Journal of Al-Qadisiyah for Computer Science and Mathematics Vol. 15(1) 2023 , pp Math. 162–173

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

On Sandwich Results of Meromorphic Univalent Functions Defined by New Hadamard Product Operator

***Waggas Galib Atshan**[a], Youssef Wali Abbas*[b]

[a]Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah- Iraq..Email: waggas.galib@qu.edu.iq

[b]Department of Mathematics, College of Computer Science and Mathematics, University of Mosul , Ninawaa, Iraq .Email: yousif.21csp31@student.uomosul.edu.iq

ARTICLE INFO

Article history:
Received: 05/02/2023
Rrevised form: 13/13/2023
Accepted : 16/03/2023
Available online: 31/03/2023

Keywords:

superordination,
subordination,
convolution,
sandwich theorems.

ABSTRACT

In the present paper, we obtain differential subordination and superordination results for meromorphic univalent functions defined by a new Hadamard product operator in a punctured open unit disk. We get a number of sandwich-type results.

MSC: 30C45

https://doi.org/ 10.29304/jqcm.2023.15.1.1180

Introduction

Let E denote the class of functions of the form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are meromorphic univalent in the punctured open unit disk $U^* = \{z: z \in \mathbb{C}, 0 < |z| < 1\}$. Several authors studied meromorphic functions for another classes and conditions see[7,9,19].

Let H be the linear space of all analytic functions in U. For a positive integer number n and $a \in \mathbb{C}$, we let

*Corresponding author

Email addresses:

Communicated by 'sub etitor'

$$H[a,n] = \{f \in H: f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

For f and g analytic functions in H, we say that f is subordinate to g *in* U and write $f(z) \prec g(z)$,if there exists a Schwarz function ω, which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U)$, such that $f(z) = g(\omega(z)), (z \in U)$.

Furthermore, if the function g is univalent in U, we have the following equivalence relationship (cf., e.g.[10,11,15,16]):

$$f(z) \prec g(z) \leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U), z \in U.$$

Definition1: ([15], *also see* [19]) Let $\Upsilon: \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h(z)$ be analytic in U. If p *and* $\Upsilon(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and If p needs to satisfy the second-order differential superordination,

$$h(z) \prec \Upsilon(p(z), zp'(z), z^2p''(z); z), \tag{1.2}$$

then p is called a solution of the differential superordination (1.2). An analytic function $q(z)$ which is called a subordinant of the solutions of differential superordination (1.2) or more simply a subordinant if $q \prec p$ for all p fulfill (1.2). A univalent subordinant $\hat{q}(z)$ that fulfills $q \prec \hat{q}$ for all subordinants q of (1.2). is said to be the best subordinant.

Definition2: [15] Let $\Upsilon: \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the second-order differential subordination

$$\Upsilon(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.3}$$

then p is called a solution of the differential subordination (1.3). The univalent function q is called a dominant of the solution of the differential subordination (1.3), or more simply dominant if $p \prec q$ for all p satisfying(1.3). A univalent dominant $\hat{q}(z)$ that satisfies $\hat{q} \prec q$ for all dominant q of (1.3) is said to be the best dominant.

Miller and Mocanu[16] and other authers [1,2,3,4,5,6,7,8,9,10,12] and also [13,14,17,18,19,22,23] discovered sufficient conditions for the functions h, p, and $\boldsymbol{\phi}$ for which the following result:

$$h(z) \prec \Upsilon(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z) (z \in U). \tag{1.4}$$

If $f_1 \in E$ is given by (1.1) and $f_2 \in E$ given by

$$f_2(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k.$$

The Hadamard product (or convolution)of f_1 *and* f_2 is given by

$$(f_1 * f_2)(z) = z^{-1} + \sum_{k=1}^{\infty} a_k b_k z^k = (f_2 * f_1)(z).$$

Using the results,(see [1,2,4,5,6,7,13,14,17,18,20,21,22,23]) to obtain adequate criteria for the satisfaction of normalized analytic functions

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 *and* q_2 are given univalent functions in$U$ with $q_1(0) = q_2(0) = 1$.

Shanmugam et al. [20][21], as well as Goyal et al. [12], sandwich results for analytic function classes were recently obtained. (See also [1,3,4,5,11]).

