = H_1 \oplus H_2$, then $H \in DGRS$.
Proof. Suppose that $T \hookrightarrow H$. As $(T + H_1) \cap H_2 \hookrightarrow H_2$, then $\exists\ B \hookrightarrow^{\oplus} H$ such that $((T + H_1) \cap H_2) + B = H_2$ and $(T + H_1) \cap B \hookrightarrow Rad_g(B)$. We deduce that $H = H_1 + ((T + H_1) \cap H_2) + B = H_1 + ((T + B) + H_1) \cap H_2 = (T + B) + H_1$. Also, since $(T + B) \cap H_1 \hookrightarrow H_1$, then $\exists\ L \hookrightarrow^{\oplus} H_1$ such that $((T + B) \cap H_1) + L = H_1$ and $(T + B) \cap L \hookrightarrow Rad_g(L)$. It follows that $H = (T + B) + H_1 = (T + B) + ((T + B) \cap H_1) + L = T + (B + L)$, however, it is easy to see that $T \cap (B + L) \hookrightarrow ((T + L) \cap B) + ((T + B) \cap L)$, and hence $T \cap (B + L) \hookrightarrow Rad_g(B) \oplus Rad_g(L) = Rad_g(B \oplus L)$. Thus $B \oplus L \hookrightarrow^{\oplus} H$. □

Corollary 5.8. The category $DGRS$ is closed under finite direct sums.

Using Corollaries 2.10 and 2.11, one can immediately establish a finite direct sums of g-lifting, generalized hollow, or g-local modules that belong to $DGRS$.

Proposition 5.9. Let $M \in DGRS$. If $H \hookrightarrow M$ suct that M/H projective, then $H \in DGRS$.
Proof. Let $T \hookrightarrow H$. As $M \in DGRS$, then $\exists\ B \hookrightarrow^{\oplus} M$ such that $M = T + B$ and $T \cap B \hookrightarrow Rad_g(B)$. From Lemma 2.11, we have $H \cap B \hookrightarrow^{\oplus} M$. Therefore $H = T + (H \cap B)$. Also $T \cap (H \cap B) \hookrightarrow (H \cap B) \cap Rad_g(M) = Rad_g(H \cap B)$, by Lemma 2.12($ii$). And it is clearly $H \cap B \hookrightarrow^{\oplus} H$, therefore $H \in DGRS$. □

