On Sandwich Results of Meromorphic Univalent Functions Defined by New Hadamard Product Operator

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ABSTRACT

In the present paper, we obtain differential subordination and superordination results for meromorphic univalent functions defined by a new Hadamard product operator in a punctured open unit disk. We get a number of sandwich-type results.

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Introduction

Let $E$ denote the class of functions of the form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k,$$  \hspace{1cm} (1.1)

which are meromorphic univalent in the punctured open unit disk $U^* = \{z; z \in \mathbb{C}, 0 < |z| < 1\}$. Several authors studied meromorphic functions for another classes and conditions see[7,9,19].

Let $H$ be the linear space of all analytic functions in $U$. For a positive integer number $n$ and $\alpha \in \mathbb{C}$, we let

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For \( f \) and \( g \) analytic functions in \( H \), we say that \( f \) is subordinate to \( g \) in \( U \) and write \( f(z) < g(z) \), if there exists a Schwarz function \( \omega \), which is analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1(z \in U) \), such that \( f(z) = g(\omega(z)), (z \in U) \).

Furthermore, if the function \( g \) is univalent in \( U \), we have the following equivalence relationship (cf., e.g,[10,11,15,16]):

\[
f(z) < g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U), z \in U.
\]

**Definition1:** ([15], also see [19]) Let \( Y: \mathbb{C}^3 \times U \rightarrow \mathbb{C} \) and let \( h(z) \) be analytic in \( U \). If \( p \) and \( Y(p(z), zp'(z), z^2p''(z); z) \) are univalent in \( U \) and if \( p \) needs to satisfy the second-order differential superordination,

\[
h(z) < Y(p(z), zp'(z), z^2p''(z); z),
\]

then \( p \) is called a solution of the differential superordination (1.2). An analytic function \( q(z) \) which is called a subordinant of the solutions of differential superordination (1.2) or more simply a subordinant if \( q < p \) for all \( p \) fulfill (1.2). A univalent subordinant \( \tilde{q}(z) \) that fulfills \( q < \tilde{q} \) for all subordinants \( q \) of (1.2), is said to be the best subordinant.

**Definition2:** [15] Let \( Y: \mathbb{C}^3 \times U \rightarrow \mathbb{C} \) and let \( h \) be univalent in \( U \). If \( p \) is analytic in \( U \) and satisfies the second-order differential subordination

\[
Y(p(z), zp'(z), z^2p''(z); z) < h(z),
\]

then \( p \) is called a solution of the differential subordination (1.3). The univalent function \( q \) is called a dominant of the solution of the differential subordination (1.3), or more simply a dominant if \( p < q \) for all \( p \) satisfying (1.3). A univalent dominant \( \tilde{q}(z) \) that satisfies \( \tilde{q} < q \) for all dominant \( q \) of (1.3) is said to be the best dominant.

Miller and Mocanu[16] and other authors [1,2,3,4,5,6,7,8,9,10,12] and also [13,14,17,18,19,22,23] discovered sufficient conditions for the functions \( h, p, \) and \( \Phi \) for which the following result:

\[
h(z) < Y(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) < p(z)(z \in U).
\]

If \( f_1 \in E \) is given by (1.1) and \( f_2 \in E \) given by

\[
f_2(z) = z^{-1} + \sum_{k=1}^{\infty} b_kz^k.
\]

The Hadamard product (or convolution) of \( f_1 \) and \( f_2 \) is given by

\[
(f_1 \ast f_2)(z) = z^{-1} + \sum_{k=1}^{\infty} a_kb_kz^k = (f_2 \ast f_1)(z).
\]

Using the results (see [1,2,4,5,6,7,13,14,17,18,20,21,22,23]) to obtain adequate criteria for the satisfaction of normalized analytic functions

\[
q_1(z) \leq \frac{zf'(z)}{f(z)} < q_2(z),
\]

where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = q_2(0) = 1 \).

Shanmugam et al. [20][21], as well as Goyal et al. [12], sandwich results for analytic function classes were recently obtained. (See also [1,3,4,5,11]).
The new integral operator was introduced and investigated by Atshan et al. [7],

\[ R^n(\beta, \alpha, \gamma, v, r) : E \to E, \]

which is defined as follows:

\[ R^n f(z) = z^{-1} + \sum_{k=1}^{\infty} \left( \frac{\delta + v - 1}{\delta + v - 1 + (k + 1)(\alpha + \gamma + r)} \right)^n a_k z^k, \quad (1.5) \]

where \((\delta > 1, \alpha > 1, \gamma > 0, v > 0, 0 < r < 1, z \in U)\).

