On Sandwich Results of Meromorphic Multivalent Functions Defined by a New Hadamard Product Operator

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\textbf{Abstract}

The goal of this research is to establish differential subordination and superordination findings for meromorphic multivalent functions defined by a new operator in a punctured open unit disk. We get a number of sandwich-type results.

\textbf{Keywords:}

superordination, subordination, convolution, sandwich theorems.

\textbf{MSC:} 30C45

1. Introduction

Let $\Sigma_p$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k,$$

which are meromorphic multivalent in the punctured open unit disk $U^* = \{ z \in \mathbb{C}, 0 < |z| < 1 \}$. Several authors studied meromorphic functions for another classes and conditions, see [7, 9, 12, 20]. Let $H$ be the linear space of all holomorphic functions in $U$. For a positive integer number $n$ and $a \in \mathbb{C}$, we let

$$H[a, n] = \{ f \in H : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.$$
For $f$ and $F$ holomorphic function in $H$, we say that $f$ is subordinate to $F$ in $U$ and write $f(z) < F(z)$, if there exists a Schwarz function $w$, which is holomorphic in $U$ with $w(0) = 0$ and $|w(z)| < 1, (z \in U)$, such that $f(z) = F(w(z)), (z \in U)$.

Furthermore, if the function $F$ is holomorphic in $U$, we have the following equivalence relationship (cf., e.g. [10,11,16,17]):

$$f(z) < F(z) \iff f(0) = F(0) \text{ and } f(U) \subset F(U), (z \in U).$$

**Definition 1:** [16] Also see [20] Let $Y: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and let $h(z)$ be holomorphic in $U$. If $p$ and $Y(p(z), zp'(z), z^2p''(z); z)$ are univalent in $U$ and if $p$ needs to satisfy the second-order differential superordination,

$$h(z) < Y(p(z), zp'(z), z^2p''(z); z), \quad (1.2)$$

then $p$ called a solution of the differential superordination (1.2). An holomorphic function $q(z)$ which is called a subordinant of the solutions of the differential superordination (1.2) or more simply, a subordinant if $q < p$ for all $p$ fulfill (1.2). A univalent subordinant $\tilde{q}(z)$ that fulfills $q < \tilde{q}$ for all subordinants $q$ of (1.2), is said to be the best subordinant.

**Definition 2:** [16] Let $Y: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and let $h$ be univalent in $U$. If $p$ is holomorphic in $U$ and satisfies the second-order differential subordination,

$$Y(p(z), zp'(z), z^2p''(z); z) < h(z), \quad (1.3)$$

then $p$ is called a solution of the differential subordination (1.3). The univalent function $q$ is called a dominant of the solution of the differential subordination (1.3), or more simply dominant if $p < q$ for all $p$ satisfying (1.3). A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q} < q$ for all dominant $q$ of (1.3) is said to be the best dominant.

Miller and Mocanu [17] and other authors [1,2,3,4,5,6,7,8,9,10,11] and also [14,15,18,19,20,23,24] discovered sufficient conditions for the functions $h, p$, and $\phi$ for which the following result:

$$h(z) < Y(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) < p(z)(z \in U). \quad (1.4)$$

If $f \in \Sigma_p$ is given by (1.1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^k.$$  

The Hadamard product (or convolution) of $f$ and $g$ is given by

$$(f \ast g)(z) = z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^k = (g \ast f)(z).$$

Using the results, (see [1,2,4,5,6,7,14,15,18,19,21,22,23,24]) to obtain adequate criteria for the satisfaction of normalized analytic functions

$$q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z),$$

where $q_1$ and $q_2$ are given univalent functions in $U$ with $q_1(0) = q_2(0) = 1$.

Shanmugam et al. [21][22], as well as Goyal et al. [13], sandwich results for holomorphic function classes were recently obtained. (See also [1,3,4,5,11]).
In a recent paper, E- Ashwah [12] defined the multiplier transform $Q_{\lambda, p}^{n, \gamma}$ of a function $f \in \Sigma_p$

\[ Q_{\lambda, p}^{n, \gamma} : \Sigma_p \rightarrow \Sigma_p \]

which is defined as follows:

\[ Q_{\lambda, p}^{n, \gamma} f(z) = z^{-p} + \sum_{k=0}^{\infty} \left( \frac{\lambda + \gamma (k + p)}{\lambda} \right)^m a_k z^k, \tag{1.5} \]

where ($\lambda > 0, \gamma > 0, z \in U^*; m \in N_0 \cup \{0\}; p \in \mathbb{N}$).