The new integral operator was introduced and investigated by Atshan et al. [7],

$$R^n(\beta,\alpha,\gamma,\upsilon,r): E \to E,$$

which is defined as follows:

$$R^n f(z) = z^{-1} + \sum_{k=1}^{\infty} \left(\frac{\delta+\upsilon-1}{\delta+\upsilon-1+(k+1)(\alpha+\gamma+r)}\right)^n a_k z^k, \tag{1.5}$$

where ($\delta > 1, \alpha > 1, \gamma > 0, \upsilon > 0, 0 < r < 1, z \in U$).

The Hurwtiz – Lerch Zeta function

$$\phi(z,s,y) = \sum_{k=0}^{\infty} \frac{z^k}{(k+y)^s}, \quad (y \in \mathbb{C}\backslash z_0^-\,; s \in \mathbb{C}, when\ |z| < 1; R(s) > 1, when\ |z| = 1).$$

We define the new Hadamard product operator

$$D^m_{a,b,c} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{(k+2)^c} * \left[\frac{a}{a+b(k+1)}\right]^m a_k z^k, \tag{1.6}$$

where $a = \delta + \upsilon - 1, b = \alpha + \gamma + r$ and $c \in \mathbb{C}$.

we note that from (1.6) , we have

$$z\left(D^m_{a,b,c} f(z)\right)' = \left(\frac{a}{b}\right) D^{m-1}_{a,b,c} f(z) - \frac{(a+b)}{b} D^m_{a,b,c} f(z) \tag{1.7}$$

This concept's major aim is to discover suitable conditions for specific normalized analytic functions f to satisfy:

$$q_1(z) \prec \left(\frac{(1-\sigma) z D^{m-1}_{a,b,c} f(z) + 2\sigma z D^m_{a,b,c} f(z)}{\sigma+1}\right)^\rho \prec q_2(z),$$

where $\rho \in \mathbb{C} \setminus \{0\}, \sigma \in \mathbb{R}^+$ and $z \in U$,

and

$$q_1(z) \prec \left(z D^m_{a,b,c} f(z)\right)^\rho \prec q_2(z),$$

whenever univalent functions $q_1\ (z)\ and\ q_2\ (z)$ are given in U with $q_1(0) = q_2(0) = 1$.

2-Preliminaries

The definitions and lemmas given below will assist us in proving our basic results.

Definition2.1[15]: The set of all analytic and injective functions on $\overline{U} \setminus E(f)$, where $\overline{U} = U \cup \{z \in \partial U\}$, is denoted by Q, and

$$E(f) = \{\omega \in \partial U : f(z) = \infty\}, \tag{2.1}$$

and are such that $f'(\omega) \neq 0$ for $\omega \in \partial U \setminus E(f)$. Furthermore, let $Q(a), Q(0) \equiv Q_0\ and\ Q(1) \equiv Q_1$, be the subclass of Q for which $f(0) = a$.

Lemma2.1: [16] Let $q(z)$ be convex univalent function in U, let $\alpha \in \mathbb{C},\ \beta \in \mathbb{C}\setminus\{0\}$ and suppose that

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right) > max\left\{0, -Re\left(\frac{\alpha}{\beta}\right)\right\}.$$

If $p(z)$ is analytic in U and

$\alpha p(z)+\beta zp'(z) \prec \alpha q(z)+\beta zq'(z)$, then $p(z) \prec q(z)$ and q is the best dominant.

Lemma2.2: [11]Let q be univalent in U and let ϕ *and* θ be analytic in the domain D containing $q(U)$ with $\phi(\omega)\neq 0$, when $\omega \in q(U)$. Set $Q(z)=zq'(z)\phi\big(q(z)\big)$ *and* $h(z)=\theta\big(q(z)\big)+Q(z)$,

suppose that

1) $Q$is starlike univalent in$U$,

2) $Re\left(\frac{zh\prime(z)}{Q(z)}\right)>0, z\in U$.
If p is analytic in U with $p(0)=q(0), p(U)\subseteq D$ and

$\theta\big(p(z)\big)+zp'(z)\phi\big(p(z)\big) \prec \theta\big(q(z)\big)+zq'(z)\phi\big(q(z)\big)$, then $p\prec q$, and q is the best dominant.