The Hurwitz–Lerch Zeta function

\[ \phi(z, s, y) = \sum_{k=0}^{\infty} \frac{z^k}{(k + y)^s}, \quad (y \in \mathbb{C}\setminus\mathbb{Z}^\infty; s \in \mathbb{C}, \text{when } |z| < 1; R(s) > 1, \text{when } |z| = 1). \]

We define the new Hadamard product operator

\[ D^n_{a, b, c} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{(k + 2)^c} \left[ \frac{a}{a + b(k + 1)} \right]^{m} a_k z^k, \quad (1.6) \]

where \(a = \delta + v - 1, b = \alpha + \gamma + r\) and \(c \in \mathbb{C}\).

we note that from (1.6), we have

\[ z \left( D^n_{a, b, c} f(z) \right) = \left( \frac{a}{b} \right) D^{n-1}_{a, b, c} f(z) - \frac{(a + b)}{b} D^n_{a, b, c} f(z) \quad (1.7) \]

This concept’s major aim is to discover suitable conditions for specific normalized analytic functions \(f\) to satisfy:

\[ q_1(z) \prec \left( \frac{1 - \sigma) z D^{m-1}_{a, b, c} f(z) + 2z \alpha D^m_{a, b, c} f(z)}{\sigma + 1} \right) \prec q_2(z), \]

where \(\rho \in \mathbb{C} \setminus \{0\}, \sigma \in \mathbb{R}^+\) and \(z \in U\),

and

\[ q_1(z) \prec \left( z D^m_{a, b, c} f(z) \right) \prec q_2(z), \]

whenever univalent functions \(q_1(z)\) and \(q_2(z)\) are given in \(U\) with \(q_1(0) = q_2(0) = 1\).

2-Preliminaries

The definitions and lemmas given below will assist us in proving our basic results.

**Definition 2.1**[15]: The set of all analytic and injective functions on \(U \setminus \mathbb{E}(f)\), where \(U = \overline{U} \cup \{z \in \partial U\}\), is denoted by \(Q\), and

\[ E(f) = \{\omega \in \partial U : f(\omega) = \infty\}, \quad (2.1) \]

and are such that \(f'(\omega) \neq 0\) for \(\omega \in \partial U \setminus E(f)\). Furthermore, let \(Q(a), Q(0) = Q_0\) and \(Q(1) = Q_1\), be the subclass of \(Q\) for which \(f(0) = a\).
Lemma 2.1: [16] Let \( q(z) \) be convex univalent function in \( U \), let \( \alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \{0\} \) and suppose that
\[
\Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{a}{\beta} \right) \right\}.
\]
If \( p(z) \) is analytic in \( U \) and
\[
ap(z) + \beta zp'(z) < ap(z) + \beta zq'(z), \text{ then } p(z) < q(z) \text{ and } q \text{ is the best dominant.}
\]

Lemma 2.2: [11] Let \( q \) be univalent in \( U \) and let \( \phi \) and \( \theta \) be analytic in the domain \( D \) containing \( q(U) \) with \( \phi(\omega) \neq 0 \), when \( \omega \in q(U) \). Set \( Q(z) = q'(z)\phi(q(z)) \) and \( h(z) = \theta(q(z)) + Q(z) \), suppose that

1) \( Q \) is starlike univalent in \( U \),
2) \( \Re \left( \frac{\theta(q(z))}{\phi(q(z))} \right) > 0, z \in U \).

If \( p \) is analytic in \( U \) with \( p(0) = q(0), p(U) \subseteq D \) and
\[
\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)),
\]
then \( q < p \), and \( q \) is the best subordinant.

Lemma 2.3: [16] Let \( q(z) \) be convex univalent in the unit disk \( U \) and let \( \theta \) and \( \phi \) be analytic in a domain \( D \) containing \( q(U) \). Suppose that

1) \( \Re \left( \frac{\theta(q(z))}{\phi(q(z))} \right) > 0 \text{ for } z \in U \),
2) \( zq'(z)\phi(q(z)) \) is starlike univalent in \( z \in U \).