Ali, Ravichandran and Seenivasagan [25] defined the operator $K_p^{t, \gamma}$ of a function $g \in \Sigma_p$

\[ K_p^{t, \gamma} : \Sigma_p \rightarrow \Sigma_p \]

which is defined as follows:

\[ K_p^{t, \gamma} g(z) = z^{-p} + \sum_{k=0}^{\infty} \left( \frac{k + \gamma t}{\gamma - p} \right) b_k z^k, \quad (\gamma > 1, t \in N_0; p \in \mathbb{N}). \tag{1.6} \]

We define the new Hadamard product operator

\[ F_{t, \lambda, p}^{m, \gamma, \rho} (f * g)(z) = Q_{\lambda, p}^{n, \gamma} f(z) * K_p^{t, \gamma} f(z) \]

\[ F_{t, \lambda, p}^{m, \gamma, \rho} (f * g)(z) = z^{-p} + \sum_{k=0}^{\infty} \left( \frac{k + \gamma t}{\gamma - p} \right) \left( \frac{\lambda + \gamma (k + p)}{\lambda} \right)^m a_k b_k z^k, \tag{1.7} \]

we note that from (1.7), we have

\[ z F_{t, \lambda, p}^{m, \gamma, \rho} (f * g)(z) = \frac{\lambda}{\gamma} (F_{t, \lambda, p}^{m+1, \gamma, \rho} (f * g)(z)) - \left( \frac{\lambda + \gamma p}{\gamma} \right) (F_{t, \lambda, p}^{m, \gamma, \rho} (f * g)(z)). \tag{1.8} \]

This concept's major aim is to discover suitable conditions for specific normalized holomorphic functions $f$ to satisfy:

\[ q_1(z) \left< \left( 1 - \sigma \right) z F_{t, \lambda, p}^{m, \gamma, \rho} (f * g)(z) + 2 \sigma z F_{t, \lambda, p}^{m+1, \gamma, \rho} (f * g)(z) \right)^\rho < q_2(z), \]

where ($\rho \in \mathbb{C} \setminus \{0\}, \sigma \in \mathbb{R}^+, z \in \mathbb{U}$ and $f, g \in \Sigma_p$)

and

\[ q_1(z) < \left( z F_{t, \lambda, p}^{m, \gamma, \rho} (f * g)(z) \right)^\rho < q_2(z), \]

whenever univalent functions $q_1(z)$ and $q_2(z)$ are given in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$

2-Preliminaries:

The definitions and lemmas given below will assist us in proving our basic results.
**Definition 2.1**[16]: The set of all holomorphic and injective functions on \( U \setminus E(f) \), where \( U = U \cup \{z \in \partial U\} \), is denoted by \( Q \), and

\[
E(f) = \{ \omega \in \partial U : f(z) = \infty \},
\]

and are such that \( f'(\omega) \neq 0 \) for \( \omega \in \partial U \setminus E(f) \). Furthermore, let \( Q(a), Q(0) = q_0 \) and \( Q(1) = q_1 \), be the subclass of \( Q \) for which \( f(0) = a \).

**Lemma 2.1**: [17] Let \( q(z) \) be convex univalent function in \( U \), let \( \alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \{0\} \) and suppose that

\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\text{Re} \left( \frac{a}{b} \right) \right\}.
\]

If \( p(z) \) is holomorphic in \( U \) and

then \( p(z) < q(z) \) and \( q \) is the best dominant. \( a p(z) + \beta z p'(z) < a q(z) + \beta z q'(z) \).

**Lemma 2.2**: [11] Let \( q \) be univalent in \( U \) and let \( \phi \) and \( \theta \) be holomorphic in the domain \( D \) containing \( q(U) \) with \( \phi(\omega) \neq 0 \), when \( \omega \in q(U) \). Set \( Q(z) = zq'(z)\phi(q(z)) \) and \( h(z) = \theta(q(z)) + Q(z) \), suppose that

1. is starlike univalent in \( U, Q \)
2. \( \text{Re} \left( \frac{2h'(z)}{q'(z)} \right) > 0, z \in U. \)

If \( p \) is holomorphic in \( U \) with \( p(0) = q(0), p(U) \subseteq D \) and

then \( p < q \), and \( q \) is the best dominant. \( \theta(p(z)) + z p'(z) \phi(p(z)) < \theta(q(z)) + z q'(z) \phi(q(z)) \).

