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On Sandwich Results of Meromorphic Multivalent Functions Defined by a New Hadamard Product Operator

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ABSTRACT

The goal of this research is to establish differential subordination and superordination findings for meromorphic multivalent functions defined by a new operator in a punctured open unit disk. We get a number of sandwich-type results.

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1. Introduction

Let ∑_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are meromorphic multivalent in the punctured open unit disk $U^* = \{z: z \in \mathbb{C}, 0 < |z| < 1\}$.Several authors studied meromorphic functions for another classes and conditions, see [7, 9, 12, 20].Let H is the linear space of all holomorphic functions in U. For a positive integernumber n and $a \in \mathbb{C}$, we let

$$H[a,n] = \{f \in H: f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

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For f and F holomorphic function in H, we say that f is subordinate to F in U and write $f(z) \prec F(z)$, if there exists a Schwarz function w, which is holomorphic in U with $w(0) = 0$ and $|w(z)| < 1, (z \in U)$, such that $f(z) = F\big(w(z)\big)$, $(z \in U)$.

Furthermore, if the function F is holomorphic in U, we have the following equivalence relationship (cf. , e.g. [10,11,16,17]):

$$f(z) \prec F(z) \leftrightarrow f(0) = F(0) \; and \; f(U) \subset F(U), (z \in U).$$

Definition1:([16],also see [20]) Let $\Upsilon: \mathbb{C}^3 \times U \to \mathbb{C}$ and let h(z) be holomorphic in U. If $p \; and \; \Upsilon(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and if p needs to satisfy the second-order differential superordination,

$$h(z) \prec \Upsilon(p(z), zp'(z), z^2p''(z); z), \tag{1.2}$$

then p called a solution of the differential superordination (1.2). An holomorphic function $q(z)$ which is called a subordinant of the solutions of the differential superordination (1.2) or more simply, a subordinant if $q \prec p$ for all p fulfill (1.2). A univalent subordinant $\widehat{q}(z)$ that fulfills $q \prec \widehat{q}$ for all subordinants q of (1.2), is said to be the best subordinant.

Definition 2 : [16] Let $\Upsilon: \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is holomorphic in U and satisfies the second-order differential subordination,

$$\Upsilon(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.3}$$

then p is called a solution of the differential subordination (1.3). The univalent function q is called a dominant of the solution of the differential subordination (1.3), or more simply dominant if $p \prec q$ for all p satisfying (1.3). A univalent dominant $\hat{q}(z)$ that satisfies $\hat{q} \prec q$ for all dominant q of (1.3) is said to be the best dominant.

Miller and Mocanu [17] and other authors [1,2,3,4,5,6,7,8,9,10,13] and also [14,15,18,19,20,23,24] discovered sufficient conditions for the functions h, p, and ϕ for which the following result:

$$h(z) \prec \Upsilon(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z) (z \in U). \tag{1.4}$$

If $\mathrm{f} \in \sum_p$ is given by (1.1) and $\mathrm{g} \in \sum_p$ given by

$$\mathrm{g}(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^k.$$

The Hadamard product (or convolution) of f and g is given by

$$(\mathrm{f} * \mathrm{g})(z) = z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^k = (\mathrm{g} * \mathrm{f})(z).$$

Using the results, (see [1,2,4,5,6,7,14,15,18,19,21,22,23,24]) to obtain adequate criteria for the satisfaction of normalized analytic functions

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

Shanmugam et al. [21][22], as well as Goyal et al. [13], sandwich results for holomorphic function classes were recently obtained. (See also [1,3,4,5,11]).

In a recent paper, E- Ashwah [12] defined the multiplier transform $Q_{\lambda,p}^{n,\lambda}$ of a function $f \in \sum_p$

$$Q_{\lambda,p}^{n,\lambda}: \sum_p \to \sum_p$$

which is defined as follows:

$$Q_{\lambda,p}^{n,\gamma} f(z) = z^{-p} + \sum_{k=0}^{\infty} \left(\frac{\lambda + \gamma(k+p)}{\lambda}\right)^m a_k z^k, \tag{1.5}$$

where ($\lambda > 0, \gamma > 0, z \in U^*; m \in N_0 \; := N \cup \{0\}\,; p \in N$).

