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About e-gH modules

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ABSTRACT

This article introduced and explored the concept of the e-gH module and its relation to many other module types.

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1. Introduction

Throughout this paper, all modules are unitary left R-modules and R is an associative ring with identity. A nonzero submodule $S \leq M$ is said to be essential in M denoted by $S \trianglelefteq M$, if $N \cap S \neq 0$ for every nonzero submodule N of M [2]. A submodule E of M is called small (e-small) denoted by $E \ll M$ (resp. $E \ll_e M$) if for every (essential) submodule N of M with the property $M = E + N$ implies $N = M$ [13]. A module $M \neq 0$ is called uniform if for every submodule E of M with $E \neq 0$, then E is essential [2]. M is called generalized hollow if any proper submodule of M is e-small in M [4]. The endomorphisms of modules it has been studied in many authors. V.A. Hiremath introduced the concept of Hopfian module, defined as a module M is called Hopfian if for every surjective R-endomorphism of M is an isomorphism [5]. In [1] Gorbani and Haghany introduced generalized for Hopfian called generalized Hopfian (gH), a module, is said to be gH if it has a small kernel for every surjective R-endomorphism of M. K. Varadarajan in 1992 introduced the concept of co-Hopfian module, defined as a module M is called Hopfian if for every injective R-endomorphism of M is an isomorphism [12]. In [8] introduced a proper generalized for Hopfian called e-gH module. A module is said to be e-gH if for every surjective R-endomorphism of M has an e-small kernel. In section 2. We proved some relation between e-gH module and some other concepts. We show that every semisimple module is e-gH, we give a case that make the concepts e-gH and gH modules are identical. Theorem 2.12 showed the equivalent between Hopfian and e-gH. Also introduced a new definition in section 2 called it right e-domin ring defined as, a nonzero ring R is called a right e-domain if, $r_R(x) \ll_e R_R$ for any nonzero element $x \in R$ where $r_R(x)$ denote the right annihilator of x in R. In the same section showed the relation between Hopfian module, semi-Hopfian (see [10]) and e-gH. We also showed the e-gH property of $M[x]$ as an $R[x]$ module. Finally, we investigate the behavior of e-gH module under localization.

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2. e-gH modules and adjacent concepts

Start with these results.

Proposition 2.1. Every generalized hollow module is e-gH.

Proof: Let M be a generalized hollow module and $\varphi: M \to M$ an epimorphism. So we have that $ker\varphi \ll_e M$ and hence M is an e-gH module.

Remark 2.2. The reverse of Proposition 2.1, is not necessary, in general, as the following example shows: From [8, Remarks and Examples 2.2(6)], the $\mathbb{Z}$-module $\mathbb{Z}$ is e-gH, but it is not generalized hollow, since (0) is the only e-small submodule of $\mathbb{Z}$ as a $\mathbb{Z}$-module.

In following, we will give a case that make the concepts e-gH and gH modules are identical.

Proposition 2.3. Let M be an indecomposable module, then M is gH if and only if it is e-gH.

Proof: The necessity is follows by [8, Remarks and Examples 2.2(1)]. Conversely, let M be an e-gH module and $\varphi: M \to M$ an epimorphism. Therefore, $ker\varphi \ll_e M$. It follows that $ker\varphi \neq M$, since if, $ker\varphi = M$ then $\varphi = 0$ which implies φ is not an epimorphism, a contradiction. By [3, Proposition 3.7], $ker\varphi \ll M$ and hence M is gH.

Corollary 2.4. If M is a uniform module, then M is gH if and only if it is e-gH.

Proof: By [7, Examples 3.51(1)], since any uniform module is indecomposable. □

Proposition 2.5. If M is a semisimple module, then M is e-gH.

Proof: Let M be a semisimple module and let $f \in End(M)$ be an epimorphism. To prove $kerf \ll_e M$, let $L \trianglelefteq M$ such that $kerf + L = M$. Since M is semisimple, so the only essential submodule of M is itself, that is $L = M$. Thus, $ker\varphi \ll_e M$ and hence M is e-gH.

The reverse of Proposition 2.5, is generally incorrect, as the following example shows.

Example 2.6. For all prime number p, it is well known that $\mathbb{Z}_{p^2}$ as a $\mathbb{Z}$-module is Noetherian, thus it is e-gH by [8, Remarks and Examples 2.2(3)]. While $\mathbb{Z}_{p^2}$ is not semisimple as a $\mathbb{Z}$-module, because $< p >$ is not a direct summand of $\mathbb{Z}_{p^2}$ for all prime p.

