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The Cubic Rank Transmuted Gumbel Distribution

Doaa Abed ELHertaniy[1] and Abdel Rahim Bashir Hamid[2]

Department of Mathematics of the university of the Holy Quran and Taseel of Science,sudan .Email: dalhirtani@gmail.com[1]

Associate professor, University of Gazira,sudan Email:, d.abdelrahim@gmail.com.[2]

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ABSTRACT

A Cubic Rank Transmuted Gumbel distribution (CTGD) in this research is extend the work of cubic transmuted distribution families. CTGD improves the flexibility of transmuted distributions and allows for the modeling of more complex data. Its hazard rate function, moment-generating function, moments, quantile function, entropy and order statistics, are only a few of the key statistical characteristics that we examine. Finally, the Cubic Transmuted Gumbel Distribution is applied to three real datasets to test its applicability and evaluate how well estimate approaches function for the CTGD, Gumbel(G), and transmuted Gumbel (TG) distributions. The observed results demonstrated that, for the used data sets, CTGD provides a superior fit than G and TG distributions.

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1. Introduction

The Gumbel distribution is named after Emil Julius Gumbel (1891–1966), based on his original papers describing the distribution. The Gumbel distribution is a particular case of the generalized extreme value distribution (also known as the Fisher-Tippett distribution). It is also known as the log-Weibull distribution and the double exponential distribution[8]. Perhaps the most frequently used statistical distribution for engineering challenges is the Gumbel distribution. It is sometimes referred to as the type I extreme value distribution. Its most recent engineering applications have been in the fields of network engineering, nuclear engineering, offshore engineering, risk-based engineering, space engineering, software reliability engineering, structural engineering, and wind engineering. The more than fifty uses listed in a recent book by Kotz and Nadarajah [10], which describes this distribution, range from accelerated life testing to earthquakes, floods, horse racing, rains, lines at supermarkets, sea currents, wind speeds, and track race records (to mention just a few). It is one of four EVDs in common use. The Frechet Distribution, the Weibull Distribution, and the Generalized Extreme Value Distribution are the other three. The generalized extreme value (GEV) distribution is a family of continuous probability distributions developed within extreme value theory to combine the Gumbel, Frechet, and Weibull families also known as type I, II and III extreme value distributions. The probability density function (PDF) and the cumulative distribution function (CDF) for Gumbel distribution are defined as follow

∗Corresponding author

Email addresses:

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$$g(X;\mu,\sigma)=\frac{1}{\sigma}e^{-(z+e^{-z})} \quad (1)$$

where

$$z=(\frac{x-\mu}{\sigma}), \quad \sigma>0.$$

and

$$G(X;\mu,\sigma)=e^{(-e^{-(\frac{x-\mu}{\sigma})})} \quad (2)'$$

Several Gumbel distribution extensions have previously been proposed. Nadarajah et al.[13] Submitted The Beta Gumbel distribution, Nadarajah [12] developed the Exponentiated Gumbel distribution as a generalization of the basic Gumbel distribution. Considering the distribution of Exponentiated Gumbel type-2, which was investigated by Okorie et al. [14], transmuted Gumbel type-II distribution, which has applicability in many different scientific domains by Ahmad et al. [1], giving Deka et al. [6], Transmuted exponentiated Gumbel distribution (TEGD) and its application to water quality data, Aryal and Tsokos [3] used quadratic rank transmutation to investigate the transmuted Gumbel distribution (TGD) as well as numerous mathematical features. Shaw and Buckley [18] proposed a quadratic rank transmuted distribution. A random variable X is said to have a quadratic rank transmuted distribution if its cumulative distribution function is given by

$$F(x)=(1+\lambda)G(x)-\lambda[G(x)]^2, |\lambda|\leq 1 \quad (3)$$

Differentiating (3) with respect to x, it gives the probability density function (pdf) of the quadratic rank transmuted distribution as

$$f(x)=g(x)[(1+\lambda)-2\lambda G(x)], |\lambda|\leq 1 \quad (4)$$

where $G(x)$ and $g(x)$ are the cdf and pdf respectively of the base distribution. It is very important observe that at $\lambda = 0$, we have the base original distribution. The family of quadratic transmuted distributions shown in (3) expands any baseline distribution G(x), increasing its applicability. Recently, Rahman et al. [15] proposed the cubic transmuted distribution family. A random variable X is said to have cubic transmuted distribution with parameter λ_1 and λ_2 if its cumulative distribution function (cdf) is given by

$$F(x)=(1+\lambda_1)G(x)+(\lambda_2-\lambda_1)[G(x)]^2-\lambda_2[G(x)]^3 \quad (5)$$

with corresponding pdf

$$f(x)=g(x)[1+\lambda_1+2(\lambda_2-\lambda_1)G(x)-3\lambda_2 G^2(x)], x\in R \quad (6)$$

where $\lambda_i \in [-1,1]$, i=1,2 are the transmutation parameters and obey the condition $-2\leq\lambda_1+\lambda_2\leq 1$. The proofs and the further details can be found in Granzatto et al. [7]. This paper is organized as follows, in Section 2 defining the cubic transmuted Gumbel distribution. Statistical properties have been discussed like the shapes of the density and hazard rate functions, quantile function, moments and moment-generating function, Characteristic Function, and cumulant generating function in Section 3. Entropy was studied in Section 4, and order statistics in Section 5. Section 6, we address the parameters of the CTG distribution via maximum likelihood method. An application of the CTGD to three real data sets for the purpose of illustration is conducted in section7. Finally, in Section 8, some conclusions are declared.