If \( p \in H[q(0), 1] \cap Q \), with \( p(U) \subseteq D \), and \( \theta(p(z)) + zp'(z)\phi(p(z)) \) is univalent in \( U \), and
\[
\theta(q(z)) + zq'(z)\phi(q(z)) < \theta(p(z)) + zp'(z)\phi(p(z)),
\]
then \( q < p \), and \( q \) is the best subordinant.

Lemma 2.4: [16] Let \( q(z) \) be convex univalent in \( U \) and \( q(0) = 1 \). Let \( \beta \in \mathbb{C} \), that \( \Re(\beta) > 0 \). If \( p(z) \in H[q(0), 1] \cap Q \) and \( p(z) + \beta zp'(z) \) is univalent in \( U \), then
\[
q(z) + \beta zq'(z) < p(z) + \beta zp'(z), \text{ which implies that } q(z) < p(z) \text{ and } q(z) \text{ is the best subordinant.}
\]

3- Results of Differential Subordinations

Now, we discuss some differential subordination results using a new Hadamard product operator \( D_{a,b,c}^m \).

Theorem 3.1 : Let \( q(z) \) be a convex univalent function in the open unit disk \( U \), with \( q(0) = 1 \), and \( q'(z) \neq 0 \), for all \( z \in U \). Let \( \tau, \rho \in \mathbb{C} \setminus \{0\}, \sigma \in \mathbb{R}^+ \). Suppose that
\[
\Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{a}{\rho} \right) \right\},
\]
If \( f \in E \) is satisfies the subordination condition:
\[
H(z) < q(z) + \frac{\tau}{\rho} zq'(z),
\]
where
\[
H(z) = \left( \frac{1-\sigma}{\beta} D_{a,b,c}^{m-1} f(z) + \frac{2\sigma}{\beta} D_{a,b,c}^m f(z) \right)^{\frac{1}{\sigma+1}} + \frac{\tau}{\rho} \left[ \left( \frac{a}{\beta} \right)^{\frac{1}{\sigma}} D_{a,b,c}^{m-1} f(z) + \left( \frac{1-\sigma}{\beta} D_{a,b,c}^{m-1} f(z) + 2\sigma D_{a,b,c}^m f(z) \right) \right].
\]
then
\[
\left(\frac{(1 - \sigma)zD^{m-1}_{a,b,c} f(z) + 2\sigma zD^m_{a,b,c} f(z)}{\sigma + 1}\right)^\rho < q(z),
\]
(3.4)

and \(q(z)\) is the best dominant.

**Proof**: Define the \(g(z)\) function as follows:
\[
g(z) = \left(\frac{(1 - \sigma)zD^{m-1}_{a,b,c} f(z) + 2\sigma zD^m_{a,b,c} f(z)}{\sigma + 1}\right)^\rho,
\]
(3.5)

then the function \(g(z)\) is analytic in \(U\) and \(g(0) = 1\) as a result of differentiating (3.5) with respect to \(z\) and then using the identity (1.7) in the resultant equation.

Thus the subordination (3.2) is equivalent to
\[
g(z) + \frac{z}{\rho}g'(z) < q(z) + \frac{z}{\rho}q'(z).
\]

An application of Lemma(2.1) with \(\beta = \frac{\tau}{\rho}, \alpha = 1\), we obtain (3.4).

**Corollary 3.1**: Let \(\tau, \rho \in \mathbb{C} \setminus \{0\}, \sigma \in \mathbb{R}^+\) and \((-1 \leq B < A \leq 1)\). Suppose that
\[
\text{Re}\left(\frac{1 - Bz}{1 + Bz}\right) > \max\left\{0, -\text{Re}\left(\frac{\rho}{\tau}\right)\right\}.
\]

If \(f \in E\) is satisfy the following subordination condition:
\[
H(z) < \frac{1 + Az}{1 + Bz} + \frac{\tau (A - B)z}{\rho (1 + Bz)^2}
\]
when \(H(z)\) given by (3.3), then
\[
\left(\frac{(1 - \sigma)zD^{m-1}_{a,b,c} f(z) + 2\sigma zD^m_{a,b,c} f(z)}{\sigma + 1}\right)^\rho < \frac{1 + Az}{1 + Bz}
\]

where the best dominating is \(\frac{1 + Az}{1 + Bz}\).

In Corollary(3.1), we can get following result with \(A = 1\) and \(B = -1\).