Lemma 2.3: [16] Let $q(z)$ be convex univalent in the unit disk U and let θ *and* ϕ be analytic in a domain D containing $q(U)$. Suppose that

1) $Re\left\{\frac{\theta\prime(q(z))}{\phi(q(z))}\right\}>0\ \ for\ z\in U$,
2) $zq'(z)\phi(q(z))$ is starlike univalent in $z\in U$.
If $p\in H[q(0),1]\cap Q$, with $p(U)\subseteq D$, and $\theta(p(z)+zp'(z)\phi(p(z))$ is univalent in U, and

$$\theta\big(q(z)\big)+zq'(z)\phi\big(q(z)\big) \prec \theta\big(p(z)\big)+zp'(z)\phi\big(p(z)\big), \tag{2.2}$$

then $q\prec p$, and q is the best subordinant.

Lemma2.4: [16] Let $q(z)$ be convex univalent in U *and* $q(0)=1$. Let $\beta\in\mathbb{C}$, that $Re\{\beta\}>0$. If $p(z)\in H[q(0),1]\cap Q$ and $p(z)+\beta zp'(z)$is univalent in U, then

$q(z)+\beta zq'(z) \prec p(z)+\beta zp'(z)$, which implies that $q(z)\prec p(z)$ *and* $q(z)$ is the best subordinant.

3- Results of Differential Subordinations

Now, we discuss some differential subordination results using a new Hadamard product operator $D^{m}_{a,b,c}$.

Theorem3.1 : Let $q(z)$ be a convex univalent function in the open unit disk U, with $q(0)=1$, and $q'(z)\neq 0$, for all $z\in U$. Let $\tau,\rho\in\mathbb{C}\setminus\{0\}, \sigma\in\mathbb{R}^{+}$. Suppose that

$$Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re\left(\frac{\rho}{\tau}\right)\right\}. \tag{3.1}$$

If $f\in E$ is satisfies the subordination condition:

$$H(z) \prec q(z)+\frac{\tau}{\rho}zq'(z), \tag{3.2}$$

where $H(z)=\left(\frac{(1-\sigma)zD^{m-1}_{a,b,c}f(z)+2\sigma zD^{m}_{a,b,c}f(z)}{\sigma+1}\right)^{\rho}+$

$$\tau\left[\left(\frac{a}{b}\right)\left(\frac{(1-\sigma)D^{m-2}_{a,b,c}f(z)+(3\sigma-1)D^{m-1}_{a,b,c}f(z)-\sigma D^{m}_{a,b,c}f(z)}{(1-\sigma)D^{m-1}_{a,b,c}f(z)+2\sigma D^{m}_{a,b,c}f(z)}\right)\right], \tag{3.3}$$

then

$$\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho} \prec q(z), \quad (3.4)$$

and $q(z)$ is the best dominant.

Proof : Define the $g(z)$ function as follows:

$$g(z)=\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho}, \quad (3.5)$$

then the function $g(z)$ is analytic in U and $g(0)=1$ as a result of differentiating (3.5) with respect to z and then using the identity (1.7) in the resultant equation.

$$H(z)=\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho} + \tau\left[\left(\frac{a}{b}\right)\left(\frac{(1-\sigma)D_{a,b,c}^{m-2}f(z)+(3\sigma-1)D_{a,b,c}^{m-1}f(z)-2\sigma D_{a,b,c}^{m}f(z)}{(1-\sigma)D_{a,b,c}^{m-1}f(z)+2\sigma D_{a,b,c}^{m}f(z)}\right)\right]$$

$= g(z)+\frac{\tau}{\rho}zg'(z).$

Thus the subordination (3.2) is equivalent to

$$g(z)+\frac{\tau}{\rho}zg'(z) \prec q(z)+\frac{\tau}{\rho}zq'(z).$$

An application of Lemma(2.1) with $\beta=\frac{\tau}{\rho}$, $\alpha=1$, we obtain (3.4).