Ali, Ravichandran and Seenivasagan [25] defined the operator $K_p^{t,\vartheta}$ of a function $g \in \sum_p$

$$K_p^{t,\vartheta}: \sum_p \to \sum_p$$

which is defined as follows:

$$K_p^{t,\vartheta} g(z) = z^{-p} + \sum_{k=0}^{\infty} \left(\frac{k+\vartheta}{\vartheta - p}\right)^t b_k z^k, \quad (\vartheta > 1, t \in N_0 \; ; p \in N). \tag{1.6}$$

We define the new Hadamard product operator

$$F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) = Q_{\lambda,p}^{n,\lambda} f(z) * K_p^{t,\vartheta} f(z)$$

$$F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) = z^{-P} + \sum_{k=0}^{\infty} \left(\frac{k+\vartheta}{\vartheta - p}\right)^t \left(\frac{\lambda + \gamma(k+p)}{\lambda}\right)^m a_k \, b_k z^k, \tag{1.7}$$

we note that from (1.7) , we have

$$z\left(F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)\right)' = \frac{\lambda}{\gamma}\left(F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)\right) - \left(\frac{\lambda + \gamma p}{\gamma}\right)\left(F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)\right). \tag{1.8}$$

This concept's major aim is to discover suitable conditions for specific normalized holomorphic functions f to satisfy:

$$q_1(z) \prec \left(\frac{(1-\sigma) z F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) + 2\sigma z F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)}{\sigma + 1}\right)^{\rho} \prec q_2(z),$$

where ($\rho \in \mathbb{C} \setminus \{0\}, \; \sigma \in \mathbb{R}^+, \; z \in U \; and \; f, g \in \sum_p$)

and

$$q_1(z) \prec \left(z F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)\right)^{\rho} \prec q_2(z),$$

whenever univalent functions $q_1\,(z)\; and \; q_2\,(z)$ are given in U with $q_1(0) = q_2(0) = 1$

2-Preliminaries :

The definitions and lemmas given below will assist us in proving our basic results.

Definition 2.1[16]: The set of all holomorphic and injective functions on $\overline{U} \setminus E(f)$, $where\ \overline{U} = U \cup \{z \in \partial U\}$, is denoted by Q, and

$$E(f) = \{\omega \in \partial U : f(z) = \infty\}, \tag{2.1}$$

and are such that $f'(\omega) \neq 0$ for $\omega \in \partial U \setminus E(f)$. Furthermore, let $Q(a), Q(0) = Q_0$ and $Q(1) = Q_1$, be the subclass of Q for which $f(0) = a$.

Lemma 2.1: [17] Let $q(z)$ be convex univalent function in U, let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \{0\}$ and suppose that

$$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > max\left\{0, -Re\left(\frac{\alpha}{\beta}\right)\right\}.$$

If $p(z)$ is holomorphic in U and

then $p(z) \prec q(z)$ and q is the best dominant.$\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z)$,

Lemma 2.2: [11] Let q be univalent in U and let $\phi\ and\ \theta$ be holomorphic in the domain D containing $q(U)$ with $\phi(\omega) \neq 0$, when $\omega \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))\ and\ h(z) = \theta(q(z)) + Q(z)$, suppose that

1- is starlike univalent in U,Q
2- $Re\left(\frac{zh\prime(z)}{Q(z)}\right) > 0, z \in U$.

If p is holomorphic in U with $p(0) = q(0), p(U) \subseteq D$ and

then $p \prec q$, and q is the best dominant.$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$,

Lemma 2.3: [17] Let $q(z)$ be convex univalent in the unit disk U and let $\theta\ and\ \phi$ be holomorphic in a domain D containing $q(U)$. Suppose that

1- $Re\left\{\frac{\theta\prime(q(z))}{\phi(q(z))}\right\} > 0\ for\ z \in U$;
2- is starlike univalent in $z \in U$.$zq'(z)\phi(q(z))$

If $p \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z) + zp'(z)\phi(p(z))$ is univalent in U , and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{2.2}$$

then $q \prec p$, and q is the best subordinant.