2. Cubic Rank Transmuted Gumbel Distribution

The cubic rank transmuted Gumbel distribution is defined as follows: The CDF of a cubic rank transmuted Gumbel distribution is obtained by using (2) in (5)

$$F(x) = (1+\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})} + (\lambda_2-\lambda_1)[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2 - \lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3 \quad (7)$$

where $x \in R$, μ, $\sigma > 0$, $\lambda_i \in$ [-1,1], i=1,2 and $-2 \le \lambda_1 + \lambda_2 \le 1$.

It is very important note observe that at value $\lambda_1 = \lambda_2 = 0$,the cubic rank transmuted Gumbel distribution reduce to Gumbel distribution according to the transmutation map.

The probability density function (pdf) of a cubic rank transmuted Gumbel distribution is given by

$$f(x) = \frac{1}{\sigma}e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 3\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2] \quad (8)$$

where

$$x \in R, \mu, \sigma > 0$$

Fig. 1.and Fig. 2, show different selected values of the model parameters $\lambda_1$and λ_2 where μ=3 and σ=2. for the pdf and cdf of the cubic rank transmuted Gumbel distribution .

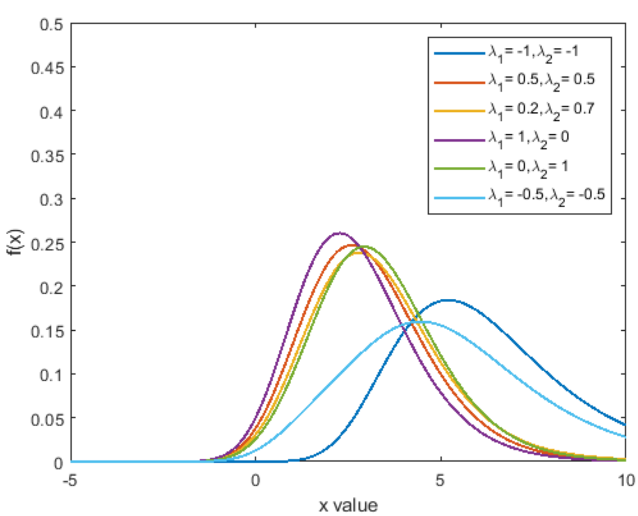


Fig. 1: Plots of pdf plots of the CTGD.

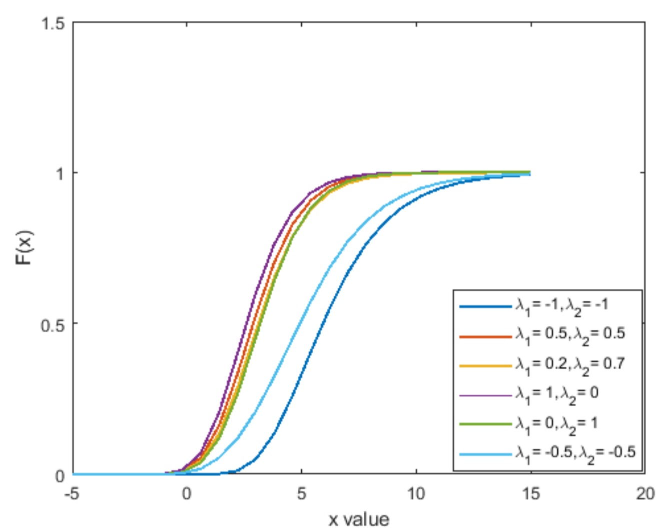


Fig. 2: Plots of cdf of the CTGD.

3. STATISTICAL PROPERTIES

3.*1 Shapes of the density and hazard rate functions*

The reliability function of the cdf F(x) of distribution is defined by $R(x) = 1 - F(x)$. For the cubic rank transmuted Gumbel (CTG) distribution, the reliability function is given as,

$$R(X) = 1 - [(1+\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})} + (\lambda_2-\lambda_1)[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2 - \lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3] \quad (9)$$

The hazard rate function of GTF distribution by (7) and (9):

$$h(x) = \frac{\frac{1}{\sigma}e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]}{1-[(1+\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})}+(\lambda_2-\lambda_1)[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2-\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3]}$$

The cumulative hazard function is defined by

$$H(x) = -lnR(x),$$

so the cumulative hazard function of the CTG distribution is

$$H(x) = -ln\left\{1 - [(1+\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})} + (\lambda_2 - \lambda_1)[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2 - \lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3]\right\}. \quad (10)$$

The reverse hazard function is

$$r(x) = \frac{f(x)}{F(X)} \quad (11)$$

We can define the reverse hazard function of the CTG distribution by using (11) as $r(x) =$

$$\frac{\frac{1}{\sigma}e^{-(\frac{x-\mu}{\sigma})}[(1+\lambda_1)+2(\lambda_2-\lambda_1)[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]-3\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]}{(1+\lambda_1)+(\lambda_2-\lambda_1)[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]-\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2} \quad (12)$$

Fig. 3 and Fig.4 , show the Hazard function and the Reliability function of the cubic rank transmuted Gumbel distribution for different values of parameters $\lambda_1$and λ_2 where μ=3 and σ=2

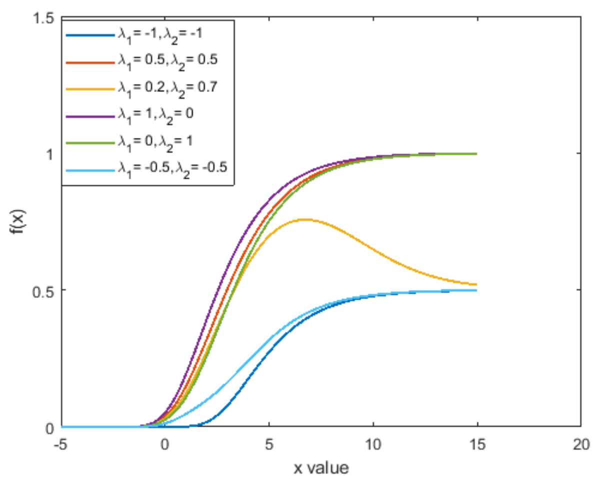


Fig. 3: Plots of hazard function of the CTGD.