Corollary3.1 : Let $\tau,\rho \in \mathbb{C}\setminus\{0\}, \sigma \in \mathbb{R}^{+}$ and $(-1\le B<A\le 1)$. Suppose that

$$Re\left(\frac{1-Bz}{1+Bz}\right) > max\left\{0,-Re\left(\frac{\rho}{\tau}\right)\right\}.$$

If $f \in E$ is satisfy the following subordination condition:

$$H(z) \prec \frac{1+Az}{1+Bz}+\frac{\tau}{\rho}\frac{(A-B)z}{(1+Bz)^{2}},$$

when $H(z)$ given by (3.3) , then

$$\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho} \prec \frac{1+Az}{1+Bz},$$

where the best dominating is $\frac{1+Az}{1+Bz}$.

In Corollary(3.1), we can get following result with $A=1$ and $B=-1$.

Corollary3.2: Let $\tau,\rho \in \mathbb{C}\setminus\{0\}, \sigma \in \mathbb{R}^{+}$ and suppose that

$$Re\left(\frac{1+z}{1-z}\right) > max\left\{0,-Re\left(\frac{\rho}{\tau}\right)\right\}.$$

If $f \in E$ fulfill the following subordination necessity:

$$H(z) \prec \frac{1+z}{1-z} + \frac{\tau}{\rho}\frac{2z}{(1-z)^2},$$

when $H(z)$ given by (3.3), then

$$\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z) + 2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho} \prec \frac{1+z}{1-z},$$

Theorem3.2: In unit disk U, let $q(z)$ be convex univalent function in the open unit disk U with $q(0)=1, q'(z)\neq 0$ and $z\frac{q'(z)}{q(z)}$ is starlike univalent in U. Let $\text{ȩ}, \text{ƫ}, \text{ꝑ} \in \mathbb{C}\setminus\{0\}, \text{Ч}, \text{ȵ}, \lambda, \text{ư} \in \mathbb{C}, f \in E$, and suppose that q satisfy the following conditions

$$Re\left\{\frac{\lambda}{\text{ȩ}}q(z) + \frac{2\text{ư}\text{Ч}}{\text{ȩ}}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right\} > 0, \tag{3.6}$$

and if $f \in E$ satisfies:

$$zD_{a,b,c}^{m}f(z) \neq 0. \tag{3.7}$$

If

$$e(z) \prec \text{ȵ} + \lambda q(z) + \text{ư}\text{Ч}q^2(z) + \text{ȩ}\frac{zq'(z)}{q(z)}, \tag{3.8}$$

where

$$e(z) = \left(zD_{a,b,c}^{m}f(z)\right)^{\rho}\left[\lambda + \text{ư}\text{Ч}\left(zD_{a,b,c}^{m}f(z)\right)^2 + \text{ȩ}\rho\left(\frac{a}{b}\right)\left[\frac{D_{a,b,c}^{m-1}f(z)}{D_{a,b,c}^{m}f(z)} - 1\right]\right], \tag{3.9}$$

then $\left(zD_{s,r,y}^{\eta+1}f(z)\right)^{\rho} \prec q(z)$, where the best dominating is $q(z)$.

Proof : As follows, define the analytic function $g(z)$:

$$g(z) = \left(zD_{a,b,c}^{m}f(z)\right)^{\rho}, \tag{3.10}$$

then the function $g(z)$ is analytic in U and $g(0)=1$. By differentiating (3.10) with respect to z, and using identity (1.7) in the resulting equation, we get

$$\frac{zg'(z)}{g(z)} = \rho\left(\frac{a}{b}\right)\left[\frac{D_{a,b,c}^{m-1}f(z)}{D_{a,b,c}^{m}f(z)} - 1\right]. \tag{3.11}$$

Setting $\theta(\omega) = \text{ȵ} + \lambda\omega + \text{ư}\text{Ч}\omega^2$ *and* $\phi(\omega) = \frac{\text{ȩ}}{\omega}, \omega \neq 0$ reveals the $\theta(\omega)$ is analytic function in $\mathbb{C}$, *and* $\phi(\omega)$ is analytic in $\mathbb{C}\setminus\{0\}$ *and* $\phi(\omega) \neq 0$, $\omega \in \mathbb{C}\setminus\{0\}$.