Theorem 5.10. Let $M \in DGRS$, and $T \hookrightarrow M$. Then,
(*i*) If for any $B \hookrightarrow^{\oplus} M$, we have $(T + B)/T \hookrightarrow^{\oplus} M/T$, then $M/T \in DGRS$.
(*ii*) If for any decomposition $M = M_1 \oplus M_2$, $T = (T \cap M_1) \oplus (T \cap M_2)$, then $M/T \in DGRS$.
(*iii*) If T is fully invariant in M, $M/T \in DGRS$. Furthermore, the factor module of each duo module in $DGRS$ is so in $DGRS$.
(*iv*) If T is distributive in M, $M/T \in DGRS$. Furthermore, the factor module of each distributive module in $DGRS$ is also in $DGRS$.
Proof. **(*i*)** Consider $T \hookrightarrow U \hookrightarrow M$. Since $M \in DGRS$, then $M = U + B$ and $U \cap B \hookrightarrow Rad_g(B)$ for some $B \hookrightarrow^{\oplus} M$. It follows $\frac{M}{T} = \frac{U}{T} + \frac{T+B}{T}$. Also, $\frac{U}{T} \cap \frac{T+B}{T} = \frac{U \cap (T+B)}{T} = \frac{T+(U \cap B)}{T}$. Consider the canonical epimorphism $\pi : B \longrightarrow \frac{T+B}{T}$. Since $U \cap B \hookrightarrow Rad_g(B)$, $\pi(U \cap B) \hookrightarrow \pi(Rad_g(B))$ implies $\frac{T+(U \cap B)}{T} \hookrightarrow Rad_g(\frac{T+B}{T})$, see [6, Corollary 2.11(1)]. By hypothesis, we get $\frac{T+B}{T} \hookrightarrow^{\oplus} \frac{M}{T}$. Hence $\frac{M}{T} \in DGRS$.
(*ii*) To prove that, we will use property in part (i). Assume that $T \hookrightarrow M$ and let $B \hookrightarrow^{\oplus} M$, $M = B \oplus \grave{B}$ for some $\grave{B} \hookrightarrow M$. So $\frac{M}{T} = \frac{B+T}{T} + \frac{\grave{B}+T}{T}$. Also $(B + T) \cap (\grave{B} + T) \hookrightarrow (B + T + \grave{B}) \cap T + (B + T + T) \cap \grave{B} \hookrightarrow T + (B + T) \cap \grave{B}$. But we have that $T = (T \cap B) \oplus (T \cap \grave{B})$ implies $(B + T) \cap (\grave{B} + T) \hookrightarrow T + (B + T \cap B + T \cap \grave{B}) \cap \grave{B}$, thus $(B + T) \cap (\grave{B} + T) \hookrightarrow T$, and so $\frac{M}{T} = \left(\frac{B+T}{T}\right) \oplus \left(\frac{\grave{B}+T}{T}\right)$. Therefore $\frac{T+B}{T} \hookrightarrow^{\oplus} \frac{M}{T}$. This completes the proof.
(*iii*) and **(*iv*)** are consequences directly from (ii). □

For $M \in$ Mod-$\mathcal{R}$, if $\exists\ T \hookrightarrow^{\oplus} M$ such that $T \hookrightarrow U$ and $\frac{U}{T} \hookrightarrow Rad_g(\frac{M}{T})$, for ecah $U \hookrightarrow M$, then we say that M has property (P_g^*). It is simple to demonstrate that M has (P_g^*) if and only if for any $U \hookrightarrow M$, $\exists$ a decomposition $M = T \oplus B$ with $T \hookrightarrow U$ and $U \cap B \hookrightarrow Rad_g(B)$. It is evident that each module with (P_g^*) is in $DGRS$. Each finite direct sum of modules with (P_g^*) is in $DGRS$, according to Corollary 5.8. However, g-lifting for modules is stronger than (P_g^*) property.

Next we will then demonstrate similar characterizations for the category of $\oplus$-g-radical supplemented modules.

Theorem 5.11. Assume $M \in DGRS$ and verify each one of the listed cases:
(*i*) M is duo.
(*ii*) M is distributive.
(*iii*) M is π-projective.
(*iv*) M is refinable and have the SIP.
(*v*) M is indecomposable.
Then M has the property (P_g^*).
Proof. $(\boldsymbol{i})$, $(\boldsymbol{ii})$, $(\boldsymbol{iii})$ and $(\boldsymbol{iv})$ are all the same proof Theorem 2.18.
$(\boldsymbol{v})$ Suppose $H \hookrightarrow M$. If $H = M$, the proof is evidently. Assume $H \neq M$, so $M = H + \grave{F}$ and $H \cap \grave{F} \hookrightarrow Rad_g(\grave{F})$ where $M = F \oplus \grave{F}$, as $M \in DGRS$. From (v), $F = 0$ and $\grave{F} = M$. Hence $F \hookrightarrow H$ and $H \cap \grave{F} \hookrightarrow Rad_g(\grave{F})$, as required. □