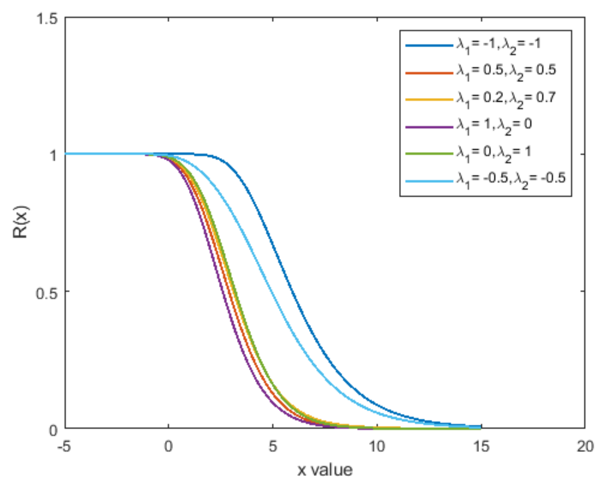


Fig. 4: Plots of Reliability function of the CTGD.

The Odd function of a distribution with cdf $F(x)$ is defined as

$$O(x) = \frac{F(x)}{1-F(x)} \quad (13)$$

Then the Odd function of the CTG distribution is given as

$$O(x) = \left\{\left[(1+\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})} + (\lambda_2 - \lambda_1)[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2 - \lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3\right]^{-1} - 1\right\}^{-1} \quad (14)$$

3.2 Quantile function

Here we will compute the quantile function of the cubic rank transmuted Gumbel probability distribution.

Theorem 3.1 Let X be random variable from the cubic rank transmuted Gumbel probability distribution with parameters $\sigma > 0, \mu > 0, \lambda_i \in [-1,1], i = 1,2$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$. Then the quantile function of X, is given by

$$x_q = \mu - \sigma \log(-\log \delta(q, \lambda_1, \lambda_2)) \tag{15}$$

Proof. To calculate the quantile function of the cubic rank transmuted Gumbel probability distribution, we substitute x by x_q and $F(x)$ by q in (7) to get the equation

$$q = (1+\lambda_1)e^{(-e^{-(\frac{x_q-\mu}{\sigma})})} + (\lambda_2 - \lambda_1)[e^{(-e^{-(\frac{x_q-\mu}{\sigma})})}]^2 - \lambda_2[e^{(-e^{-(\frac{x_q-\mu}{\sigma})})}]^3 \tag{16}$$

Then, we solve the equation (15) for x_q. So, let $y = e^{(-e^{-(\frac{x_q-\mu}{\sigma})})}$. Thus, (15) becomes

$$q = (1+\lambda_1)y + (\lambda_2 - \lambda_1)y^2 - \lambda_2 y^3$$

and hence,

$$\lambda_2 y^3 + (\lambda_1 - \lambda_2)y^2 + (-1-\lambda_1)y + q = 0 \tag{17}$$

Let a = λ_2, b = $(\lambda_1 - \lambda_2)$, c = (-1 - λ_1) and d = q, then the equation (16) becomes

$$ay^3 + by^2 + cy + d = 0.$$

Then,

$$y = -\frac{b}{3a} - \frac{\sqrt[3]{2}\beta_1}{3a(\beta_2 + \sqrt{4\beta_1^3 + \beta_2^2})^{\frac{1}{3}}} + \frac{(\xi_2 + \sqrt{4\beta_1^3 + \beta_2^2})^{\frac{1}{3}}}{3\sqrt[3]{2}a} \tag{18}$$

where $\beta_1 = -b^2 + 3ac,\ \beta_2 = -2b^3 + 9abc - 27a^2 d,$ and $d = q.$

Now, let the function $\delta(q, \lambda_1, \lambda_2)$ be defined by

$$\delta(q, \lambda_1, \lambda_2) = -\frac{b}{3a} - \frac{\sqrt[3]{2}\beta_1}{3a(\beta_2 + \sqrt{4\beta_1^3 + \beta_2^2})^{\frac{1}{3}}} + \frac{(\beta_2 + \sqrt{4\beta_1^3 + \beta_2^2})^{\frac{1}{3}}}{3\sqrt[3]{2}a}$$

Hence,

$$y = e^{(-e^{-(\frac{x_q-\mu}{\sigma})})} = \delta(q, \lambda_1, \lambda_2) \tag{19}$$

Take natural Logarithm to both sides to get

$$-e^{-(\frac{x_q-\mu}{\sigma})} = \log \delta(q, \lambda_1, \lambda_2)$$

Then, we have the equation

$$x_q = \mu - \sigma \log(-\log \delta(q, \lambda_1, \lambda_2))$$

The first quartile, median and third quartile can be obtained by setting q = 0.25, 0.50 and 0.75 in (15) respectively.