**Lemma 2.3**: [17] Let \( q(z) \) be convex univalent in the unit disk \( U \) and let \( \theta \) and \( \phi \) be holomorphic in a domain \( D \) containing \( q(U) \). Suppose that

1. \( \text{Re} \left( \frac{\theta(q(z))}{\phi(q(z))} \right) > 0 \) for \( z \in U; \)
2. is starlike univalent in \( z \in U, zq'(z)\phi(q(z)) \)

If \( p \in H[q(0), 1] \cap Q, p(U) \subseteq D, \) and \( \theta(p(z)) + z p'(z) \phi(p(z)) \) is univalent in \( U \), and

\[
\theta(q(z)) + z q'(z) \phi(q(z)) < \theta(p(z)) + z p'(z) \phi(p(z)),
\]

then \( q < p \), and \( q \) is the best subordinant.

**Lemma 2.4**: [17] Let \( q(z) \) be convex univalent in \( U \) and \( q(0) = 1 \). Let \( \beta \in \mathbb{C} \), that \( \text{Re} \beta > 0 \). If \( p(z) \in H[q(0), 1] \cap Q \) and \( p(z) + \beta z p'(z) \) is univalent in \( U \), then \( q(z) + \beta z q'(z) < p(z) + \beta z p'(z) \), which implies that \( q(z) < p(z) \) and \( q(z) \) is the best subordinant.

**3- Results of Differential Subordinations**

Now, we discuss some differential subordination results using a new Hadamard product operator \( F^{m,\delta,\gamma}_{\tau,\rho} \).

**Theorem 3.1**: Let \( q(z) \) be a convex univalent in the open unit disk \( U \) with \( q(0) = 1 \), and \( q'(z) \neq 0 \), for all \( z \in U \). Let \( \tau, \rho \in \mathbb{C} \setminus \{0\}, \sigma \in \mathbb{R}^+. \) Suppose that

\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\text{Re} \left( \frac{\rho}{\tau} \right) \right\}.
\]
If $f \in \Sigma_p$ is satisfies the subordination condition:

$$H(z) < q(z) + \frac{\tau}{\rho} q'(z). \quad (3.2)$$

where

$$H(z) = \left( \frac{(1 - \sigma)zF^{m,\delta,y}_{t,\lambda,p}(f \ast g)(z) + 2\sigma zF^{m+1,\delta,y}_{t,\lambda,p}(f \ast g)(z)}{\sigma + 1} \right)\rho$$

$$+ \tau \left[ \frac{1}{\gamma} \left( \frac{2\sigma\lambda z^{m+2,\delta,y}(f \ast g)(z) + \lambda(1-3\sigma) - 2\sigma y \rho - 1)F^{m+1,\delta,y}_{t,\lambda,p}(f \ast g)(z) - (1-\sigma)\lambda(y+p)F^{m,\delta,y}_{t,\lambda,p}(f \ast g)(z)}{(1-\sigma)F^{m,\delta,y}_{t,\lambda,p}(f \ast g)(z) + 2\sigma F^{m+1,\delta,y}_{t,\lambda,p}(f \ast g)(z)} \right) \right], \quad (3.3)$$

then

$$\left( \frac{(1 - \sigma)zF^{m,\delta,y}_{t,\lambda,p}(f \ast g)(z) + 2\sigma zF^{m+1,\delta,y}_{t,\lambda,p}(f \ast g)(z)}{\sigma + 1} \right)\rho < q(z), \quad (3.4)$$

where the best dominating is $q(z)$.

**Proof**: Define the $g(z)$ function as follows:

$$g(z) = \left( \frac{(1 - \sigma)zF^{m,\delta,y}_{t,\lambda,p}(f \ast g)(z) + 2\sigma zF^{m+1,\delta,y}_{t,\lambda,p}(f \ast g)(z)}{\sigma + 1} \right)\rho, \quad (3.5)$$

then the function $g(z)$ is holomorphic in $U$ and $g(0) = 1$ as a result of differentiating (3.5) with respect to $z$ and then using the identity (1.8) in the resultant equation.