Lemma 2.4: [17] Let $q(z)$ be convex univalent in $U\ and\ q(0) = 1$. Let $\beta \in \mathbb{C}$, that $Re\{\beta\} > 0$. If $p(z) \in H[q(0), 1] \cap Q$ and $p(z) + \beta z p'(z)$ is univalent in U, then $q(z) + \beta z q'(z) \prec p(z) + \beta z p'(z)$, which implies that $q(z) \prec p(z)\ and\ q(z)$ is the best subordinant.

3- Results of Differential Subordinations

Now, we discuss some differential subordination results using a new Hadamard product operator $F_{t,\lambda,p}^{m,\vartheta,\gamma}$.

Theorem 3.1 : Let $q(z)$ be a convex univalent in the open unit disk U with $q(0) = 1$, and $q'(z) \neq 0$, for all $z \in U$. Let $\tau, \rho \in \mathbb{C} \setminus \{0\}$, $\sigma \in \mathbb{R}^+$. Suppose that

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re\left(\frac{\rho}{\tau}\right)\right\}. \tag{3.1}$$

If $f \in \sum_p$ is satisfies the subordination condition:

$$H(z) \prec q(z) + \frac{\tau}{\rho} zq'(z), \tag{3.2}$$

where

$$H(z) = \left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)}{\sigma + 1} \right)^{\rho}$$

$$+\tau \left[\left(\frac{1}{\gamma}\right) \left(\frac{2\sigma\lambda F_{t,\lambda,p}^{m+2,\vartheta,\gamma}(\mathrm{f*g})(z) + \lambda(1-3\sigma) - 2\sigma\gamma(p-1) F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f*g})(z) - (1-\sigma)(\lambda+\gamma p) F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f*g})(z)}{(1-\sigma)F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f*g})(z) + 2\sigma F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f*g})(z)} \right) \right], \tag{3.3}$$

then

$$\left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)}{\sigma + 1} \right)^{\rho} \prec q(z), \tag{3.4}$$

where the best dominating is q(z).

Proof : Define the $g(z)$ function as follows:

$$g(z) = \left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)}{\sigma + 1} \right)^{\rho}, \tag{3.5}$$

then the function $g(z)$ is holomorphic in U and $g(0) = 1$ as a result of differentiating (3.5) with respect to z and then using the identity (1.8) in the resultant equation.

$$H(z) = \left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) + 2\sigma F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)}{\sigma + 1} \right)^{\rho}$$

$$+\tau \left[\left(\frac{1}{\gamma}\right) \left(\frac{2\sigma\lambda F_{t,\lambda,p}^{m+2,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) + \lambda(1-3\sigma) - 2\sigma\gamma(p-1) F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) - (1-\sigma)(\lambda+\gamma p) F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)}{(1-\sigma)F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) + 2\sigma F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)} \right) \right]$$

$$= g(z) + \frac{\tau}{\rho} zg'(z).$$

Thus the subordination (3.2) is equivalent to

$$g(z) + \frac{\tau}{\rho} zg'(z) \prec q(z) + \frac{\tau}{\rho} zq'(z).$$

An application of Lemma(2.1) with $\beta = \frac{\tau}{\rho}$, $\alpha = 1$, we obtain (3.4).

Corollary 3.1 : Let $\tau, \rho \in \mathbb{C} \setminus \{0\},\ \sigma \in \mathbb{R}^{+}$ and $(-1 \leq B < A \leq 1)$. Suppose that

$$Re\left(\frac{1-Bz}{1+Bz}\right) > max\left\{0, -Re\left(\frac{\rho}{\tau}\right)\right\}.$$

If $f \in \sum_p$ is satisfy the following subordination condition:

$$H(z) \prec \frac{1+Az}{1+Bz} + \frac{\tau}{\rho}\frac{(A-B)z}{(1+Bz)^2},$$

when $H(z)$ given by (3.3) , then

$$\left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\right)^{\rho} \prec \frac{1+Az}{1+Bz},$$

where the best dominating is $\frac{1+Az}{1+Bz}$.