Corollary 2.7. Every simple module is e-gH.

Proof: It follows directly by Proposition 2.5.

Remark 2.8. The reverse of Corollary 2.7, is generally incorrect, as example shows: by [8, Remarks and Examples 2.2(6)] the $\mathbb{Z}$-module $\mathbb{Z}$ is e-gH and it is not simple. □

Theorem 2.9. Let M be an e-gH module. If $g: M \to M \oplus \acute{M}$ is an epimorphism for some module $\acute{M}$, then $\acute{M}$ is semisimple.

Proof. Let $g: M \to M \oplus \acute{M}$ be an epimorphism for some module $\acute{M}$. Consider the projection map $\rho: M \oplus \acute{M} \to M$. Then $\rho g \in End(M)$ and $ker\,(\rho g) = g^{-1}(ker\rho) = g^{-1}(0 \oplus \acute{M})$. Since M is e-gH, then $g^{-1}(0 \oplus \acute{M}) \ll_e M$. By [13, Proposition 2.5,(2)], we have that $0 \oplus \acute{M} = g\left(g^{-1}(0 \oplus \acute{M})\right) \ll_e M \oplus \acute{M}$. Thus, $\acute{M} \ll_e \acute{M}$ by [13, Proposition 2.5,(3)]. Therefore $\acute{M}$ is a semisimple module, by [11, Lemma 2.4]. □

By compare between Proposition 2.5 and Theorem 2.9, we have:

Corollary 2.10. Let M be an e-gH module. If $g: M \to M \oplus \acute{M}$ is an epimorphism for some module $\acute{M}$, then $\acute{M}$ is e-gH.

Theorem 2.11. Let R be a ring. Then the following are equivalent.

(1) Each R-module is e-gH.
(2) Each projective R-module is e-gH.
(3) Each free R-module is e-gH.
(4) R is semisimple.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) holds.

(3) $\Longrightarrow$ (4) By [6, Lemma 4.4.3], $R^{\mathbb{N}}$ is a free R-module and so is e-gH, by (3). As $R^{\mathbb{N}} \cong R^{\mathbb{N}} \oplus R^{\mathbb{N}}$, Theorem 2.9, implies $R^{\mathbb{N}}$ is semisimple. Therefore R is semisimple.

(4) $\Longrightarrow$ (1) Suppose that R be a semisimple ring. From [6, Corollary 8.2.2], any R-module M is semisimple, and hence by Proposition 2.5, M is e-gH. □

Theorem 2.12. Consider the following for an R-module M:
(1) M is Hopfian.
(2) M is e-gH.
(3) M is Dedekind finite.
Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). If M is quasi-projective, then (3) $\Longrightarrow$ (1).
Proof. (1) $\Longrightarrow$ (2) [8, Remarks and Examples 2.2(2)].
(2) $\Longrightarrow$ (3) Assume $f, g \in End(M)$ such that $fg = 1$. It follows that g is injective and f is surjective. Thus, $M = fg(M) = f(g(M))$. By [8, Theorem 2.3], we have that $M = g(M)$, that means g is invertible. Hence $gf = 1$. Therefore $End(M)$ is a Dedekind finite ring. So (3), holds.
(3) $\Longrightarrow$ (1) Let $g \in End(M)$ be a surjective. As M is quasi-projective, then there is an $h \in End(M)$ such that $gh = 1$. By (**3**), $hg = 1$. Thus g is an injective and this complete the proof. □

Proposition 2.13. Every nonzero e-small quasi-Dedekind module is e-gH.
Proof. Let $M \neq 0$ be an e-small quasi-Dedekind module, and let $f \in End(M)$ be a surjective. Then $f \neq 0$. By assumption, $kerf \ll_e M$ and hence M is e-gH. □

Example 2.14. The convers of Proposition 2.13, is generally incorrect. Consider $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module. Then the set of all surjective endomorphisms of $\mathbb{Z}_{12}$ are $f_1(\bar{x}) = \bar{x}$, $f_2(\bar{x}) = 5\bar{x}$, $f_3(\bar{x}) = 7\bar{x}$ and $f_4(\bar{x}) = 11\bar{x}$ which has zero kernel (i.e., $kerf_i = 0 \ll_e \mathbb{Z}_{12}$ for all $i = 1,2,3,4$). Therefore $\mathbb{Z}$-module $\mathbb{Z}_{12}$ is e-gH. Now, let $g: \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by $f(\bar{x}) = 4\bar{x}$ for all $\bar{x} \in \mathbb{Z}_{12}$. Then $0 \neq g \in End(\mathbb{Z}_{12})$, but $kerg = 3\mathbb{Z}_{12}$ is not e-small in $\mathbb{Z}_{12}$ (in fact $3\mathbb{Z}_{12} + 2\mathbb{Z}_{12} = \mathbb{Z}_{12}$ while $2\mathbb{Z}_{12}$ is a proper essential in $\mathbb{Z}_{12}$). This mean $\mathbb{Z}_{12}$ is not e-small quasi-Dedekind $\mathbb{Z}$-module.