Proposition 5.12. Let $M \in$ Mod-$\mathcal{R}$ is refinable, then $M \in DGRS$ if and only if M is g-radical supplemented.
Proof. The necessity is evident. Now, if $T \hookrightarrow M$, then $\exists\ B \hookrightarrow M$ such that $M = T + B$ and $T \cap B \hookrightarrow Rad_g(B)$. Since M is refinable, then $\exists\ \grave{B} \hookrightarrow^{\oplus} M$ such that $M = T + \grave{B}$ and $\grave{B} \hookrightarrow B$. It follows $T \cap \grave{B} = (T \cap B) \cap \grave{B} \hookrightarrow Rad_g(M) \cap \grave{B} = Rad_g(\grave{B})$, according to Lemma 2.12(ii), and hence $M \in DGRS$. □

Corollary 5.13. Assume the next for a refinable $M \in$ Mod-$\mathcal{R}$ with $Rad_g(M) \hookrightarrow^{gs} M$.
(*i*) M is g-radical supplemented.
(*ii*) M is g-supplemented.
(*iii*) $M \in DGRS$.
(*iv*) $M \in DGS$.
(*v*) $M/Rad_g(M)$ is semisimple.
Then $(\boldsymbol{i}) \Leftrightarrow (\boldsymbol{ii}) \Leftrightarrow (\boldsymbol{iii}) \Leftrightarrow (\boldsymbol{iv})$ and $(\boldsymbol{v}) \Longrightarrow (\boldsymbol{iv})$. Also $(\boldsymbol{iv}) \Longrightarrow (\boldsymbol{v})$ if, M is distributive.
Proof. $(\boldsymbol{i}) \Leftrightarrow (\boldsymbol{iii})$ Follows from Proposition 5.12.
$(\boldsymbol{ii}) \Leftrightarrow (\boldsymbol{iv})$ Follows directly from Proposition 2.16.
$(\boldsymbol{iii}) \Leftrightarrow (\boldsymbol{iv})$ Follows directly from Proposition 5.3.
$(\boldsymbol{iv}) \Longrightarrow (\boldsymbol{v})$ Follows directly from Theorem 2.19. □

Proposition 5.14. Assume the next for a projective $M \in$ Mod-$\mathcal{R}$.
$(\boldsymbol{i})$ $M \in DGRS$.
$(\boldsymbol{ii})$ M is g-semiperfect.
Then $(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$; and if $Rad_g(M) \hookrightarrow^{gs} M$ then $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$.
Proof. Directly from Proposition 5.3 and Theorem 4.2. □

6. Main results

This part looks at the direct summands and decompositions of modules that have the property of ⊕-g-radical supplemented.

Theorem 6.1. Let $\{H_t |\ t \in (\tau\ \ is\ finite)\}$ be a family of relatively projective modules. Then $H = \oplus_{t\in\tau} H_t \in DGRS$ if and only if $H_t \in DGRS$, for all $t \in \tau$.
Proof. Corollary 5.8 implies the sufficiency. We will prove $H_1 \in DGRS$. Suppose $U \hookrightarrow H_1$, then $M = U + K$ and $U \cap K \hookrightarrow Rad_g(K)$ for some $B \hookrightarrow^{\oplus} H$. Since $H = H_1 + B$, so by [10, Lemma 4.47] $\exists\ B_1 \hookrightarrow B$ with $H = H_1 \oplus B_1$ that gives $B = B_1 \oplus (H_1 \cap B)$. As $H_1 \cap B \hookrightarrow^{\oplus} B$, then $U \cap (H_1 \cap B) = U \cap B \hookrightarrow (H_1 \cap B) \cap Rad_g(B) = Rad_g(H_1 \cap B)$, form Lemma 2.12($ii$). Easily show $H_1 = U + (H_1 \cap B)$ and $H_1 \cap B \hookrightarrow^{\oplus} H_1$, as required. □