$$H(z) = \left( \frac{(1 - \sigma)zF^{m,\delta,y}_{t,\lambda,p}(f \ast g)(z) + 2\sigma zF^{m+1,\delta,y}_{t,\lambda,p}(f \ast g)(z)}{\sigma + 1} \right)\rho$$

$$+ \tau \left[ \frac{1}{\gamma} \left( \frac{2\sigma\lambda z^{m+2,\delta,y}(f \ast g)(z) + \lambda(1-3\sigma) - 2\sigma y \rho - 1)F^{m+1,\delta,y}_{t,\lambda,p}(f \ast g)(z) - (1-\sigma)\lambda(y+p)F^{m,\delta,y}_{t,\lambda,p}(f \ast g)(z)}{(1-\sigma)F^{m,\delta,y}_{t,\lambda,p}(f \ast g)(z) + 2\sigma F^{m+1,\delta,y}_{t,\lambda,p}(f \ast g)(z)} \right) \right]zg'(z).

Thus the subordination (3.2) is equivalent to

$$g(z) + \frac{\tau}{\rho} zg'(z) < q(z) + \frac{\tau}{\rho} zq'(z).$$

An application of Lemma(2.1) with $\beta = \frac{\tau}{\rho}, \alpha = 1$, we obtain (3.4).

**Corollary 3.1**: Let $\tau, \rho \in \mathbb{C} \setminus \{0\}$, $\sigma \in \mathbb{R}^+$ and $(-1 \leq B < A \leq 1)$. Suppose that

$$\text{Re} \left( \frac{1-Bz}{1+Bz} \right) > \max \left\{ 0, -\text{Re} \left( \frac{\rho}{\tau} \right) \right\}.$$

If $f \in \Sigma_p$ is satisfy the following subordination condition:
when $H(z)$ given by (3.3), then

\[
\left(\frac{(1 - \sigma)zF_{t,\lambda,p}^{m,\delta,\gamma}(f \ast g)(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\delta,\gamma}(f \ast g)(z)}{\sigma + 1}\right)^{\rho} < \frac{1 + Az}{1 + Bz'}
\]

where the best dominating is $\frac{1 + Az}{1 + Bz}$.

In Corollary(3.1), we can get following result with $A = 1$ and $B = -1$.

**Corollary 3.2:** Let $\tau, \rho \in \mathbb{C} \setminus \{0\}, \sigma \in \mathbb{R}^+$ and suppose that

\[
\text{Re}\left(\frac{1 + z}{1 - z}\right) > \max\{0, -\text{Re}\left(\frac{\rho}{\tau}\right)\},
\]

If $f \in \Sigma_p$ fulfill the following subordination condition:

\[
H(z) < \frac{1 + z}{1 - z} + \frac{\tau}{\rho} \frac{2z}{(1 - z)^2},
\]

when $H(z)$ given by (3.3), then

\[
\left(\frac{(1 - \sigma)zF_{t,\lambda,p}^{m,\delta,\gamma}(f \ast g)(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\delta,\gamma}(f \ast g)(z)}{\sigma + 1}\right)^{\rho} < \frac{1 + z}{1 - z},
\]

and $\frac{1 + z}{1 - z}$ is the best dominant.

**Theorem 3.2:** In unit disk $U$, let $q(z)$ be convex univalent function in the open unit disk $U$ with $q(0) = 1$, $q'(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. Let $\rho \in \mathbb{C} \setminus \{0\}, \xi, \alpha, \lambda, \mu \in \mathbb{C}, f \in \Sigma_p$, and suppose that $q$ satisfy the following conditions

\[
\text{Re}\left(\frac{\lambda}{q} q(z) + \frac{2\mu}{\xi} q^2(z) + 1 + z \frac{q''(z)}{q(z)} - \frac{q'(z)}{q(z)}\right) > 0,
\]

and if $f, g \in \Sigma_p$ satisfies:

\[
zF_{t,\lambda,p}^{m,\delta,\gamma}(f \ast g)(z) \neq 0.
\]

If

\[
e(z) < a + \lambda q(z) + \mu \xi q^2(z) + q \frac{zq'(z)}{q(z)},
\]

where

\[
e(z) = \left(zF_{t,\lambda,p}^{m,\delta,\gamma}(f \ast g)(z)\right)^{\rho} \times \left[\lambda + \mu \xi \left(zF_{t,\lambda,p}^{m,\delta,\gamma}(f \ast g)(z)\right)^{\mu} + \rho \left(1 - \gamma\right) \left[\lambda zF_{t,\lambda,p}^{m+1,\delta,\gamma}(f \ast g)(z) - \gamma \left(zF_{t,\lambda,p}^{m,\delta,\gamma}(f \ast g)(z) - (\lambda - \gamma(1 + p))\right)\right]\right],
\]

\[\text{pp Math. 150–161}
\]
then \( (zF_{t,\lambda,p}^{m,\gamma}(f*g)(z))^{\rho} < q(z) \), where the best dominating is \( q(z) \).