**Corollary 3.2**: Let \(\tau, \rho \in \mathbb{C} \setminus \{0\}, \sigma \in \mathbb{R}^+\) and suppose that
\[
\text{Re}\left(\frac{1 + z}{1 - z}\right) > \max\left\{0, -\text{Re}\left(\frac{\rho}{\tau}\right)\right\}.
\]

If \(f \in E\) fulfill the following subordination necessity:
\[ H(z) < \frac{1 + z}{1 - z} + \frac{\tau}{\rho} \frac{2z}{(1 - z)^2} \]

when \( H(z) \) given by (3.3), then

\[
\left( \frac{1 - \sigma z D_{\alpha,\beta,\gamma}^{m-1} f(z) + 2\sigma z D_{\alpha,\beta,\gamma}^m f(z)}{\sigma + 1} \right)^\rho < \frac{1 + z}{1 - z}.
\]

**Theorem 3.2:** In unit disk \( U \), let \( q(z) \) be convex univalent function in the open unit disk \( U \) with \( q(0) = 1, q'(z) \neq 0 \) and \( \frac{q(z)}{q'(z)} \) is starlike univalent in \( U \). Let \( \eta, \tau, \rho \in \mathbb{C} \setminus \{0\}, \zeta, n, \lambda, \mu \in \mathbb{C}, f \in \mathbb{E} \), and suppose that \( q \) satisfy the following conditions

\[
Re \left( \frac{\lambda}{\eta} q(z) + \frac{2\eta q}{\zeta} q^2(z) + 1 + z \frac{q''(z)}{q'(z)} - z \frac{q'(z)}{q(z)} \right) > 0,
\]

and if \( f \in \mathbb{E} \) satisfies:

\[
z D_{\alpha,\beta,\gamma}^m f(z) \neq 0.
\]

If

\[
e(z) < n + \lambda q(z) + \eta q^2(z) + \nu z q'(z) q(z),
\]

where

\[
e(z) = \left( z D_{\alpha,\beta,\gamma}^m f(z) \right)^\rho \left[ \lambda + \eta q \left( z D_{\alpha,\beta,\gamma}^m f(z) \right)^2 + \nu q \left( \frac{a}{b} D_{\alpha,\beta,\gamma}^{m-1} f(z) - 1 \right) \right]^ho
\]

then \( \left( z D_{\alpha,\beta,\gamma}^{m+1} f(z) \right)^\rho < q(z) \), where the best dominating is \( q(z) \).

**Proof:** As follows, define the analytic function \( g(z) \):

\[
g(z) = \left( z D_{\alpha,\beta,\gamma}^m f(z) \right)^\rho,
\]

then the function \( g(z) \) is analytic in \( U \) and \( g(0) = 1 \). By differentiating (3.10) with respect to \( z \), and using identity (1.7) in the resulting equation, we get

\[
\frac{z g'(z)}{g(z)} = \rho \left( \frac{q}{b} \right) \left[ D_{\alpha,\beta,\gamma}^{m-1} f(z) - 1 \right].
\]

Setting \( \theta(\omega) = n + \lambda \omega + \eta q \omega^2 \) and \( \phi(\omega) = \zeta \omega, \omega \neq 0 \) reveals the \( \theta(\omega) \) is analytic function in \( \mathbb{C} \), and \( \phi(\omega) \) is analytic in \( \mathbb{C} \setminus \{0\} \) and \( \phi(\omega) \neq 0, \omega \in \mathbb{C} \setminus \{0\} \).

If, we let

\[ Q(z) = z q'(z) q(z) = \frac{z q'(z)}{q(z)} \text{ and } h(z) = \theta(q(z)) + Q(z) = n + \lambda q'(z) + \nu q^2(z) + \frac{z q'(z)}{q(z)}, \]

we find that \( Q(z) \) is starlike univalent in \( U \), we have

\[
h'(z) = \lambda q'(z) + 2\mu \xi q(z) q'(z) + \frac{q'(z)}{q(z)} + q z \frac{q''(z)}{q'(z)} - q z \left( \frac{q'(z)}{q(z)} \right)^2,
\]
and

\[ \frac{zh'(z)}{Q(z)} = \frac{\lambda}{\xi} q(z) + 2u\xi q^2(z) + 1 + z \frac{q''(z)}{q'(z)} - z \frac{q'(z)}{q(z)}, \]

hence that

\[ \text{Re} \left( \frac{zh'(z)}{Q(z)} \right) = \text{Re} \left( \frac{\lambda}{\xi} q(z) + 2u\xi q^2(z) + 1 + z \frac{q''(z)}{q'(z)} - z \frac{q'(z)}{q(z)} \right) > 0. \]