In Corollary(3.1), we can get following result with $A = 1$ *and* $B = -1$.

Corollary 3.2: Let $\tau, \rho \in \mathbb{C} \setminus \{0\}$, $\sigma \in \mathbb{R}^{+}$ and suppose that

$$Re\left(\frac{1+z}{1-z}\right) > max\left\{0, -Re\left(\frac{\rho}{\tau}\right)\right\}.$$

If $f \in \sum_p$ fulfill the following subordination condition:

$$H(z) \prec \frac{1+z}{1-z} + \frac{\tau}{\rho}\frac{2z}{(1-z)^2},$$

when $H(z)$ given by (3.3), then

$$\left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z) + 2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\right)^{\rho} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant.

Theorem 3.2: In unit disk U, let $q(z)$ be convex univalent function in the open unit disk U with, $q(0) = 1, q'(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $\rho \in \mathbb{C} \setminus \{0\}, \xi, a, \lambda, \mu \in \mathbb{C}, f \in \sum_p$, and suppose that q satisfy the following conditions

$$Re\left\{\frac{\lambda}{\varrho}q(z) + \frac{2\mu\xi}{\varrho}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right\} > 0, \tag{3.6}$$

and if $f, g \in \sum_p$ satisfies :

$$zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z) \neq 0. \tag{3.7}$$

If

$$e(z) \prec a + \lambda q(z) + \mu\xi q^2(z) + \varrho\frac{zq'(z)}{q(z)}, \tag{3.8}$$

where

$$e(z) = \left(zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)\right)^{\rho}$$

$$\times\left[\lambda + \mu\xi\left(zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)\right)^2 + \varrho\rho\left(\frac{1}{\gamma}\right)\left[\frac{\lambda zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)} - (\lambda - \gamma(1+p))\right]\right], \tag{3.9}$$

then $\left(zF_{t,\lambda,p}^{m,\vartheta}(\mathrm{f}*\mathrm{g})(z)\right)^{\rho} \prec q(z)$, where the best dominating is $q(z)$.

Proof : As follows, define the holomorphic function $g(z)$:

$$g(z)=\left(zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)\right)^{\rho}. \tag{3.10}$$

then the function $g(z)$ is holomorphic in U and $g(0)=1$. By differentiating (3.10) with respect to z, and using identity (1.8) in the resulting equation, we get

$$\frac{zg'(z)}{g(z)}=\rho\left(\frac{1}{\gamma}\right)\left[\frac{\lambda F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}-(\lambda-\gamma(1+p))\right]. \tag{3.11}$$

Setting $\theta(\omega)=a+\lambda\omega+\mu\xi\omega^2$ *and* $\phi(\omega)=\frac{\varrho}{\omega},\omega\neq 0$, reveals the $\theta(\omega)$ is holomorphic function in $\mathbb{C}$, *and* $\phi(\omega)$ is holomorphic in $\mathbb{C}\setminus\{0\}$ *and* $\phi(\omega)\neq 0$, $\omega\in\mathbb{C}\setminus\{0\}$.

If, we let

$$Q(z)=zq'(z)\phi(z)=\varrho\frac{zq'(z)}{q(z)} \quad and \quad h(z)=\theta(q(z))+Q(z)=a+\lambda q'(z)+\mu\xi q^2(z)+\varrho\frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in U , we have

$$h'(z)=\lambda q'(z)+2\mu\xi q(z)q'(z)+\varrho\frac{q'(z)}{q(z)}+\varrho z\frac{q''(z)}{q'(z)}-\varrho z\left(\frac{q'(z)}{q(z)}\right)^2,$$

and

$$\frac{zh'(z)}{Q(z)}=\frac{\lambda}{\varrho}q(z)+\frac{2\mu\xi}{\varrho}q^2(z)+1+z\frac{q''(z)}{q'(z)}-z\frac{q'(z)}{q(z)},$$

hence that

$$Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\lambda}{\varrho}q(z)+\frac{2\mu\xi}{\varrho}q^2(z)+1+z\frac{q''(z)}{q'(z)}-z\frac{q'(z)}{q(z)}\right)>0.$$