**Proof:** As follows, define the holomorphic function \( g(z) \):

\[
g(z) = (zF_{t,\lambda,p}^{m,\gamma}(f*g)(z))^{\rho}.
\]

then the function \( g(z) \) is holomorphic in \( U \) and \( g(0) = 1 \). By differentiating (3.10) with respect to \( z \), and using identity (1.8) in the resulting equation, we get

\[
\frac{zg'(z)}{g(z)} = \rho \left( \frac{1}{\gamma} \right) \left[ \frac{\lambda F_{t,\lambda,p}^{m+1,\gamma}(f*g)(z)}{F_{t,\lambda,p}^{m,\gamma}(f*g)(z)} - (\lambda - \gamma(1 + p)) \right].
\]

(3.11)

Setting \( \theta(\omega) = a + \lambda \omega + \mu \xi \omega^2 \) and \( \phi(\omega) = \frac{q}{\omega} \), \( \omega \neq 0 \), reveals the \( \theta(\omega) \) is holomorphic function in \( \mathbb{C} \), and \( \phi(\omega) \) is holomorphic in \( \mathbb{C} \setminus \{0\} \) and \( \phi(\omega) \neq 0 \), \( \omega \in \mathbb{C} \setminus \{0\} \).

If, we let

\[
Q(z) = zq'(z)\phi(z) = \frac{zq'(z)}{q(z)} \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z) = a + \lambda q'(z) + \mu \xi q^2(z) + \rho \frac{zq'(z)}{q(z)},
\]

we find that \( Q(z) \) is starlike univalent in \( U \), we have

\[
h'(z) = \lambda q'(z) + 2\mu \xi q(z)q'(z) + \rho \frac{q'(z)}{q(z)} + \rho z \frac{q''(z)}{q'(z)} - \rho z \left( \frac{q'(z)}{q(z)} \right)^2,
\]

and

\[
\frac{zh'(z)}{Q(z)} = \frac{\lambda}{\rho} q(z) + \frac{2\mu \xi}{\phi} q^2(z) + 1 + z \frac{q''(z)}{q'(z)} - z \frac{q'(z)}{q(z)},
\]

hence that

\[
\text{Re} \left( \frac{zh'(z)}{Q(z)} \right) = \text{Re} \left( \frac{\lambda}{\rho} q(z) + \frac{2\mu \xi}{\phi} q^2(z) + 1 + z \frac{q''(z)}{q'(z)} - z \frac{q'(z)}{q(z)} \right) > 0.
\]

By using (3.11), we obtain

\[
\lambda g(z) + \mu \xi g^2(z) + \rho \frac{zg'(z)}{g(z)} = \left( zF_{t,\lambda,p}^{m,\gamma}(f*g)(z) \right)^\rho \left[ \lambda + \mu \xi \left( zF_{t,\lambda,p}^{m+1,\gamma}(f*g)(z) \right)^\rho \right]
\]

\[
+ \rho \rho \left( \frac{1}{\gamma} \right) \left[ \frac{\lambda F_{t,\lambda,p}^{m+1,\gamma}(f*g)(z)}{F_{t,\lambda,p}^{m,\gamma}(f*g)(z)} - (\lambda - \gamma(1 + p)) \right].
\]

By using (3.8), we have

\[
\lambda g(z) + \mu \xi g^2(z) + \rho \frac{zg'(z)}{g(z)} < \lambda q(z) + \mu \xi q^2(z) + \rho \frac{zq'(z)}{q(z)},
\]

we can infer that subordination (3.8) implies that \( g(z) < q(z) \), and that the function \( q(z) \) is the best domain by using Lemma 2.2.
Taking the function \( q(z) = \frac{1+Az}{1+Bz} \) \((-1 \leq B < A \leq 1)\), in Theorem 3.2, the condition (3.6) becomes

\[
Re \left\{ \frac{\lambda}{\mu} \Phi \left( \frac{1+Az}{1+Bz} \right) + \frac{2\mu \xi}{\mu} \Phi \left( \frac{(1+Ax)^2}{1+Bz} \right) + 1 + \frac{(A-B)z}{(1+Bz)(1+Ax)} - \frac{2Bz}{1+Bz} \right\} > 0 \quad (q \in \mathbb{C} \setminus \{0\}), \quad (3.12)
\]
as a result, we may deduce the following conclusion.