By using (3.11), we obtain

\[ \lambda g(z) + u\eta g^2(z) + \epsilon \frac{z g'(z)}{g(z)} = \left(zD_{\infty, r; z}^\eta f(z)\right)^\rho \left[ \lambda + \mu \epsilon \left( zD_{a,b,c}^n f(z) \right)^2 \right] + \epsilon \rho \left( a \right), \]

By using (3.8), we have

\[ \lambda g(z) + \mu \epsilon g^2(z) + \epsilon \frac{z g'(z)}{g(z)} \lambda q(z) + \mu \epsilon q^2(z) + \epsilon \frac{z q'(z)}{q(z)} , \]

We can infer that subordination(3.8) implies that \( g(z) < q(z) \), and that the function \( q(z) \) is the best domain by using Lemma 2.2.

Taking the function \( q(z) = \frac{1 + Az}{1 + Bz} \) \((-1 \leq B < A \leq 1)\, , in \, Theorem\, 3.2, \, the \, condition \,(3.6) \, becomes

\[ \text{Re} \left( \frac{\lambda}{\xi} \frac{1 + Az}{1 + Bz} \right)^\rho \left[ \lambda + \mu \epsilon \left( \frac{1 + Az}{1 + Bz} \right)^2 \right] + \epsilon \rho \left( \frac{A - B)z}{1 + Bz} \right) > 0 \, (\xi \in \mathbb{C} \setminus \{0\} ), \]

(3.12)
as a result, we may deduce the following conclusion.

**Corollary 3.3:** Let \((-1 \leq B < A \leq 1)\, , \xi, \rho, \alpha, \beta \in \mathbb{C} \setminus \{0\} \, , \xi, \alpha, \lambda, \mu \in \mathbb{C} \, assume \, that \,(3.12) \, holds \, . \, If \, f \in E \, and

\[ e(z) < a + \lambda \left( \frac{1 + Az}{1 + Bz} \right) + \mu \epsilon \left( \frac{1 + Az}{1 + Bz} \right)^2 + \epsilon \rho \left( \frac{(A - B)z}{1 + Bz} \right) , \]

where \( e(z) \) is defined in (3.9), then

\[ \left( zD_{a,b,c}^n f(z) \right)^\rho < \frac{1 + Az}{1 + Bz} , \, and \, \frac{1 + Az}{1 + Bz} \, is \, the \, best \, dominant. \]

Taking the function \( q(z) = \left( \frac{1 + z}{1 - z} \right)^i \) \((0 < i \leq 1)\, , in \, Theorem\,(3.2), \, the \, condition \,(3.6) \, becomes

\[ \text{Re} \left( \frac{\lambda}{\xi} \frac{1 + z}{1 - z} \right)^i + \frac{2u\xi}{\xi} \left( \frac{1 + z}{1 - z} \right)^2i + \frac{2z^2}{1 - z^2} > 0 , \, (\nu \in \mathbb{C} \setminus \{0\} ). \]

(3.13)

As a result, we may deduce the following conclusion.

**Corollary 3.4:** Let \( 0 < i \leq 1, \xi, \rho, \alpha, \beta \in \mathbb{C} \setminus \{0\} \, , \xi, \eta, \nu, \lambda, \mu \in \mathbb{C} \, Assume \, that \,(3.13) \, holds \, . \, If \, f \in E \, and

\[ e(z) < \nu + \lambda \left( \frac{1 + z}{1 - z} \right)^i + \mu \epsilon \left( \frac{1 + z}{1 - z} \right)^2i + \epsilon \rho \left( \frac{2iz}{1 - z^2} \right) , \]

where \( e(z) \) is defined in (3.9), then \( \left( zD_{a,b,c}^m f(z) \right)^\rho < \left( \frac{1 + z}{1 - z} \right)^i, \) and \( \left( \frac{1 + z}{1 - z} \right)^i \) is the best dominant.