Proposition 6.2. Let $H = H_1 \oplus H_2 \in$ Mod-$\mathcal{R}$. Then $H_1 \in DGRS$ if and only if for each $T/H_2 \hookrightarrow H/H_2$, $\exists\ B \hookrightarrow^{\oplus} H$ such that $B \hookrightarrow H_1$, $H = T + B$ and $T \cap B \hookrightarrow Rad_g(H)$.
Proof. Assume $H_1 \in DGRS$. Let $T/H_2 \hookrightarrow H/H_2$. As $T \cap H_1 \hookrightarrow H_1$, then $H_1 = (T \cap H_1) + B$ and $T \cap B \hookrightarrow Rad_g(B)$ for some $B \hookrightarrow^{\oplus} H_1$. Hence $H = (T \cap H_1) + B + H_2 = T + B$ and $T \cap B \hookrightarrow Rad_g(H)$. Conversely, suppose $U \hookrightarrow H_1$. Consider $(U \oplus H_2)/H_2 \hookrightarrow H/H_2$. From the hypothesis, $H = (U + B) \oplus H_2$ and $(U + H_2) \cap B \hookrightarrow Rad_g(H)$ for some $B \hookrightarrow^{\oplus} H$ with $B \hookrightarrow H_1$. Thus $H_1 = U + B$, and by Lemma 2.12(ii) $U \cap B \hookrightarrow B \cap Rad_g(H) = Rad_g(B)$. Therefore, B is a generalized radical supplement of U in H_1 and $B \hookrightarrow^{\oplus} H_1$, hence $H_1 \in DGRS$. □

Proposition 6.3. If $M \in DGRS$ with (D_3), then $T \in DGRS$ for each $T \hookrightarrow^{\oplus} M$.
Proof. Let $T \hookrightarrow^{\oplus} M$, and $U \hookrightarrow T$. Since $M \in DGRS$, $M = U + H$ and $U \cap H \hookrightarrow Rad_g(H)$ for some $H \hookrightarrow^{\oplus} M$. So, $T = U + (T \cap H)$. As M has (D_3) with $M = T + H$, then $T \cap H \hookrightarrow^{\oplus} M$. From Lemma 2.12($ii$), we deduce $U \cap (T \cap H) \hookrightarrow (T \cap H) \cap Rad_g(M) = Rad_g(T \cap H)$, as required. □

For $M \in$ Mod-$\mathcal{R}$ is referred to as having the summand sum property, we write SSP, if for each $U, V \hookrightarrow^{\oplus} M$ implies $U + V \hookrightarrow^{\oplus} M$. Thus, we deduce:

Proposition 6.4. Let $M \in DGRS$ has the SIP, or SSP. Then $T \in DGRS$, for each $T \hookrightarrow^{\oplus} M$.

Proof. Let $M \in DGRS$. Assume M has the SIP, then it has (D_3), Proposition 6.3 implies the result. If M has the SSP. Let $T \hookrightarrow^{\oplus} M$, then $M = T \oplus \grave{T}$ for some $\grave{T} \hookrightarrow M$. To show $M/\grave{T} \in DGRS$. Let $U \hookrightarrow^{\oplus} M$, then also $U + \grave{T} \hookrightarrow^{\oplus} M$, as M has the SSP. Thus, $M = (U + \grave{T}) \oplus B$ for some $B \hookrightarrow M$. Then $\frac{M}{\grave{T}} = (\frac{U+\grave{T}}{\grave{T}}) \oplus (\frac{B+\grave{T}}{\grave{T}})$. By Theorem 5.10($i$), the proof is end. □

Proposition 6.5. Let $M \in DGRS$ whose any generalized radical supplement in M is a d.s., then $T \in DGRS$, for each $T \hookrightarrow^{\oplus} M$.