3.3 *Moments and Moment-generating function*

Moments function

Theorem 3.2 Let X ~ CTGD(μ,σ). Then the rth moment of X is given by

$$E(x^r) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sigma^k \mu^{r-k} \left[(1+\lambda_1)\frac{\partial k}{\alpha^k}\Gamma(\alpha) + 2(\lambda_2-\lambda_1)\frac{\partial k}{\alpha^k}2^{-\alpha}\Gamma(\alpha) - 3\lambda_2\frac{\partial k}{\alpha^k}3^{-\alpha}\Gamma(\alpha)\right]|\alpha = 1 \quad (20)$$

Proof. The rth moment of the positive random variable X with probability density function is given by

$$E(x^r) = \int_0^\infty x^r f(x)dx \quad (21)$$

Substituting from (8) in to (21),

$$E(x^r) = \int_0^\infty x^r \frac{1}{\sigma} e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}[(1+\lambda_1) + 2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 3\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]dx \quad (22)$$

Using the transformation $y = e^{-(\frac{x-\mu}{\sigma})}$

Then $dy = \frac{-e^{-(\frac{x-\mu}{\sigma})}}{\sigma}dx,\ x = \mu - \sigma lny$

With substitution by this transformation in (22) then

$$E(x^r) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sigma^k \mu^{r-k} \left[(1+\lambda_1)\frac{\partial k}{\alpha^k}\Gamma(\alpha) + 2(\lambda_2-\lambda_1)\frac{\partial k}{\alpha^k}(2^{-\alpha}\Gamma(\alpha)) - 3\lambda_2\frac{\partial k}{\alpha^k}(3^{-\alpha}\Gamma(\alpha))\right]|\alpha = 1$$

Moment Generating Function

Theorem 3.3 The moment generating function $M_x(t)$ of a random variable X CTGD (μ,σ) is given by

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\sum_{k=0}^{r} (-1)^k \binom{r}{k} \sigma^k \mu^{r-k} \left[(1+\lambda_1)\frac{\partial k}{\alpha^k}\Gamma(\alpha) + 2(\lambda_2-\lambda_1)\frac{\partial k}{\alpha^k}(2^{-\alpha}\Gamma(\alpha)) - 3\lambda_2\frac{\partial k}{\alpha^k}(3^{-\alpha}\Gamma(\alpha))\right]|\alpha = 1 \quad (23)$$

Proof. We know that

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$

Using series expansion of e^{tx},

$$M_x(t) = \sum_{r=0}^{n} \frac{t^r}{r!}\int_0^\infty x^r f(x)dx = \sum_{r=0}^{n} \frac{t^r}{r!}E(x^r)$$

Then

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\sum_{k=0}^{r} (-1)^k \binom{r}{k} \sigma^k \mu^{r-k} \left[(1+\lambda_1)\frac{\partial k}{\alpha^k}\Gamma(\alpha) + 2(\lambda_2-\lambda_1)\frac{\partial k}{\alpha^k}(2^{-\alpha}\Gamma(\alpha)) - 3\lambda_2\frac{\partial k}{\alpha^k}(3^{-\alpha}\Gamma(\alpha))\right]|\alpha = 1$$

3.4 *Characteristic Function*

The cubic transmuted Gumbel distribution's characteristic function theorem is stated as follows:

Theorem 3.4 Assume that the random variable X have the CTGD ($\mu,\sigma,\lambda_1,\lambda_2$), then characteristic function, $\phi_x(t)$ is

$$\phi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sigma^k \mu^{r-k} \left[(1+\lambda_1)\frac{\partial k}{\alpha^k}\Gamma(\alpha) + 2(\lambda_2-\lambda_1)\frac{\partial k}{\alpha^k}(2^{-\alpha}\Gamma(\alpha)) - 3\lambda_2 \frac{\partial k}{\alpha^k}(3^{-\alpha}\Gamma(\alpha))\right] |\alpha = 1 \quad (24)$$

Where $i = \sqrt{-1}$ and $t \in \Re$

3.5 Cumulant Generating Function

The cumulant generating function is defined by

$$K_x(t) = \log_e M_x(t)$$

Cumulant generating function of cubic rank transmuted Gumbel distribution is given by

$$K_x(t) = \log_e \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sigma^k \mu^{r-k} \left[(1+\lambda_1)\frac{\partial k}{\alpha^k}\Gamma(\alpha) + 2(\lambda_2-\lambda_1)\frac{\partial k}{\alpha^k}(2^{-\alpha}\Gamma(\alpha)) - 3\lambda_2 \frac{\partial k}{\alpha^k}(3^{-\alpha}\Gamma(\alpha))\right] |\alpha = 1 \quad (25)$$

4. ENTROPY

4.1 Rényi Entropy

If X is a non-negative continuous random variable with pdf $f(x)$, then the Renyi entropy of order δ (Renyi [16]) of X is defined as

$$H_\delta(x) = \frac{1}{1-\delta} log \int_0^\infty [f(x)]^\delta dx, \forall\delta > 0, (\delta \neq 1) \quad (26)$$

Theorem 4.1 The Rényi entropy of a random variable X ~ CTGD (μ,σ), with $\lambda_1 \neq 1$, $\lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$. is given by

$$H_\delta(x) = \frac{1}{1-\delta} log \left[\sum_{j=0}^{\infty} \sum_{k=0}^{j} c(j,k,\delta) \frac{-1}{\sigma^{\delta-1}} \lambda_2^k (1+\lambda_1)^{\delta-j} (\lambda_2-\lambda_1)^{j-k} \frac{\Gamma(\delta)}{(\delta+j+k)^\delta}\right]$$

Proof. Assume X has the pdf in (8). Then, can compute

$$[f(x)]^\delta = \frac{1}{\sigma^\delta} e^{-\delta(\frac{x-\mu}{\sigma} + e^{-(\frac{x-\mu}{\sigma})})} \left[[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 3\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]\right]^\delta \quad (27)$$