By using (3.11), we obtain

$$\lambda g(z)+\mu\xi g^2(z)+\varrho\frac{zg'(z)}{g(z)}=\left(zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)\right)^{\rho}\left[\lambda+\mu\xi\left(zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)\right)^{2}\right]$$
$$+\varrho\rho\left(\frac{1}{\gamma}\right)\left[\frac{\lambda F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}-(\lambda-\gamma(1+p))\right].$$

By using (3.8), we have

$$\lambda g(z)+\mu\xi g^2(z)+\varrho\frac{zg'(z)}{g(z)}\prec\lambda q(z)+\mu\xi q^2(z)+\varrho\frac{zq'(z)}{q(z)},$$

we can infer that subordination (3.8) implies that $g(z)\prec q(z)$, and that the function $q(z)$ is the best domain by using Lemma2.2.

Taking the function $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$, in Theorem3.2, the condition (3.6) becomes

$$Re\left\{\frac{\lambda}{\varrho}\left(\frac{1+Az}{1+Bz}\right)+\frac{2\mu\xi}{\varrho}\left(\frac{1+Az}{1+Bz}\right)^{2}+1+\frac{(A-B)z}{(1+Bz)(1+Az)}-\frac{2Bz}{1+Bz}\right\}>0\ (\varrho\in\mathbb{C}\setminus\{0\}), \tag{3.12}$$

as a result, we may deduce the following conclusion..

Corollary 3.3: Let $(-1\le B<A\le 1), \varrho,\ \rho\in\mathbb{C}\setminus\{0\}$, $\xi, a, \lambda, \mu\ \in\mathbb{C}$, assume that (3.12) holds .If $f\in\ \sum_p$ and

$$e(z)\prec a+\lambda\left(\frac{1+Az}{1+Bz}\right)+\mu\xi\left(\frac{1+Az}{1+Bz}\right)^{2}+\varrho\frac{(A-B)z}{(1+Bz)(1+Az)},$$

where $e(z)$ is defined in (3.9), then

is the best dominant.$\left(zF_{t,\lambda,p}^{m,\vartheta}(\mathrm{f}*\mathrm{g})(z)\right)^{\rho}\prec\frac{1+Az}{1+Bz}$, *and* $\frac{1+Az}{1+Bz}$

Taking the function $q(z)=\left(\frac{1+z}{1-z}\right)^{\iota}$ $(0<\iota\le 1)$, in Theorem (3.2), the condition (3.6) becomes

$$Re\left\{\frac{\lambda}{\varrho}\left(\frac{1+z}{1-z}\right)^{\iota}+\frac{2\mu\xi}{\varrho}\left(\frac{1+z}{1-z}\right)^{2\iota}+\frac{2z^{2}}{1-z^{2}}\right\}>0\,,(\varrho\in\mathbb{C}\setminus\{0\}\ ). \tag{3.13}$$

As a result, we may deduce the following conclusion.

Corollary 3.4: Let $0<\iota\le 1, \varrho,\ \rho\in\mathbb{C}\setminus\{0\}$, $\xi, a, \lambda, \mu\in\mathbb{C}$. Assume that (3.13) holds. If $f\in\ \sum_p$ and

$$e(z)\prec a+\lambda\left(\frac{1+z}{1-z}\right)^{\iota}+\mu\xi\left(\frac{1+z}{1-z}\right)^{2\iota}+\varrho\frac{2\iota z}{1-z^{2}},$$

where $e(z)$ is defined in (3.9) , then $\left(zF_{t,\lambda,p}^{m,\vartheta}(\mathrm{f}*\mathrm{g})(z)\right)^{\rho}\prec\left(\frac{1+z}{1-z}\right)^{\iota}$, and $\left(\frac{1+z}{1-z}\right)^{\iota}$ is the best dominant.