If, we let

$$Q(z) = zq'(z)\phi(z) = \text{ȩ}\frac{zq'(z)}{q(z)} \quad and \quad h(z) = \theta(q(z)) + Q(z) = \text{ȵ} + \lambda q'(z) + \text{ư}\text{Ч}q^2(z) + \text{ȩ}\frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in U , we have

$$h'(z) = \lambda q'(z) + 2\mu\xi q(z)q'(z) + \varrho\frac{q'(z)}{q(z)} + \varrho z\frac{q''(z)}{q'(z)} - \varrho z\left(\frac{q'(z)}{q(z)}\right)^2,$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\lambda}{ç} q(z) + \frac{2ɯЧ}{\varrho} q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)},$$

hence that

$$Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\lambda}{\varrho} q(z) + \frac{2\mu\xi}{\varrho} q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right) > 0.$$

By using (3.11), we obtain

$$\lambda g(z) + ɯЧ g^2(z) + ç\frac{zg'(z)}{g(z)} = \left(zD_{s,r,y}^{\eta+1} f(z)\right)^{\rho}\left[\lambda + \mu\xi\left(zD_{a,b,c}^{m} f(z)\right)^2\right] + ç\rho\left(\frac{a}{b}\right)\left[\frac{D_{a,b,c}^{m-1} f(z)}{D_{a,b,c}^{m} f(z)} - 1\right].$$

By using (3.8), we have

$$\lambda g(z) + \mu\xi g^2(z) + \varrho\frac{zg'(z)}{g(z)} \prec \lambda q(z) + \mu\xi q^2(z) + \varrho\frac{zq'(z)}{q(z)},$$

We can infer that subordination(3.8) implies that $g(z) \prec q(z)$, and that the function $q(z)$ is the best domain by using Lemma2.2.

Taking the function $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, in Theorem3.2, the condition (3.6) becomes

$$Re\left\{\frac{\lambda}{ç}\left(\frac{1+Az}{1+Bz}\right) + \frac{2ɯЧ}{ç}\left(\frac{1+Az}{1+Bz}\right)^2 + 1 + \frac{(A-B)z}{(1+Bz)(1+Az)} - \frac{2Bz}{1+Bz}\right\} > 0 \; (ç \in C \setminus \{0\}), \tag{3.12}$$

as a result, we may deduce the following conclusion..

Corollary3.3: Let $(-1 \le B < A \le 1), \varrho, \rho \in \mathbb{C} \setminus \{0\}$, $\xi, a, \lambda, \mu \in \mathbb{C}$, assume that (3.12) holds .If $f \in E$ and

$$e(z) \prec a + \lambda\left(\frac{1+Az}{1+Bz}\right) + \mu\xi\left(\frac{1+Az}{1+Bz}\right)^2 + \varrho\frac{(A-B)z}{(1+Bz)(1+Az)},$$

where $e(z)$ is defined in (3.9), then

$$\left(zD_{a,b,c}^{m} f(z)\right)^{\rho} \prec \frac{1+Az}{1+Bz} \;, \; and \; \frac{1+Az}{1+Bz} \text{ is the best dominant.}$$

Taking the function $q(z) = \left(\frac{1+z}{1-z}\right)^{\iota}$ $(0 < \iota \le 1)$, in Theorem(3.2), the condition (3.6) becomes

$$Re\left\{\frac{\lambda}{ç}\left(\frac{1+z}{1-z}\right)^{\iota} + \frac{2ɯЧ}{ç}\left(\frac{1+z}{1-z}\right)^{2\iota} + \frac{2z^2}{1-z^2}\right\} > 0 \,, (\varrho \in C \setminus \{0\}). \tag{3.13}$$

As a result, we may deduce the following conclusion.

Corollary3.4: Let $0 < \iota \le 1, ç, \rho \in \mathbb{C} \setminus \{0\}$, Ч, ɲ, λ, ɯ $\in \mathbb{C}$. Assume that (3.13) holds . If $f \in E$ and

$$e(z) \prec ɲ + \lambda\left(\frac{1+z}{1-z}\right)^{\iota} + ɯЧ\left(\frac{1+z}{1-z}\right)^{2\iota} + ç\frac{2\iota z}{1-z^2},$$

where $e(z)$ is defined in (3.9) , then $\left(zD_{a,b,c}^{m} f(z)\right)^{\rho} \prec \left(\frac{1+z}{1-z}\right)^{\iota}$, and $\left(\frac{1+z}{1-z}\right)^{\iota}$ is the best dominant.