By the general binomial expansion, we have

$$\left[[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 3\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]\right]^\delta$$

$$= \sum_{j=0}^{\infty} \binom{\delta}{j} (1+\lambda_1)^{\delta-j} \left[2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 3\lambda_2 e^{-2(e^{-(\frac{x-\mu}{\sigma})})}\right]^j \quad (28)$$

by the Binomial Theorem,

$$\left[2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 3\lambda_2 e^{-2(e^{-(\frac{x-\mu}{\sigma})})}\right]^j$$

$$= \sum_{k=0}^{j} \binom{j}{k} \left[2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})}\right]^{j-k} \left[-3\lambda_2 e^{-2(e^{-(\frac{x-\mu}{\sigma})})}\right]^k$$

$$= \sum_{k=0}^{j} \binom{j}{k} 2^{j-k} (\lambda_2 - \lambda_1)^{j-k} e^{-(j-k)(e^{-(\frac{x-\mu}{\sigma})})} (-3)^k \lambda_2^k e^{-2k(e^{-(\frac{x-\mu}{\sigma})})}$$

$$= \sum_{k=0}^{j} \binom{j}{k} 2^{j-k} (-3)^k \lambda_2^k (\lambda_2 - \lambda_1)^{j-k} e^{-(j+k)(e^{-(\frac{x-\mu}{\sigma})})} \quad (29)$$

Substitute from(29) in (28), to get

$$\left[[1 + \lambda_1 + 2(\lambda_2 - \lambda_1) e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 3\lambda_2 [e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]\right]^{\delta}$$

$$= \sum_{j=0}^{\infty} \binom{\delta}{j} (1+\lambda_1)^{\delta-j} \left[\sum_{k=0}^{j} \binom{j}{k} 2^{j-k} (-3)^k \lambda_2^k (\lambda_2 - \lambda_1)^{j-k} e^{-(j+k)(e^{-(\frac{x-\mu}{\sigma})})}\right]$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{\delta}{j} \binom{j}{k} 2^{j-k} (-3)^k \lambda_2^k (1+\lambda_1)^{\delta-j} (\lambda_2 - \lambda_1)^{j-k} e^{[-(j+k)(e^{-(\frac{x-\mu}{\sigma})})]} \quad (30)$$

Now, substitute from(29) in (26), to get

$$[f(x)]^{\delta} = \frac{1}{\sigma^{\delta}} e^{-\delta(\frac{x-\mu}{\sigma} + e^{-(\frac{x-\mu}{\sigma})})} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{\delta}{j} \binom{j}{k} 2^{j-k} (-3)^k \lambda_2^k (1+\lambda_1)^{\delta-j} (\lambda_2 - \lambda_1)^{j-k} e^{[-(j+k)(e^{-(\frac{x-\mu}{\sigma})})]}$$

Let c(j, k, δ)= $\binom{\delta}{j}\binom{j}{k} 2^{j-k}(-3)^k$, then

$$[f(x)]^{\delta} = \frac{1}{\sigma^{\delta}} \sum_{j=0}^{\infty} \sum_{k=0}^{j} c(j,k,\delta) \lambda_2^k (1+\lambda_1)^{\delta-j} (\lambda_2 - \lambda_1)^{j-k} e^{[-(\delta+j+k)(e^{-(\frac{x-\mu}{\sigma})}) + (-\delta\frac{(x-\mu)}{\sigma})]} \quad (31)$$

To find $H_{\delta}(x)$, we substitute from (31) in (26)

$$H_{\delta}(x) = \frac{1}{1-\delta} log\left[\sum_{j=0}^{\infty} \sum_{k=0}^{j} c(j,k,\delta) \frac{1}{\sigma^{\delta}} \lambda_2^k (1+\lambda_1)^{\delta-j} (\lambda_2 - \lambda_1)^{j-k} \times \int_0^{\infty} e^{[-(\delta+j+k)(e^{-(\frac{x-\mu}{\sigma})}) + (-\delta\frac{(x-\mu)}{\sigma})]} dx\right]. \quad (32)$$

We can evaluate the integration by using the transformation

$$\int_0^{\infty} e^{[-(\delta+j+k)(e^{-(\frac{x-\mu}{\sigma})}) + (-\delta\frac{(x-\mu)}{\sigma})]} dx = \int_0^{\infty} (e^{(-\frac{(x-\mu)}{\sigma})})^{\delta} \times e^{-(\delta+j+k)(e^{-(\frac{x-\mu}{\sigma})})} dx \quad (33)$$

let $y = e^{-(\frac{x-\mu}{\sigma})}$, and $\delta \geq 1$, then $dy = \frac{-e^{-(\frac{x-\mu}{\sigma})}}{\sigma} dx$ and $0 < y < \infty$

With substitution with this transformation in (33) then,

$$-\sigma \int_0^{\infty} y^{\delta-1} e^{-(\delta+j+k)y} dy = -\sigma \frac{\Gamma(\delta)}{(\delta+j+k)^{\delta}} \quad (34)$$

After solving the integral, we get the Rényi entropy of the CTGD (μ,σ) by substitute from (34) in (32)

$$H_{\delta}(x) = \frac{1}{1-\delta} \log\left[\sum_{j=0}^{\infty} \sum_{k=0}^{j} c(j,k,\delta) \frac{-1}{\sigma^{\delta-1}} \lambda_2^k (1+\lambda_1)^{\delta-j} (\lambda_2 - \lambda_1)^{j-k} \frac{\Gamma(\delta)}{(\delta+j+k)^{\delta}}\right]$$

4.2 q-Entropy

The q-entropy was introduced by Havrda and Charvat [9]. It is the one parameter generalization of the Shannon entropy. Ullah[20] defined the q-entropy as

$$I_H(q) = \frac{1}{1-q}\left[1 - \int_0^\infty f(x)^q dx\right], where q > 0, and q \neq 1 \quad (35)$$

Theorem 4.2 The q-entropy of a random variable X ~ CTGD (μ,σ), with $\lambda_1 \neq 1$, $\lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$. is given by

$$I_H(q) = \frac{1}{1-q}\left[1 - \sum_{j=0}^{\infty} \sum_{k=0}^{j} c(j,k,q) \frac{-1}{\sigma^{q-1}} \lambda_2^k (1+\lambda_1)^{q-j} (\lambda_2-\lambda_1)^{j-k} \frac{\Gamma(q)}{(q+j+k)^q}\right]$$

Proof. To find $I_H(q)$, we substitute (31) in (36).