4- Results of Differential Superordinations :

Theorem 4.1: Assume that the function $q(z)$ is a convex univalent in U with $q(0)=1,\ \rho\in\mathbb{C}\setminus\{0\},\ Re\{\tau\}>0,\ \sigma\in\mathbb{R}^{+}$, if $f\in\ \sum_p$, such that

$$\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)+2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\neq 0,\quad and$$

$$\left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)+2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\right)^{\rho}\in H[q(0),1]\cap Q, g\in\ \sum_p . \tag{4.1}$$

If the function $H(z)$ in (3.3) is univalent and the superordination criterion is fulfilled:

$$q(z)+\frac{\tau}{\rho}zq'(z)\prec\ H(z), \tag{4.2}$$

holds, then

$$q(z)\prec\left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)+2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\right)^{\rho}, \tag{4.3}$$

where the best subordinant is q(z).

Proof: Define a function $g(z)$ by

$$g(z)=\left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)+2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\right)^{\rho}. \quad (4.4)$$

Differentiating (4.4) with respect to z, we get

$$\frac{zg'(z)}{g(z)}=\rho\left[\frac{(1-\sigma)z\left(F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)\right)'+2\sigma z\left(F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)\right)'}{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)+2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}+1\right]. \quad (4.5)$$

A simple computation and using (1.8), from (4.5), we will get

$$H(z)=\left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)+2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\right)^{\rho}$$

$$+\tau\left[\left(\frac{1}{\gamma}\right)\left(\frac{2\sigma\lambda F_{t,\lambda,p}^{m+2,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)+\lambda(1-3\sigma)-2\sigma\gamma(p-1)F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)-(1-\sigma)(\lambda+\gamma p)F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{(1-\sigma)F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)+2\sigma F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}\right)\right],$$

$$=g(z)+\frac{\tau}{\rho}zg'(z).$$

Now, by using Lemma2.4, we get the desired result.

Taking $q(z)=\frac{1+Az}{1+Bz}$, $(-1\le B<A\le 1)$, we obtain the following conclusion from Theorem 4.1.

Corollary 4.1: Let $Re\{\tau\}>0, \rho\in\mathbb{C}\setminus\{0\}$, $\sigma\in\mathbb{R}^{+}$ and $(-1\le B<A\le 1)$, such that

$$\left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)+2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\right)^{\rho}\in H[q(0),1]\cap Q.$$

If $H(z)$ in (3.3) is univalent in U, and $f\in\sum_p$ fulfills the superordination condition,

$$\frac{1+Az}{1+Bz}+\frac{\tau}{\rho}\frac{(A-B)z}{(1+Bz)^2}\prec \mathrm{F}(z),$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)+2\sigma zF_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\right)^{\rho},$$

the best subordinant is the function $\frac{1+Az}{1+Bz}$.

Theorem 4.2: Let $q(z)$ be a convex univalent function in the open unit disk U with $q(0)=1$, $q'(z)\neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $\varrho,\rho\in\mathbb{C}\setminus\{0\}$, $\xi,a,\lambda,\mu\in\mathbb{C}$. Suppose that q satisfy the condition $Re\left\{\frac{q(z)}{\varrho}(2\mu\xi+\lambda)\right\}q'(z)>0$. Let $f\in\sum_p$ and satisfies the next conditions

$$\left(zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)\right)^{\rho}\in H[q(0),1]\cap Q,\qquad g\in\textstyle\sum_p \quad (4.6)$$

and

If the function $e(z)$ is given by (3.9), is univalent in U ,$zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z) \neq 0$.

$$a + \lambda q(z) + \mu\xi q^2(z) + \varrho \frac{zq'(z)}{q(z)} \prec F(z)\,, \tag{4.7}$$

implies

where the best subordinant is q(z).$q(z) \prec \left(zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)\right)^{\rho}$,

Proof : Allow $g(z)$ to be defined on U by (3.10).