Recall that an R-module M is anti-Hopfian if M is non-simple and all nonzero factor modules of M are isomorphic to M

Proposition 2.15. Let M be an anti-Hopfian module. Then M is e-gH if and only if M is generalized Hollow.
Proof. Assume that M is an e-gH module and N any proper submodule of M. If $N = 0$, then N is e-small in M. Let $N \neq 0$. We have $M/N \neq N$, so by assumption $M/N \cong M$. By [8, Theorem 2.19], N is an e-small submodule of M. Therefore M is generalized Hopfian. The converse is proved in Proposition 2.1. □

Proposition 2.16. Let M be a quasi-projective module. If M is co-Hopfian then it is Hopfian, and so e-gH.
Proof. Assume $g \in End(M)$ is an epimorphism. As M is a quasi-projective module, then there exists an $h \in End(M)$ such that $gh = 1$, then gh is an injective and so is h. As M is co-Hopfian, so h is automorphism, i.e., h is a surjective. Now, if $x \in kerg$, then $x \in M$ and $g(x) = 0$. As $h: M \to M$ is a surjective, then $h(a) = x$ for some $a \in M$. Then $0 = g(x) = g\big(h(a)\big) = gh(a)$, so $a \in kergh = 0$ implies $x = h(0) = 0$. Hence $kerg = 0$, i.e., g is an injective and hence M is Hopfian. By [8, Remarks and Examples 2.2(2)], M is e-gH. □

Theorem 2.17. Let M be a module, consider the following cases:
(1) if $g \in End(M)$ is a surjective, then $kerg$ is semisimple.
(2) if $g \in End(M)$ has a right inverse, then $kerg$ is semisimple.
(3) if $g \in End(M)$ has a right inverse, then $kerg \ll_e M$.
(4) M is e-gH.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). If M is quasi-projective, then (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2) Let $g \in End(M)$ has a right inverse. Thus $gf = 1$ for some $f \in End(M)$. Therefore $g \in End(M)$ is a surjective, and so $kerg$ is semisimple, by (1).
(2) $\Longrightarrow$ (3) Let $g \in End(M)$ has a right inverse. By (2), $kerg$ is semisimple. To show that $kerg \ll_e M$. Let $kerg + L = M$ for some $L \leq M$. As $kerg \cap L \leq kerg$, then $(kerg \cap L) \oplus T = kerg$ for some $T \leq kerg$. Then $M = T + L$. Also, $T \cap L \subseteq kerg \cap L$ and $T \cap L \subseteq T$ implies $T \cap L \subseteq (kerg \cap L) \cap T = 0$. Then $T \oplus L = M$. Thus, $L \leq^{\oplus} M$ and $M/L \cong T$ is semisimple, and hence $kerg \ll_e M$, by [13, Proposition 2.3].
(3) $\Longrightarrow$ (4) Assume $g \in End(M)$ is a surjective. Since M is quasi-projective, then there is an $h \in End(M)$ such that $gh = 1$, i.e., $g \in End(M)$ has a right inverse. By (3), $kerg \ll_e M$ and hence M is e-gH.

(4) $\Longrightarrow$ (1) Assume $g \in End(M)$ is a surjective. So by (4), $kerg \ll_e M$. Since M is quasi-projective, then g has a right inverse $h \in End(M)$, i.e., $gh = 1$. Thus, we have that $kerg = (1 - hg)(M)$, to see this; if $x \in kerg$ then $g(x) = 0$ and henec $hg(x) = 0$. So, $x = x - hg(x) \in M - hg(M) = (1 - hg)(M)$, therefore $kerg \subseteq (1 - hg)(M)$. Now, let $y \in (1 - hg)(M)$ then $y = m - hg(m)$ for some $m \in M$. So, $g(y) = g(m) - gh\big(g(m)\big) = g(m) - g(m) = 0$, then $y \in kerg$. This means $(1 - hg)(M) \subseteq kerg$ and then $kerg = (1 - hg)(M)$. Then $kerg + hg(M) = M$. Also, if $x \in kerg \cap hg(M)$ then $g(x) = 0$ and $x = hg(m)$ for some $m \in M$. So $0 = g(x) = gh(g(m)) = g(m)$, thus $x = h\big(g(m)\big) = h(0) = 0$. It follows that $kerg \oplus hg(M) = M$. By [13, Proposition 2.3], $kerg$ is semisimple. □