4- Results of Differential Superordinations:
**Theorem 4.1:** Assume that the function $q(z)$ is a convex univalent in $U$ with $q(0) = 1, \rho \in \mathbb{C} \setminus \{0\}, \text{Re}(\tau) > 0$, $\sigma \in \mathbb{R}^+$, if $f \in E$, such that
\[
\frac{(1 - \sigma)zD_{a,b,c}^{m-1} f(z) + 2\sigma zD_{a,b,c}^m f(z)}{\sigma + 1} \neq 0, \quad \text{and}
\]
\[
\left(\frac{(1 - \sigma)zD_{a,b,c}^{m-1} f(z) + 2\sigma zD_{a,b,c}^m f(z)}{\sigma + 1}\right)^\rho \in H[q(0), 1] \cap Q. \tag{4.1}
\]

If the function $H(z)$ in (3.3) is univalent and the superordination criterion is fulfilled:
\[
q(z) + \frac{\tau}{\rho} q'(z) < H(z), \tag{4.2}
\]
holds, then
\[
q(z) \prec \left(\frac{(1 - \sigma)zD_{a,b,c}^{m-1} f(z) + 2\sigma zD_{a,b,c}^m f(z)}{\sigma + 1}\right)^\rho, \tag{4.3}
\]
where the best subordinant is $q(z)$.

**Proof:** Define a function $g(z)$ by
\[
g(z) = \left(\frac{(1 - \sigma)zD_{a,b,c}^{m-1} f(z) + 2\sigma zD_{a,b,c}^m f(z)}{\sigma + 1}\right)^\rho. \tag{4.4}
\]
Differentiating (4.4) with respect to $z$, we get
\[
\frac{z g'(z)}{g(z)} = \rho \left[\frac{(1 - \sigma)z\left(D_{a,b,c}^{m-1} f(z)\right)' + 2\sigma z\left(D_{a,b,c}^m f(z)\right)'}{(1 - \sigma)D_{a,b,c}^{m-1} f(z) + 2\sigma D_{a,b,c}^m f(z)} + 1\right]. \tag{4.5}
\]
A simple computation and using (1.7), from (4.5), we will get
\[
H(z) = \left(\frac{(1 - \sigma)zD_{a,b,c}^{m-1} f(z) + 2\sigma zD_{a,b,c}^m f(z)}{\sigma + 1}\right)^\rho + \tau \left[\left(\frac{a}{b}\right)^{1 - \sigma - 1} \left(D_{a,b,c}^{m-1} f(z) + (3\sigma - 1)D_{a,b,c}^{m-1} f(z) - \sigma D_{a,b,c}^m f(z)\right)\right] = g(z) + \frac{\tau}{\rho} z g'(z).
\]
Now, by using Lemma 2.4, we get the desired result.

Taking $q(z) = \frac{1 + Az}{1 + Bz}, (-1 \leq B < A \leq 1)$, we obtain the following conclusion from Theorem 4.1.

**Corollary 4.1:** Let $\text{Re}(\tau) > 0, \rho \in \mathbb{C} \setminus \{0\}, \sigma \in \mathbb{R}^+$ and $(-1 \leq B < A \leq 1)$, such that
\[
\left(\frac{(1 - \sigma)zD_{a,b,c}^{m-1} f(z) + 2\sigma zD_{a,b,c}^m f(z)}{\sigma + 1}\right)^\rho \in H[q(0), 1] \cap Q.
\]
If $H(z)$ in (3.3) is univalent in $U$, and $f \in E$ fulfills the superordination condition,
\[
\frac{1 + Az}{1 + Bz} + \frac{\tau (A - B)z}{\rho (1 + Bz)^2} < F(z),
\]
then
the best subordinant is the function $\frac{1 + Aq}{1 + Bz}$.

**Theorem 4.2:** Let $q(z)$ be a convex univalent function in the open unit disk $U$ with $q(z) = 1$, $q'(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. Let $q, p \in \mathbb{C} \setminus \{0\}$, $\xi, \alpha, \lambda, \mu \in \mathbb{C}$. Suppose that $q$ satisfy the condition $\Re\left\{\frac{q(z)}{q(z)}\right\}(2\mu \xi + \lambda)q'(z) > 0$. Let $f \in E$ satisfies the next conditions:

$$\left(zD_{a,b,c}^m f(z)\right)^\theta \in H[q(0), 1] \cap Q, \tag{4.6}$$

and

$$zD_{a,b,c}^m f(z) \neq 0. \text{ If the function } e(z) \text{ is given by (3.9), is univalent in } U,$n$$

$$a + \lambda q(z) + \mu \xi q^2(z) + q \frac{zq'(z)}{q(z)} < e(z), \tag{4.7}$$

implies

$$q(z) < \left(zD_{a,b,c}^m f(z)\right)^\theta,$$

where the best subordinant is $q(z)$.