4.3 Shannon Entropy

In a non-negative continuous random variable X with pdf f(x), the Shannons entropy [17] is

$$H(f) = E[-\log f(x)] = -\int_0^\infty f(x)\log(f(x))dx \quad (36)$$

The Expansion of the Logarithm function will be used below (Taylor series at 1), $\log(x) = \sum_{m=0}^{\infty} (-1)^{m-1} \frac{(x-1)^m}{m}, |x| < 1$ (37)

The Shannon entropy of a random variable X ~ CTGD (μ,σ), with $\lambda_1 \neq 1$, $\lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$. is given by

$$H(f) = \sum_{m=1}^{\infty} \sum_{n=0}^{m} \sum_{j=0}^{n+1} \sum_{k=0}^{j} (-1)^n \binom{m}{n} \frac{1}{m} c(j,k,n+1) \frac{-1}{\sigma^n} \lambda_2^k \quad (38)$$

$$\times (1+\lambda_1)^{n+1-j} (\lambda_2-\lambda_1)^{j-k} \frac{\Gamma(n+1)}{(n+1+j+k)^{n+1}}$$

Proof. Using the logarithm function's expansion (36)

$$\log(f(x)) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{(f(x)-1)^m}{m},$$

and by Binomial Theorem,

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{m} \left\{\sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} f^n(x)\right\}$$

$$\log(f(x)) = \sum_{m=1}^{\infty} \sum_{n=0}^{m} (-1)^{n+1} \binom{m}{n} \frac{1}{m} f^n(x) \quad (39)$$

To compute the Shannon's entropy of X, substitute from (39) in (36)

$$H(f) = -\int_0^\infty f(x)\log(f(x))dx = -\int_0^\infty f(x) \sum_{m=1}^{\infty} \sum_{n=0}^{m} (-1)^{n+1} \binom{m}{n} \frac{1}{m} f^n(x)dx$$

$$H(f) = \int_0^\infty \sum_{m=1}^{\infty} \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{1}{m} f^{n+1}(x)dx \quad (40)$$

substituting from (31) in (40), to get

$$H(f) = \int_0^\infty \sum_{m=1}^{\infty} \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{1}{m} \sum_{j=0}^{n+1} \sum_{k=0}^{j} c(j,k,n+1) \frac{1}{\sigma^{n+1}} \lambda_2^k$$

$$\times (1+\lambda_1)^{n+1-j} (\lambda_2-\lambda_1)^{j-k} e^{[-(n+1+j+k)(e^{-(\frac{x-\mu}{\sigma})}) + (-(n+1)\frac{(x-\mu)}{\sigma})]} dx$$

$$H(f) = \sum_{m=1}^{\infty} \sum_{n=0}^{m} \sum_{j=0}^{n+1} \sum_{k=0}^{j} (-1)^n \binom{m}{n} \frac{1}{m} c(j,k,n+1) \frac{1}{\sigma^{n+1}} \lambda_2^k$$

$$\times (1+\lambda_1)^{n+1-j}(\lambda_2-\lambda_1)^{j-k}\int_0^\infty e^{[-(n+1+j+k)(e^{-(\frac{x-\mu}{\sigma})})+(-(n+1)\frac{(x-\mu)}{\sigma})]}dx$$

Now, we use (34) to fined the value of the integration, so

$$H(f)=\sum_{m=1}^{\infty}\sum_{n=0}^{m}\sum_{j=0}^{n+1}\sum_{k=0}^{j}(-1)^n\binom{m}{n}\frac{1}{m}c(j,k,n+1)\frac{-1}{\sigma^n}\lambda_2^k$$

$$\times (1+\lambda_1)^{n+1-j}(\lambda_2-\lambda_1)^{j-k}\frac{\Gamma(n+1)}{(n+1+j+k)^{n+1}}$$

5. ORDER STATISTICS

Let $X_1, X_2, \ldots, X_n$ be a random sample of size k from the CTG distribution with parameters $\mu>0, \sigma>0, 0\le \lambda_1\le 1$ and $\lambda_1-1\le\lambda_2\le\lambda_1$ From (Casella and Berger [5]), the pdf of the kth order statistics is obtain by

$$f_{X_{(k)}}(x)=k\binom{n}{k}f(x)[F(x)]^{k-1}[1-F(x)]^{n-k} \qquad (41)$$

Let X_k be the kth order statistic from $X\sim$ CTGD (μ,σ) with $\lambda_1\neq 0$ and $\lambda_2\neq 0$. Then, the pdf of the k th order statistic is given by

$$f_{X_{(k)}}(x)=k\binom{n}{k}f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}, \text{byBinomialTheorem},$$

$$=k\binom{n}{k}f(x)[F(x)]^{k-1}\left[\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}[F(x)]^j\right]$$

$$=\sum_{j=0}^{n-k}(-1)^j k\binom{n}{k}\binom{n-k}{j}f(x)[F(x)]^{k+j-1}$$

$$f_{X_{(k)}}(x)=\sum_{j=0}^{n-k}(-1)^j k\binom{n}{k}\binom{n-k}{j}\frac{1}{\sigma}e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]$$

$$\times\left[(1+\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})}+(\lambda_2-\lambda_1)[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2-\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3\right]^{k+j-1}$$