After that, a calculation reveals that

$$\frac{zg'(z)}{g(z)} = \rho\left(\frac{1}{\gamma}\right)\left[\frac{F_{t,\lambda,p}^{m+1,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{F_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)} - (\lambda - \gamma(1+p))\right]. \tag{4.8}$$

By setting $\Theta(\omega) = a + \lambda\omega + \mu\xi\omega^2, and\ \phi = \frac{\varrho}{\omega}$, $\omega \neq 0$. It can be easily observed that $\Theta(\omega)$ is holomorphic in $\mathbb{C}$, $and\ \phi(\omega)$ is holomorphic in $\mathbb{C} \setminus \{0\}$, that $\phi(\omega) \neq 0$ $(\omega \in \mathbb{C} \setminus \{0\})$. Also , we get

it was discovered that $Q(z)$ is a starlike univalent in U.$Q(z) = zq'(z)\phi(q(z)) = \varrho \frac{zq'(z)}{q(z)}$,

Because $q(z)$ is convex, we may deduce that

$$Re\left(\frac{z\Theta'(q(z))}{\phi(q(z))}\right) = Re\left\{\frac{q(z)}{\varrho}(2\mu\xi q(z) + \lambda\right\}q'(z) > 0\,.$$

By making use (4.8) the hypothesis (4.7) can by equivalently

$$\Theta(q(z)) + zq'(z)\phi(q(z)) \prec \Theta(g(z)) + zg'(z)\phi(g(z)).$$

The proof is therefore completed by utilizing the Lemma 2.3.

5- Sandwich Results:

Theorem 5.1: Let q_1 and q_2 be convex univalent functions in U with $q_1(0) = q_2(0) = 1$ and q_2 satisfies (3.1). Suppose that $Re\{\tau\} > 0$, $\tau, \rho \in \mathbb{C} \setminus \{0\}$, $\sigma \in \mathbb{R}^+$. If $f \in \sum_p$, such that

$$\left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z) + 2\sigma zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\right)^{\rho} \in H[q(0),1] \cap Q,$$

and the univalent function $H(z)$, defined by (3.3), satisfies

$$q_1(z) + \frac{\tau}{\rho}zq_1{}'(z) \prec H(z) \prec q_2(z) + \frac{\tau}{\rho}zq_2{}'(z), \tag{5.1}$$

then

$$q_1(z) \prec \left(\frac{(1-\sigma)zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z) + 2\sigma zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f}*\mathrm{g})(z)}{\sigma+1}\right)^{\rho} \prec q_2(z),$$

where q_1 *and* q_2 are the best subordinant and dominant of the pair, respectively (5.1).

We obtain the following sandwich theorem by merging Theorems 3.2 and 4.2:

Theorem 5.2: Let q_j be two univalent convex functions in U, with $q_j(0) = 1, q_j'(z) \neq 0, (j = 1,2)$. Assume that q_1 *and* q_2 satisfy the conditions (3.8) and(4.8), respectively.

If $f \in \sum_p$, and suppose that f satisfies the next condition:

$$\left(zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)\right)^{\rho} \in H[q(0),1] \cap Q,$$

and $zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z) \neq 0$, *and* $e(z)$ is univalent in U, and given by (3.9), then

$$a + \lambda q_1(z) + \mu\xi {q_1}^2(z) + \varrho \frac{z{q_1}'(z)}{q_1(z)} \prec e(z) \prec a + \lambda q_2(z) + \mu\xi {q_2}^2(z) + \varrho \frac{z{q_2}'(z)}{q_2(z)}, \qquad (5.2)$$

implies

$$q_1(z) \prec \left(zF_{t,\lambda,p}^{m,\vartheta,\gamma}(\mathrm{f} * \mathrm{g})(z)\right)^{\rho} \prec q_2(z),$$

where the best subordinant and dominant are q_1 *and* q_2, respectively.

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