**Proof:** Allow $g(z)$ to be defined on $U$ by (3.10).

After that, a calculation reveals that

$$\frac{zg'(z)}{g(z)} = \rho \left(\frac{q}{\phi}\right) \left[D_{a,b,c}^{m-1} f(z)\right] \left[D_{a,b,c}^m f(z)\right] - 1. \tag{4.8}$$

By setting $\theta(\omega) = a + \lambda \omega + \mu \xi \omega^2$, and $\phi = \frac{q}{\phi}$, $\omega \neq 0$. It can be easily observed that $\theta(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C} \setminus \{0\}$, that $\phi(\omega) \neq 0$ ($\omega \in \mathbb{C} \setminus \{0\}$). Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = q \frac{zq'(z)}{q(z)}.$$

It was discovered that $Q(z)$ is a starlike univalent in $U$.

Because $q(z)$ is convex, we may deduce that

$$\Re\left(\frac{z\theta'(q(z))}{\phi(q(z))}\right) = \Re\left(\frac{q(z)}{q(z)}(2\mu \xi q(z) + \lambda)q'(z) > 0.\right.$$

By making use (4.8) the hypothesis (4.7) can by equivalently

$$\theta(q(z)) + zq'(z)\phi(q(z)) < \theta(g(z)) + zg'(z)\phi(g(z)).$$

The proof is therefore completed by utilizing the Lemma 2.3.

**5- Sandwich Results:**

By combining Theorems 3.1 and 4.1, we have the following sandwich Theorem:

**Theorem 5.1:** Let $q_1$ and $q_2$ be convex univalent functions in $U$ with $q_1(0) = q_2(0) = 1$ and $q_2$ satisfies (3.1). Suppose that $\Re(\tau) > 0$, $\tau, \rho \in \mathbb{C} \setminus \{0\}$, $\sigma \in \mathbb{R}^*$. If $f \in E$, such that
\[
\left( \frac{(1 - \sigma)D_{a,b,c}^{m-1}f(z) + 2\sigma D_{a,b,c}^mf(z)}{\sigma + 1} \right) ^\rho \in H[q(0), 1 ] \cap Q,
\]

and the univalent function \( H(z) \), defined by (3.3), satisfies

\[
q_1(z) + \frac{\tau}{\rho} z q_1'(z) < H(z) < q_2(z) + \frac{\tau}{\rho} z q_2'(z), \tag{5.1}
\]

then

\[
q_1(z) < \left( \frac{(1 - \sigma)D_{a,b,c}^{m-1}f(z) + 2\sigma D_{a,b,c}^mf(z)}{\sigma + 1} \right) ^\rho < q_2(z),
\]

where \( q_1 \) and \( q_2 \) are the best subordinant and dominant, respectively (5.1).

We obtain the following sandwich theorem by merging Theorems 3.2 and 4.2:

**Theorem 5.2:** Let \( q_j \) and be two univalent convex functions in \( U \), in condition for \( q_j(0) = 1, q'_j(z) \neq 0, (j = 1,2) \).
Assume that \( q_1 \) and \( q_2 \) satisfy the conditions (3.8) and (4.8), respectively.

If \( f \in E \), and suppose that \( f \) satisfies the next condition

\[
\left( zD_{a,b,c}^mf(z) \right) ^\rho \in H[q(0), 1 ] \cap Q,
\]

and \( zD_{a,b,c}^mf(z) \neq 0 \), and \( e(z) \) is univalent in \( U \), and given by (3.9), then

\[
a + \lambda q_1(z) + \mu z q_1'(z) + \frac{qzq_1'(z)}{q_1(z)} < e(z) < a + \lambda q_2(z) + \mu z q_2'(z) + \frac{qzq_2'(z)}{q_2(z)}, \tag{5.2}
\]

Implies

\[
q_1(z) < \left( zD_{a,b,c}^mf(z) \right) ^\rho < q_2(z),
\]

where the best subordinant and dominant are \( q_1 \) and \( q_2 \), respectively.

**References**