**Corollary 3.3:** Let \((-1 \leq B < A \leq 1), q, \rho \in \mathbb{C} \setminus \{0\}\), \(\xi, a, \lambda, \mu \in \mathbb{C}\), assume that (3.12) holds. If \(f \in \Sigma_p\) and

\[
e(z) < a + \lambda \Phi \left( \frac{1+Az}{1+Bz} \right) + \mu \xi \Phi \left( \frac{1+Az}{1+Bz} \right)^2 + q \frac{(A-B)z}{(1+Bz)(1+Az)}
\]
where \(e(z)\) is defined in (3.9), then

is the best dominant. \((zF_{\lambda,\mu}^m(f \circ g)(z))^\rho < \frac{1+Az}{1+Bz}, \) and \(\frac{1+Az}{1+Bz}\)

Taking the function \( q(z) = \left( \frac{1+Az}{1+Bz} \right)^\iota \) \((0 < \iota \leq 1)\), in Theorem (3.2), the condition (3.6) becomes

\[
Re \left\{ \frac{\lambda}{\mu} \Phi \left( \frac{1+Az}{1+Bz} \right)^\iota + \frac{2\mu \xi}{\mu} \Phi \left( \frac{1+Az}{1+Bz} \right)^{2\iota} + \frac{2\iota z}{1-z^\iota} \right\} > 0 \quad (q \in \mathbb{C} \setminus \{0\}). \quad (3.13)
\]

As a result, we may deduce the following conclusion.

**Corollary 3.4:** Let \(0 < \iota \leq 1, q, \rho \in \mathbb{C} \setminus \{0\}\), \(\xi, a, \lambda, \mu \in \mathbb{C}\). Assume that (3.13) holds. If \(f \in \Sigma_p\) and

\[
e(z) < a + \lambda \Phi \left( \frac{1+Az}{1+Bz} \right)^\iota + \mu \xi \Phi \left( \frac{1+Az}{1+Bz} \right)^{2\iota} + q \frac{2\iota z}{1-z^\iota},
\]
where \(e(z)\) is defined in (3.9), then \((zF_{\lambda,\mu}^m(f \circ g)(z))^\rho < \left( \frac{1+Az}{1-z} \right)^\iota\), and \(\left( \frac{1+Az}{1-z} \right)^\iota\) is the best dominant.

### 4- Results of Differential Superordinations:

**Theorem 4.1:** Assume that the function \( q(z) \) is a convex univalent in \( U \) with \( q(0) = 1, \rho \in \mathbb{C} \setminus \{0\}, Re(\tau) > 0, \sigma \in \mathbb{R}^+ \), if \( f \in \Sigma_p \), such that

\[
\frac{(1-\sigma)zF_{\lambda,\mu}^{m,\nu}(f \circ g)(z) + 2\sigma zF_{\lambda,\mu}^{m+1,\nu}(f \circ g)(z)}{\sigma + 1} \neq 0, \quad \text{and}
\]

\[
\left( \frac{(1-\sigma)zF_{\lambda,\mu}^{m,\nu}(f \circ g)(z) + 2\sigma zF_{\lambda,\mu}^{m+1,\nu}(f \circ g)(z)}{\sigma + 1} \right)^\rho \in H[q(0),1] \cap Q, g \in \Sigma_p. \quad (4.1)
\]

If the function \( H(z) \) in (3.3) is univalent and the superordination criterion is fulfilled:

\[
q(z) + \frac{\tau}{\rho} zq'(z) < H(z), \quad (4.2)
\]
holds, then

\[
q(z) < \left( \frac{(1-\sigma)zF_{\lambda,\mu}^{m,\nu}(f \circ g)(z) + 2\sigma zF_{\lambda,\mu}^{m+1,\nu}(f \circ g)(z)}{\sigma + 1} \right)^\rho, \quad (4.3)
\]