Proof. Suppose that $T \hookrightarrow^{\oplus} M$. Then $M = T \oplus B$ for some $B \hookrightarrow M$. We have M is a g-radical supplemented module, as $M \in DGRS$. It follows $M/B \cong T$ is g-radical supplemented, from [9, Lemma 2.7]. Let $U \hookrightarrow T$, U has a generalized radical supplement V in T. To show that $V \hookrightarrow^{\oplus} T$. Notice that $M = T \oplus B = (U + B) + V$ and $(U + B) \cap V \hookrightarrow (U + V) \cap B + (V + B) \cap U = (V + B) \cap U \hookrightarrow U$. Hence $(U + B) \cap V \hookrightarrow U \cap V \hookrightarrow Rad_g(V)$. It follows that $U + B$ has a generalized radical supplement V in M, so by assumption, $\exists\, \grave{V} \hookrightarrow M$ with $M = V \oplus \grave{V}$. Therefore $T = V \oplus (T \cap \grave{V})$.□

Proposition 6.6. If $M \in$ Mod-$\mathcal{R}$ has property $(P_g^{\ *})$, then $T \in DGRS$, for each $T \hookrightarrow^{\oplus} M$.

Proof. Let $T \hookrightarrow^{\oplus} M$, and $B \hookrightarrow T$. Since M has $(P_g^{\ *})$, then $\exists\, U, \grave{U} \hookrightarrow M$ such that $U \hookrightarrow B$ and $B \cap \grave{U} \hookrightarrow Rad_g(\grave{U})$ where $M = U \oplus \grave{U}$. Therefore $T = U \oplus (T \cap \grave{U})$, i.e., $T \cap \grave{U} \hookrightarrow^{\oplus} T$. Hence $T = X + (T \cap \grave{U})$. Assume $a \in B \cap \grave{U}$, then $a \in Rad_g(M)$ and so $aR \hookrightarrow^{gs} M$, see [7, Lemma 5]. Lemma 2.12(i) implies that $aR \hookrightarrow^{gs} T \cap \grave{U}$, and then $a \in Rad_g(T \cap \grave{U})$. Therefore $B \cap (T \cap \grave{U}) = B \cap \grave{U} \hookrightarrow Rad_g(T \cap \grave{U})$, and so $T \cap \grave{U}$ is a generalized radical supplement of B in T. □

Using the two outcomes Theorem 5.11 and Proposition 6.6, we can conclude the following:

Corollary 6.7. If $M \in$ Mod-$\mathcal{R}$ is π-projective, then $M \in DGRS$ if and only if $T \in DGRS$, for each $T \hookrightarrow^{\oplus} M$.

Proposition 6.8. Let $M \in$ Mod-$\mathcal{R}$, and $T \hookrightarrow M$ is (fully invariant or distributive) d.s. of M. Then $M \in DGRS$ if and only if $T \in DGRS$ and $M/T \in DGRS$.

Proof. ⟹) From Theorem 5.10(iii), if $T \hookrightarrow M$ is fully invariant, then $M/T \in DGRS$. Assume that $T \hookrightarrow^{\oplus} M$ is any fully invariant, and $B \hookrightarrow T$. Because $M \in DGRS$, then $M = B + A$ and $B \cap A \hookrightarrow Rad_g(A)$ for some decomposition $M = A \oplus \grave{A}$. Thus $T = B + (T \cap A)$. From [20, Lemma 2.1] $T = (T \cap A) \oplus (T \cap \grave{A})$, that means $T \cap A \hookrightarrow^{\oplus} T$. Lemma 2.12($ii$) implies that $B \cap (T \cap A) \hookrightarrow Rad_g(M) \cap (T \cap A) = Rad_g(T \cap A)$. Thus, $T \in DGRS$. Similarly, when T is distributive.

⟸) Evidently, from Proposition 5.7. □

The following corollaries of proposition 6.8 directly follow.

Corollary 6.9. Assume $M \in$ Mod-$\mathcal{R}$ is weak duo or distributive, and let $T \hookrightarrow^{\oplus} M$. Then $M \in DGRS$ if and only if $T \in DGRS$ and $M/T \in DGRS$.

Corollary 6.10. Assume $M \in$ Mod-$\mathcal{R}$ is weak duo or distributive. Then $M \in DGRS$ if and only if $T \in DGRS$, for each $T \hookrightarrow^{\oplus} M$.