4- Results of Differential Superordinations:

Theorem4.1: Assume that the function $q(z)$ is a convex univalent in U with $q(0)=1, \rho \in \mathbb{C} \setminus \{0\}, Re\{\tau\}>0,\ \sigma \in \mathbb{R}^{+}, if\ f \in E$, such that

$$\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1} \neq 0, \quad and$$

$$\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho} \in H[q(0),1] \cap Q. \tag{4.1}$$

If the function $H(z)$ in (3.3) is univalent and the superordination criterion is fulfilled:

$$q(z)+\frac{\tau}{\rho}zq'(z) \prec H(z), \tag{4.2}$$

holds, then

$$q(z) \prec \left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho}, \tag{4.3}$$

where the best subordinant is q(z).

Proof: Define a function $g(z)$ by

$$g(z)=\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho}. \tag{4.4}$$

Differentiating (4.4) with respect to z, we get

$$\frac{zg'(z)}{g(z)}=\rho\left[\frac{(1-\sigma)z\left(D_{a,b,c}^{m-1}f(z)\right)'+2\sigma z\left(D_{a,b,c}^{m}f(z)\right)'}{(1-\sigma)D_{a,b,c}^{m-1}f(z)+2\sigma D_{a,b,c}^{m}f(z)}+1\right]. \tag{4.5}$$

A simple computation and using (1.7), from (4.5), we will get

$$H(z)=\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho} + \tau\left[\left(\frac{a}{b}\right)\left(\frac{(1-\sigma)D_{a,b,c}^{m-2}f(z)+(3\sigma-1)D_{a,b,c}^{m-1}f(z)-\sigma D_{a,b,c}^{m}f(z)}{(1-\sigma)D_{a,b,c}^{m-1}f(z)+2\sigma D_{a,b,c}^{m}f(z)}\right)\right]=g(z)+\frac{\tau}{\rho}zg'(z).$$

Now, by using Lemma2.4, we get the desired result.

Taking $q(z)=\frac{1+Az}{1+Bz}$, $(-1 \le B < A \le 1)$, we obtain the following conclusion from Theorem 4.1.

Corollary4.1: Let $Re\{\tau\}>0, \rho \in \mathbb{C}\setminus\{0\}, \sigma \in \mathbb{R}^{+}$ and $(-1 \le B < A \le 1)$,such that

$$\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho} \in H[q(0),1] \cap Q.$$

If $H(z)$ in (3.3) is univalent in U, and $f \in E$ fulfills the superordination condition,

$$\frac{1+Az}{1+Bz}+\frac{\tau}{\rho}\frac{(A-B)z}{(1+Bz)^{2}} \prec \mathrm{F}(z),$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho},$$

the best subordinant is the function $\frac{1+Az}{1+Bz}$.

Theorem4.2: Let $q(z)$ be a convex univalent function in the open unit disk U with $q(z)=1$, $q'(z)\neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $\varrho, \rho \in \mathbb{C}\setminus\{0\}$, $\xi, a, \lambda, \mu \in \mathbb{C}$. Suppose that q satisfy the condition $Re\left\{\frac{q(z)}{\varrho}(2\mu\xi+\lambda)\right\}q'(z)>0$.Let $f\in E$ satisfies the next conditions:

$$\left(zD_{a,b,c}^{m}f(z)\right)^{\rho} \in H[q(0),1]\cap Q, \tag{4.6}$$

and

$zD_{a,b,c}^{m}f(z)\neq 0$. If the function $e(z)$ is given by (3.9), is univalent in U,

$$a+\lambda q(z)+\mu\xi q^2(z)+\varrho\frac{zq'(z)}{q(z)} \prec e(z)\,, \tag{4.7}$$

implies

$q(z)\prec\left(zD_{a,b,c}^{m}f(z)\right)^{\rho}$, where the best subordinant is q(z).