Corollary 2.18*.* Let M be a projective module. Then the following are equivalent.
(1) if $g \in End(M)$ is a surjective, then $kerg$ is semisimple.
(2) if $g \in End(M)$ has a right inverse, then $kerg$ is semisimple.
(3) if $g \in End(M)$ has a right inverse, then $kerg \ll_e M$.
(4) M is e-gH.

Now, we present the following definition.

Definition 2.19. A nonzero ring R is called a right e-domain if, $r_R(x) \ll_e R_R$ for any nonzero element $x \in R$.

Proposition 2.20. Let R be a nonzero right e-domain ring. Then every nonzero principal right ideal I of R is e-small quasi-Dedekind.

Proof. Let $0 \neq a \in R$. Put $I = aR$, $\psi : I \longrightarrow I$ is a nonzero R-homomorphism and $x = \psi(a)$. Hence $ker\psi = \{ar \in I \colon f(ar) = 0\} = \{ar \in I \colon xr = 0\} = a.r_R(x)$. We have $0 \neq x$ (since if $x = 0$, then for all $at \in I$, $\psi(at) = xt = 0.t = 0$, then $at \in ker\psi$ and so $ker\psi = I$. Thus, $\psi(I) = \psi(ker\psi) = 0$, a contradiction). Since $x \neq 0$ and R a right e-domain, so $r_R(x) \ll_e R_R$. If $f\colon R \longrightarrow I$ is given by left multiplication by a, i.e., $f(r) = ar$ for all $r \in R$. Then $f\big(r_R(x)\big) = a.r_R(x) \ll_e I$, by [13, Proposition 2.5(2)], and hence $ker\psi \ll_e I$. Thus I is e-small quasi-Dedekind. □

Corollary 2.21. In a nonzero right e-domain ring, every nonzero principal right ideal is e-gH.

Proof. It follows directly by Propositions 2.20 and 2.13. □

Corollary 2.22. Every nonzero right e-domain ring is e-gH.

Proof. Let $R = \langle 1 \rangle$ be a nonzero right e-domain ring. By Corollary 2.21, R is an e-gH. □

Proposition 2.23. Let M be an R-module. If for all regular $f \in End(M)$ has an e-small kernel, then M is e-gH.

Proof. Let $f \in End(M)$ be a surjective. Then $fff(M) = ff\big(f(M)\big) = f\big(f(M)\big) = f(M)$, i.e., f is regular, so by assumption $kerf \ll_e M$, and hence M is e-gH. □

The converse of Proposition 2.23, is generally incorrect, as the following example shows.

Example 2.24. Suppose that $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module. By Example 2.14, $\mathbb{Z}_{12}$ is e-gH $\mathbb{Z}$-module. Now, let $g \in End(\mathbb{Z}_{12})$ defined by $g(\bar{x}) = 4\bar{x}$ for all $\bar{x} \in \mathbb{Z}_{12}$. Thus $g(\mathbb{Z}_{12}) = \{\bar{0}, \bar{4}, \bar{8}\}$, so $gg(\mathbb{Z}_{12}) = g(\{\bar{0}, \bar{4}, \bar{8}\}) = \{\bar{0}, \bar{4}, \bar{8}\}$, and hence $ggg(\mathbb{Z}_{12}) = g(gg(\mathbb{Z}_{12})) = g(\{\bar{0}, \bar{4}, \bar{8}\}) = \{\bar{0}, \bar{4}, \bar{8}\}$. Therefore g is regular. But $kerg = 3\mathbb{Z}_{12}$ is not e-small in $\mathbb{Z}_{12}$.

Proposition 2.25. Let M be a uniform R-module. Then M is Hopfian if and only if M semi-Hopfian and e-gH.