Then, the PDF of first order statistic $X_{(1)}$ of CTG distribution is given as

$$f_{X_{(1)}}(x)=n\frac{1}{\sigma}e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]$$

$$\times\left[1-\left\{(1+\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})}+(\lambda_2-\lambda_1)[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2-\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3\right\}\right]^{n-1}$$

Therefore, the of the largest order statistic $X_{(n)}$ of CTG distribution is given by

$$f_{X_{(n)}}(x)=n\frac{1}{\sigma}e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]$$

$$\times\left[(1+\lambda_1)e^{(-e^{-(\frac{x-\mu}{\sigma})})}+(\lambda_2-\lambda_1)[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2-\lambda_2[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3\right]^{n-1}$$

Not that when $\lambda_2=\lambda_1=0$ then the pdf of the kth order statistic for CTGD as follows

$$f_{X_{(k)}}(x)=\sum_{j=0}^{n-k}(-1)^j k\binom{n}{k}\binom{n-k}{j}\frac{1}{\sigma}e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}\left[e^{(-e^{-(\frac{x-\mu}{\sigma})})}\right]^{k+j-1}$$

6. MAXIMUM LIKELIHOOD ESTIMATION (MLE)

Assume $X_1, X_2, \ldots, X_n$ be a random sample of size n from CTGD(μ,σ) then the likelihood function can be written as

$$l(\mu,\sigma,\lambda_1,\lambda_2) = \prod_{i=1}^{n} f(x_i) =$$

$$\prod_{i=1}^{n}\left[\frac{1}{\sigma}e^{-(\frac{x_i-\mu}{\sigma}+e^{-(\frac{x_i-\mu}{\sigma})})}[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2]\right]$$

Then

$$l(\mu,\sigma,\lambda_1,\lambda_2) = \frac{1}{\sigma^n}e^{-\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma}+e^{-(\frac{x_i-\mu}{\sigma})})}\prod_{i=1}^{n}\left[[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2]\right] \quad (42)$$

Then, by taking the logarithm given as,

$$\log l(\mu,\sigma,\lambda_1,\lambda_2) = \log\prod_{i=1}^{n} f(x_i) = -n\log\sigma + \log(e^{-\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma}+e^{-(\frac{x_i-\mu}{\sigma})})})$$

$$+\sum_{i=1}^{n}\log\left[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2\right]$$

Then

$$\log l(\mu,\sigma,\lambda_1,\lambda_2) = -n\log(\sigma) - \sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma}+e^{-(\frac{x_i-\mu}{\sigma})})$$

$$+\sum_{i=1}^{n}\log\left[1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2\right]$$

Differentiate w.r.t parameters μ , σ , λ_1, and λ_2 we have

$$\frac{\partial \log l}{\partial\mu} = \frac{n}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{n} e^{-(\frac{x_i-\mu}{\sigma})}$$

$$+\frac{1}{\sigma}\sum_{i=1}^{n}\frac{2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}e^{-(\frac{x_i-\mu}{\sigma})}+6\lambda_2[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2e^{-(\frac{x_i-\mu}{\sigma})}}{1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2} \quad (43)$$

$$\frac{\partial \log l}{\partial\sigma} = \frac{-n}{\sigma} + \sum_{i=1}^{n}\frac{x_i-\mu}{\sigma^2} - \sum_{i=1}^{n} e^{-(\frac{x_i-\mu}{\sigma})}\frac{x_i-\mu}{\sigma^2}$$

$$+\sum_{i=1}^{n}\frac{2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}e^{-(\frac{x_i-\mu}{\sigma})}\frac{x_i-\mu}{\sigma^2}+6\lambda_2[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2e^{-(\frac{x_i-\mu}{\sigma})}\frac{x_i-\mu}{\sigma^2}}{1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2} \quad (44)$$

$$\frac{\partial \log l}{\partial\lambda_1} = \sum_{i=1}^{n}\frac{1-2e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}}{1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2} \quad (45)$$

$$\frac{\partial \log l}{\partial \lambda_2}=\sum_{i=1}^{n}\frac{2e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-3[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2}{1+\lambda_1+2(\lambda_2-\lambda_1)e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-3\lambda_2[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2} \quad (46)$$

We can obtain the estimates of the unknown parameters by the maximum likelihood method by setting these above nonlinear equations to zero

$$\frac{\partial logl(\mu,\sigma,\lambda_1,\lambda_2)}{\partial\mu}=0$$

$$\frac{\partial logl(\mu,\sigma,\lambda_1,\lambda_2)}{\partial\sigma}=0$$

$$\frac{\partial logl(\mu,\sigma,\lambda_1,\lambda_2)}{\partial\lambda_1}=0$$

$$\frac{\partial logl(\mu,\sigma,\lambda_1,\lambda_2)}{\partial\lambda_2}=0 \quad (47)$$

and solving them simultaneously. Therefore, statistical software can be employed in obtaining the numerical solution to the nonlinear equations,We can compute the maximum likelihood estimators (MLEs) of parameters (μ , σ , λ_1, λ_2) using quasi-Newton procedure,or computer packages/ softwares such as R, SAS, Ox, MATLAB, MAPLE and MATHEMATICA.

7. APPLICATIONS

In this section the (CTG) distribution applied on three data sets as follows. The first data in Table 1 the time of successive failures of the air conditioning system of jet airplanes. These data was obtained [4]. The second data is values for flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada.are shown in Table 2. [19]. The third data in Table 3 is represents the remission times (in months) of 128 bladder cancer Patients was introduced by [11].