where the best subordinant is \( q(z) \).
Proof: Define a function \( g(z) \) by
\[
g(z) = \left( \frac{(1 - \sigma)zF_{t,\lambda,p}^{m,\vartheta,y}(f*g)(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,y}(f*g)(z)}{\sigma + 1} \right)^\rho. \tag{4.4}
\]
Differentiating (4.4) with respect to \( z \), we get
\[
\frac{zg'(z)}{g(z)} = \rho \left[ \frac{(1 - \sigma)z\left(F_{t,\lambda,p}^{m,\vartheta,y}(f*g)(z)\right)' + 2\sigma z\left(F_{t,\lambda,p}^{m+1,\vartheta,y}(f*g)(z)\right)'}{(1 - \sigma)zF_{t,\lambda,p}^{m,\vartheta,y}(f*g)(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,y}(f*g)(z)} + 1 \right]. \tag{4.5}
\]
A simple computation and using (1.8), from (4.5), we will get
\[
H(z) = \left( \frac{(1 - \sigma)zF_{t,\lambda,p}^{m,\vartheta,y}(f*g)(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,y}(f*g)(z)}{\sigma + 1} \right)^\rho
+ \tau \left[ \frac{\partial}{\partial z} \left( \frac{2\sigma F_{t,\lambda,p}^{m+2,\vartheta,y}(f*g)(z) + (1 - \vartheta)(1 - \lambda) - 2\sigma(p-1)F_{t,\lambda,p}^{m+1,\vartheta,y}(f*g)(z) - (1 - \sigma)(p+1)F_{t,\lambda,p}^{m,\vartheta,y}(f*g)(z)}{(1 - \sigma)zF_{t,\lambda,p}^{m,\vartheta,y}(f*g)(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,y}(f*g)(z)} \right) \right],
\]  
which
\[= g(z) + \frac{\tau}{\rho} zg'(z). \]

Now, by using Lemma 2.4, we get the desired result.

Taking \( q(z) = \frac{1 + Az}{1 + Bz} \), \((-1 \leq B < A \leq 1)\), we obtain the following conclusion from Theorem 4.1.

Corollary 4.1: Let \( \text{Re}(\tau) > 0, \rho \in \mathbb{C} \setminus \{0\}, \vartheta \in \mathbb{R}^+ \) and \((-1 \leq B < A \leq 1)\), such that
\[
\left( \frac{(1 - \sigma)zF_{t,\lambda,p}^{m,\vartheta,y}(f*g)(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,y}(f*g)(z)}{\sigma + 1} \right)^\rho \in H[q(0),1] \cap Q.
\]
If \( H(z) \) in (3.3) is univalent in \( U \), and \( f \in \Sigma_p \) fulfills the superordination condition,
\[
\frac{1 + Az}{1 + Bz} + \frac{\tau}{\rho} \frac{(A - B)z}{(1 + Bz)^2} < F(z),
\]
then
\[
\frac{1 + Az}{1 + Bz} \frac{(1 - \sigma)zF_{t,\lambda,p}^{m,\vartheta,y}(f*g)(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,y}(f*g)(z)}{\sigma + 1} \rho,
\]
the best subordinant is the function \( \frac{1 + Az}{1 + Bz} \).

Theorem 4.2: Let \( q(z) \) be a convex univalent function in the open unit disk \( U \) with \( q(0) = 1 \), \( q'(z) \neq 0 \) and \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( U \). Let \( q, \rho \in \mathbb{C} \setminus \{0\}, \xi, \alpha, \lambda, \mu \in \mathbb{C} \). Suppose that \( q \) satisfy the condition \( \text{Re} \left\{ \frac{q(z)}{\rho} (2\mu \xi + \lambda) \right\} q'(z) > 0 \). Let \( f \in \Sigma_p \) and satisfies the next conditions
\[
\left( zF_{t,\lambda,p}^{m,\vartheta,y}(f*g)(z) \right)^\rho \in H[q(0),1] \cap Q, \quad g \in \Sigma_p \tag{4.6}
\]
and

If the function \( e(z) \) is given by (3.9), is univalent in \( U, zF_{\lambda,\mu}^{m,d,\gamma}(f \ast g)(z) \neq 0 \).

\[
a + \lambda q(z) + \mu \xi q^2(z) + q \frac{zq'(z)}{q(z)} < F(z),
\]

implies

where the best subordinant is \( q(z)q(z) < \left( zF_{\lambda,\mu}^{m,d,\gamma}(f \ast g)(z) \right)^\rho \).