Corollary 6.11. If $M \in DGRS$, then $M/Rad_g(M) \in DGRS$. Moreover, $Rad_g(M) \in DGRS$ whenever $Rad_g(M) \hookrightarrow^{\oplus} M$.

Proof. Theorem 5.10(iii) and Proposition 6.8 respectively come next. □

When $M \in$ Mod-$\mathcal{R}$ is weak duo or distributive, one can observe a finite decomposition $M = \oplus_{t=1}^{m} M_t \in DGRS$ if and only if $M_t \in DGRS$ for $t \in \{1,2, \dots, m\}$. Additionally, an effort will be made to satisfy a specific instance that requires that this feature is achieved for each category of modules, as is seen below.

Proposition 6.12. Let $\{H_t\}_{t\in\tau}$ be a family of modules with $H = \oplus_{t\in\tau} H_t$. If $H \in$ Mod-$\mathcal{R}$ is duo, then $H \in DGRS$ if and only if, for each $t \in \tau$, $H_t \in DGRS$.

Proof. Corollary 6.10 follows the necessity. Conversely, let $U \hookrightarrow H$, then $U \cap H_t \hookrightarrow H_t$ for each $t \in \tau$. By assumption, $\exists\, V_t \hookrightarrow^{\oplus} H_t$ such that $H_t = (U \cap H_t) + V_t$ and $(U \cap H_t) \cap V_t = U \cap V_t \hookrightarrow Rad_g(V_t)$ for each $t \in \tau$. From [20, Lemma 2.1] $U = \oplus_{t\in\tau} (U \cap H_t)$. Let $V = \oplus_{t\in\tau} V_t$, it is easily to show $V \hookrightarrow^{\oplus} H$. Thus, $H = U + V$ and $U \cap V = \oplus_{t\in\tau} (U \cap H_t) \cap (\oplus_{t\in\tau} V_t) = \oplus_{t\in\tau} (U \cap V_t) \hookrightarrow \oplus_{t\in\tau} Rad_g(V_t) = Rad_g(V)$ by Lemma 1.2, as required. □

Proposition 6.13. Assume $H \in DGRS$, then $H = H_1 \oplus H_2$ such that H_1 is semisimple and $H_2 \in$ Mod-$\mathcal{R}$ has essential generalized radical.

Proof. According to [9, Proposition 2.13]. □

Theorem 6.14. If $M \in$ Mod-$\mathcal{R}$ has (D_3), then the next are coincide.

(*i*) $T \in DGRS$, for each $T \hookrightarrow^{\oplus} M$.

(*ii*) $M \in DGRS$.

(*iii*) $M = M_1 \oplus M_2$ where M_1 is semisimple and $M_2 \in DGRS$ with $Rad_g(M_2) \hookrightarrow^{e} M_2$.

(*iv*) $M = M_1 \oplus M_2$ where M_1 in $DGRS$ and $M_2 \in$ Mod-$\mathcal{R}$ with $Rad_g(M_2) = M_2$.

Proof. $(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Directly from Proposition 6.3.

$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{iii})$ Directly from Proposition 6.13 and part (i).

$(\boldsymbol{iii}) \Longrightarrow (\boldsymbol{ii})$ Directly from Proposition 5.7.

$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{iv})$ As $M \in DGRS$ and $Rad_g(M) \hookrightarrow M$, then $\exists\ M_1, M_2 \hookrightarrow M$ such that $M = M_1 \oplus M_2$, $Rad_g(M) + M_1 = M$ and $Rad_g(M) \cap M_1 \hookrightarrow Rad_g(M_1)$. It follows $M_1 \in DGRS$, by (i). From Lemma 1.2, we deduce that $M = M_1 \oplus Rad_g(M_2)$ and then $Rad_g(M_2) = M_2$.

$(\boldsymbol{iv}) \Longrightarrow (\boldsymbol{ii})$ Directly from Propositions 5.2 and 5.7. □

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