Proof : Allow $g(z)$ to be defined on U by (3.10).

After that, a calculation reveals that

$$\frac{zg'(z)}{g(z)}=\rho\left(\frac{a}{b}\right)\left[\frac{D_{a,b,c}^{m-1}f(z)}{D_{a,b,c}^{m}f(z)}-1\right], \tag{4.8}$$

By setting $\Theta(\omega)=a+\lambda\omega+\mu\xi\omega^2, and\ \ \phi=\frac{\varrho}{\omega}$, $\omega\neq 0$. It can be easily observed that $\Theta(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C}\setminus\{0\}$, that $\phi(\omega)\neq 0$ $(\omega\in\mathbb{C}\setminus\{0\})$. Also , we get

$Q(z)=zq'(z)\phi(q(z))=\varrho\frac{zq'(z)}{q(z)}$, it was discovered that $Q(z)$ is a starlike univalent in U.

Because $q(z)$ is convex, we may deduce that

$$Re\left(\frac{z\Theta'(q(z))}{\phi(q(z))}\right)=Re\left\{\frac{q(z)}{\varrho}(2\mu\xi q(z)+\lambda\right\}q'(z)>0.$$

By making use (4.8) the hypothesis (4.7) can by equivalently

$$\Theta(q(z))+zq'(z)\phi(q(z))\prec\Theta(g(z))+zg'(z)\phi(g(z)).$$

The proof is therefore completed by utilizing the Lemma 2.3.

5- Sandwich Results:

By combining Theorems 3.1 and 4.1, we have the following sandwich Theorem:

Theorem5.1: I Let q_1 *and* q_2 be convex univalent functions in U with $q_1(0)=q_2(0)=1$ *and* q_2 satisfies (3.1). Suppose that $Re\{\tau\}>0$, $\tau,\rho\in\mathbb{C}\setminus\{0\}$, $\sigma\in\mathbb{R}^+$. If $f\in E$, such that

$$\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho}\in H[q(0),1]\cap Q,$$

and the univalent function $H(z)$, defined by (3.3), satisfies

$$q_1(z)+\frac{\tau}{\rho}zq_1{}'(z)\prec H(z)\prec q_2(z)+\frac{\tau}{\rho}zq_2{}'(z), \tag{5.1}$$

then

$$q_1(z)\prec\left(\frac{(1-\sigma)zD_{a,b,c}^{m-1}f(z)+2\sigma zD_{a,b,c}^{m}f(z)}{\sigma+1}\right)^{\rho}\prec q_2(z),$$

where q_1 *and* q_2 are the best subordinant and dominant, respectively (5.1).

We obtain the following sandwich theorem by merging Theorems 3.2 and 4.2:

Theorem5.2: Let q_j and be two univalent convex functions in U, in condition for $q_j(0)=1, q'_j(z)\neq 0, (j=1,2)$. Assume that q_1 *and* q_2 satisfy the conditions(3.8) and(4.8), respectively.

If $f\in E$, and suppose that f satisfies the next condition

$$\left(zD_{a,b,c}^{m}f(z)\right)^{\rho}\in H[q(0),1]\cap Q,$$

and $zD_{a,b,c}^{m}f(z)\neq 0$, *and* $e(z)$ is univalent in U, and given by (3.9), then

$$a+\lambda q_1(z)+\mu\xi q_1{}^2(z)+\varrho\frac{zq_1{}'(z)}{q_1(z)}\prec e(z)\prec a+\lambda q_2(z)+\mu\xi q_2{}^2(z)+\varrho\frac{zq_2{}'(z)}{q_2(z)}, \tag{5.2}$$

Implies

$$q_1(z)\prec\left(zD_{a,b,c}^{m}f(z)\right)^{\rho}\prec q_2(z),$$

where the best subordinant and dominant are q_1 *and* q_2, respectively.

References

[1] R. Abd Al-Sajjad and W. G. Atshan, Certain analytic function sandwich theorems involving operator defined by Mittag-Leffler function, AIP Conference Proceedings, 2398(2022),060065,1-8.