**Proof**: Allow \( g(z) \) to be defined on \( U \) by (3.10).

After that, a calculation reveals that

\[
\frac{zg'(z)}{g(z)} = \rho \left( \frac{1}{\gamma} \left[ \frac{F_{\lambda,\mu}^{m+1,d,\gamma}(f \ast g)(z)}{F_{\lambda,\mu}^{m,d,\gamma}(f \ast g)(z)} - (\lambda - \gamma(1 + p)) \right] \right),
\]

By setting \( \theta(\omega) = a + \lambda \omega + \mu \xi \omega^2, \) and \( \phi = \frac{q}{\omega}, \omega \neq 0 \). It can be easily observed that \( \theta(\omega) \) is holomorphic in \( C, \) and \( \phi(\omega) \) is holomorphic in \( C \setminus \{0\} \), that \( \phi(\omega) \neq 0 (\omega \in C \setminus \{0\}) \). Also, we get

it was discovered that \( Q(z) \) is a starlike univalent in \( U, Q(z) = zq'(z)\phi(q(z)) = q \frac{zq'(z)}{q(z)}, \)

Because \( q(z) \) is convex, we may deduce that

\[
\text{Re} \left( \frac{z\theta'(q(z))}{\phi(q(z))} \right) = \text{Re} \left( \frac{q(z)}{q(z)} (2\mu \xi q(z) + \lambda) q'(z) > 0. \right.
\]

By making use (4.8) the hypothesis (4.7) can by equivalently

\[
\theta(q(z)) + zq'(z)\phi(q(z)) < \theta(g(z)) + zg'(z)\phi(g(z)) .
\]

The proof is therefore completed by utilizing the Lemma 2.3.

5- Sandwich Results:

**Theorem 5.1**: Let \( q_1 \) and \( q_2 \) be convex univalent functions in \( U \) with \( q_1(0) = q_2(0) = 1 \) and \( q_2 \) satisfies (3.1). Suppose that \( \text{Re}(\tau) > 0, \tau, \rho \in C \setminus \{0\}, \sigma \in \mathbb{R}^+, \) if \( f \in \sum_{\nu} \) such that

\[
\left( \frac{(1 - \sigma)zF_{\lambda,\mu}^{m,d,\gamma}(f \ast g)(z) + 2\sigma zF_{\lambda,\mu}^{m,d,\gamma}(f \ast g)(z)}{\sigma + 1} \right)^\rho \in H[q(0), 1] \cap Q,
\]

and the univalent function \( H(z) \), defined by (3.3), satisfies

\[
q_1(z) + \frac{\tau}{\rho} zq_1'(z) < H(z) < q_2(z) + \frac{\tau}{\rho} zq_2'(z), \tag{5.1}
\]

then

\[
q_1(z) < \left( \frac{(1 - \sigma)zF_{\lambda,\mu}^{m,d,\gamma}(f \ast g)(z) + 2\sigma zF_{\lambda,\mu}^{m,d,\gamma}(f \ast g)(z)}{\sigma + 1} \right)^\rho < q_2(z),
\]
where \( q_1 \) and \( q_2 \) are the best subordinant and dominant of the pair, respectively (5.1).

We obtain the following sandwich theorem by merging Theorems 3.2 and 4.2:

**Theorem 5.2:** Let \( q_j \) be two univalent convex functions in \( U \), with \( q_j(0) = 1, q_j'(z) \neq 0, (j = 1, 2) \). Assume that \( q_1 \) and \( q_2 \) satisfy the conditions (3.8) and (4.8), respectively.

If \( f \in \sum_\rho \), and suppose that \( f \) satisfies the next condition:

\[
\left( z^{m,\alpha,\gamma}_{\ell,\lambda,p}(f * g)(z) \right) ^\rho \in H[q(0), 1] \cap Q,
\]

and \( z^{m,\beta,\gamma}_{\ell,\lambda,p}(f * g)(z) \neq 0 \), and \( e(z) \) is univalent in \( U \), and given by (3.9), then

\[
a + \lambda q_1(z) + \mu \xi q_1(z) + \frac{zq_1'(z)}{q_1(z)} < e(z) < a + \lambda q_2(z) + \mu \xi q_2(z) + \frac{zq_2'(z)}{q_2(z)},
\]

implies

\[
q_1(z) < \left( z^{m,\alpha,\gamma}_{\ell,\lambda,p}(f * g)(z) \right)^\rho < q_2(z),
\]

where the best subordinant and dominant are \( q_1 \) and \( q_2 \), respectively.

**References**