Proof. The necessity is clear. Assume M is a semi-Hopfian and e-gH R-module. If $f \in End(M)$ is a surjective. Then $kerf \leq^{\oplus} M$ and $kerf \ll_e M$. Thus, $kerf \oplus L = M$, i.e., $kerf + L = M$ and $kerf \cap L = 0$ for some $L \leq M$. Since M uniform, then $L \trianglelefteq M$ and hence $L = M$, as $kerf \ll_e M$. So $kerf = kerf \cap M = 0$, i.e., f is an injective. Therefore M is a Hopfian R-module. □

Suppose M is an R-module. The set $\{\sum m_i x^i \mid m_i \in M, i \in (I \text{ is any index set})\}$ is denoted by $M[x]$. Then $M[x]$ can be as a right $R[x]$-module. This module is called polynomial module.

We now need to proof the following simple fact.

Lemma 2.26. Let M be an R-module. If $N \trianglelefteq M$ as R-module, then $N[x] \trianglelefteq M[x]$ as $R[x]$-module.

Proof. Let $0 \neq f = m_\circ + m_1 x + m_2 x^2 + \cdots + m_k x^k \in M[x]$, where $k \in \mathbb{N}$. If $m_\circ \neq 0$, there exists an $r_\circ \in R$ such that $0 \neq m_\circ r_\circ \in N$ (as $N \trianglelefteq M$). If $m_i r_\circ = 0$ for all $1 \leq i \leq t$, then $fr_\circ = m_\circ r_\circ$, that is $0[x] \neq fr_\circ \in N[x]$, and the proof is ends. Let $m_1 r_\circ \neq 0$, then there exists an $r_1 \in R$ such that $0 \neq m_1 r_\circ r_1 \in N$ (as $N \trianglelefteq M$). Continuing with this argument, we get $r \in R$ such that $0[x] \neq fr \in N[x]$. Hence $N[x] \trianglelefteq M[x]$. □

Proposition 2.27. Let M be an R-module. If $M[x]$ is e-gH as $R[x]$-module, then M is e-gH as R-module.

Proof. Assume that $\psi: M \to M$ is an R-epimorphism. Define the R-homomorphism $\psi[x]: M[x] \to M[x]$ by $\psi[x](\sum m_i x^i) = \sum \psi(m_i) x^i$. Let $\sum m_i x^i \in M[x]$. Since $\psi$is a surjective map, so for any $m_i \in M$ there is $n_i \in M$ such that $\psi(n_i) = m_i$, then $\psi[x](\sum n_i x^i) = \sum \psi(n_i) x^i = \sum m_i x^i$, that means $\psi[x]$ is a surjective. Therefore $ker\,(\psi[x]) = ker\psi[x] \ll_e M[x]$. To prove that $ker\psi \ll_e M$. Let $L \trianglelefteq M$ such that $ker\psi + L = M$, then $ker\,(\psi[x]) + L[x] = (ker\psi + L)[x] = M[x]$. From Lemma 2.26, $L[x] \trianglelefteq M[x]$, then $L[x] = M[x]$ (as $ker\,(\psi[x]) \ll_e M[x]$) so $L = M$, thus $ker\psi \ll_e M$. Hence M is e-gH as R-module. □

Now, we will investigate the behavior of e-gH module under localization.

Proposition 2.28. Let M be an R-module and S is a multiplicative closed subset of R, such that $\mathcal{L}(L) \cap S = \emptyset$ for any $L \leq M$. If $S^{-1}M$ is e-gH as $S^{-1}R$-module, then M is e-gH as R-module.

Proof. Suppose that $f: M \to M$ is an R-epimorphism. Define $S^{-1}R$-endomorphism $S^{-1}f: S^{-1}M \to S^{-1}M$ by $S^{-1}f\left(\frac{m}{s}\right) = \frac{f(m)}{s}$ for all $m \in M$, $s \in S$. Then we have $Im(S^{-1}f) = S^{-1}(Imf) = S^{-1}M$, then $S^{-1}f$ is an $S^{-1}R$-epimorphism. As $S^{-1}M$ is e-gH, thus $ker(S^{-1}f) = S^{-1}(kerf) \ll_e S^{-1}M$ and so $kerf \ll_e M$, by [9, Lemma 2.3.3(2)]. Hence M is e-gH. □

Proposition 2.29. Let M be an R-module and S is a multiplicative closed subset of R, such that for all proper submodule N of M, $[N:_M s] = N$, for all $s \in S$. If $S^{-1}M$ is e-gH as $S^{-1}R$-module, then M is e-gH as R-module.

Proof. By [9, Lemma 2.3.9(2)], the proof is analogues to proof Proposition 2.28.□

Conclusion

There are many communication between the e-gH module and modules of other classes. Future desires have deeper outcomes for the questions posed in this work.

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