[2] S. A. Al-Ameedee, W. G. Atshan and F. A. Al-Maamori, On sandwich results of univalent functions defined by a linear operator, Journal of Interdisciplinary Mathematics, 23(4)(2020), 803-809.

[3] S. A. Al-Ameedee, W. G. Atshan and F. A. Al-Maamori, Some new results of differential subordinations for higher-order derivatives of multivalent functions, Journal of Physics: Conference Series, 1804 (2021) 012111, 1-11.

[4] R. M. Ali, V. Ravichandran, M. H. Khan and K. G. Subramanian, Differential sandwich theorems for certain analytic functions, Far East J. Math. Sci.,15(2004), 87-94.

[5] W. G. Atshan and A. A. R. Ali, On some sandwich theorems of analytic functions involving Noor –Sâlâgean operator, Advances in Mathematics: Scientific Journal, 9(10)(2020), 8455-8467.

[6] W. G. Atshan and A. A. R. Ali, On sandwich theorems results for certain univalent functions defined by generalized operators, Iraqi Journal of science, 62(7)(2021), PP:2376-2383.

[7] W. G. Atshan, A. H. Battor and A. F. Abass, Some sandwich theorems for meromorphic univalent functions defined by new integral operator, Journal of Interdisciplinary Mathematics , 24(3)(2021), 579-591.

[8] W. G. Atshan and R. A. Hadi, Some differential subordination and superordination results of p-valent functions defined by differential operator, Journal of Physics: Conference Series, 1664 (2020) 012043, 1-15.

[9] W. G. Atshan and S. R. Kulkarni, On application of differential subordination for certain subclass of meromorphically p-valent functions with positive coefficients defined by linear operator, Journal of Inequalities in Pure and Applied Mathematics, 10(2)(2009), Article 53, 11 pp.

[10] T. Bulboacă, Classes of first-order differential superordinations, Demonstration Math., 35(2)2002),287-292.

[11] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, (2005).

[12] S. P. Goyal, P. Goswami and H. Silverman, Subordination and superordination results for a class of analytic multivalent functions, Int. J. Math. Math. Sci., (2008), Article ID 561638, 1– 12.

[13] I. A. Kadum, W. G. Atshan and A. T. Hameed, sandwich theorems for a new class of complete homogeneous symmetric function by using cyclic operator, Symmetry, 14(10)(2022), 2223, 1-16.

[14] B. K. Mihsin, W. G. Atshan and S. S. Alhily, On new sandwich results of univalent functions defined by a linear operator, Iraqi Journal of science, 63(12)2022, PP: 5467-5475.

[15] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Text Books in Pure and Applied Mathematics, 225, Marcel Dekker, New York and Basel, (2000).

[16] S. S. Miller and P. T. Mocanu, Subordinats of differential superordinations, Complex, Var. Theory Appl., 48(2003), $815 - 826$.

[17] M. A. Sabri, W. G. Atshan and E. El-Seidy, On sandwich-type results for a subclass of certion univalent functions using a new Hadamard product operator, Symmetry, 14(5)(2022), 931, 1-11.

[18] F. O. Salman and W. G. Atshan, New results on integral operator for a subclass of analytic functions using differential subordinations and superordinations, Symmetry,15(2)(2023), 295, 1-10.

[19] N. Seenivasagan, Differential Subordination and Superordination for Analytic and Meromorphic Functions Defined by Linear Operator, Doctoral Dissertation, University Sains Malaysia, (2007).

[20] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, Differential sandwich theorems for subclasses of analytic functions, Aust. J. Math. Anal. Appl., 3 (2006), Article 8, 1– 11.

[21] T. N. Shanmugam, S. Shivasubramanian and H. Silverman, On sandwich theorems for some classes of analytic functions, Int. J. Math. Math. Sci. (2006), Article ID 29684, $1 - 13$.

[22] S. D. Theyab, W. G. Atshan, A. A. Lupas and H. K. Abdullah, New results on higher-order differential subordination and superordination for univalent analytic functions using a new operator, Symmetry, 14(8)(2022), 1576,1-12.

[23] S. D. Theyab, W. G. Atshan and H. K. Abdullah, On some sandwich results of univalent functions related by differential operator, Iraqi Journal of science, 63(11)(2022), PP:4928-4936.