Table 1- the time of successive failures of the air conditioning system of jet airplanes data

149	413	90	74	55	23	97	50	359	50	130
487	102	15	14	10	57	320	261	51	44	9
254	493	18	209	41	58	60	48	56	87	11
102	12	5	100	14	29	37	186	29	104	7
4	72	270	283	7	57	33	100	61	502	220
120	141	22	603	35	98	54	181	65	49	12
239	14	18	39	3	12	5	32	9	14	70
47	62	142	3	104	85	67	169	24	21	246
47	68	15	2	91	59	447	56	29	176	225
77	197	438	43	134	184	20	386	182	71	80
188	230	152	36	79	59	33	246	1	79	3
27	201	84	27	21	16	88	130	14	118	44
15	42	106	46	230	59	153	104	20	206	5
66	34	29	26	35	5	82	5	61	31	118
326	12	54	36	34	18	25	120	31	22	18
156	11	216	139	67	310	3	46	210	57	76
14	111	97	62	26	71	39	30	7	44	11
63	23	22	23	14	18	13	34	62	11	191
14	16	18	130	90	163	208	1	24	70	16
101	52	208	95							

Table 2- Exceedances of Wheaton River flood data

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.4	18.7	8.5
25.5	11.6	14.1	22.1	1.1	0.6	2.2	39.0	0.3	15.0	11.0
7.3	22.9	0.9	1.7	7.0	20.1	0.4	2.8	14.1	9.9	5.6
30.8	13.3	4.2	25.5	3.4	11.9	21.5	1.5	2.5	27.4	1.0
27.1	20.2	16.8	5.3	1.9	10.4	13.0	10.7	12.0	30.0	9.3
3.6	2.5	27.6	14.4	36.4	1.7	2.7	37.6	64.0	1.7	9.7
0.1	27.5	1.1	2.5	0.6	27.0					

Table 3: The remission times (in months) of 128 bladder cancer patients

0.08	0.20	0.40	0. 0.50	0.51	0.81	0.90	1.05	1.19	1.26
1.35	1.40	1.46	1.76	2.02	2.02	2.07	2.09	2.23	2.26
2.46	2.54	2.62	2.64	2.69	2.69	2.75	2.83	2.87	3.02
3.70	3.82	3.25	3.31	3.36	3.36	3.48	3.52	3.57	3.64
4.51	4.87	3.88	4.18	4.23	4.26	4.33	4.34	4.40	4.50
5.41	5.49	4.98	5.06	5.09	5.17	5.32	5.32	5.34	5.41
6.97	7.09	5.62	5.71	5.85	6.25	6.54	6.76	6.93	6.94
7.87	7.93	7.26	7.28	7.32	7.39	7.59	7.62	7.63	7.66
9.74	10.06	8.26	9.74	10.06	8.26	9.74	10.06	8.26	9.74
12.03	12.07	10.34	10.66	10.75	11.25	11.64	11.79	11.98	12.02
12.63	13.11	13.29	13.80	14.24	14.76	14.77	14.83	15.96	16.62
17.12	17.14	17.36	18.10	19.13	20.28	21.73	22.69	23.63	25.74
25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05		

Table 4 illustrates the summary statistics of the three data sets. Three different distributions have been compared using these data sets for comparative purposes: the (CTGD) model, the Gumbel distribution (GD), and the transmuted Gumbel distribution (TGD). Tables 5, 7, and 9 show the estimated values of the model parameters as well as the accompanying standard errors for selected models using the MLE method.

The goodness of fit of the (CTGD), (TGD), and (GD) has been introduced in Tables 6 , 8 and 10 using various comparison measures. Some criteria that have been taken into consideration include(-2$\mathfrak{L}(\theta)$): where is $\mathfrak{L}(\theta)$ is the maximum value of the log-likelihood function, AIC (Akaike Information Criterion), AICc (Corrected Akaike In general, the smaller value of the statistics(-2$\mathfrak{L}(\theta)$), AIC and BIC indicates a better fit of the distribution. Furthermore, the Kolmogorov-Smirnov (K-S) statistics and associated p-values were obtained along with the Cramer-von Mises (w*) and Andersen-Darling (A*) statistics. A very good fit of the model to the data is shown by reduced values for all three goodness-of-fit indicators. Large p-values also indicate a good fit for the model, which is another benefit.

Plots of the empirical and theoretical cdfs and pdfs for fitted distributions are given in Fig. 5, Fig .6 and Fig. 7, respectively. These Figures shows that: the curve of the pdf and cdf CTGD is closer to the curve of the sample of data than the curve of the pdf and cdf of TGD and GD . So, the CTGD is a better model than one based on the TGD and GD.

Table 4: Descriptive Statistics of Data Set 1, 2, 3

Data	mean	Median	Skewness	kurtosis
Set1	92.9295	57	2.13887	8.175586
Set2	12.0917	9.5	1.497109	6.252663
set3	9.365625	6.395	3.32566967	19.2259202

Table 5: MLE's of the parameters and respective SE's for various distributions for data set 1

Distribution	Parameter	Estimate	SE
CTGD	μ	70.79651	6.45060
	σ	60.12495	4.07872
	λ_1	1	0.36541
	λ_2	-1	0.47512
TGD	μ	64.54945	6.16876
	σ	64.88599	4.69145
	λ	0.40640	0.13464
GD	μ	51.7553	4.2376
	σ	59.6863	3.5792

Table 6: Goodness-of fit statistics using the selection criteria values for data set 1

Model	$-2\ell(\theta)$	AIC	BIC	W*	A*	K-S	P-value
CTGD	**2428.66**	**2436.66**	**2450.11**	**0.6657**	**4.38068**	**0.10106**	**0.0258**
TGD	2454.98	2460.98	2471.06	1.02504	6.25989	0.11316	0.00855
GD	2461.83	2465.83	2472.56	1.12753	6.81592	0.12869	0.00173

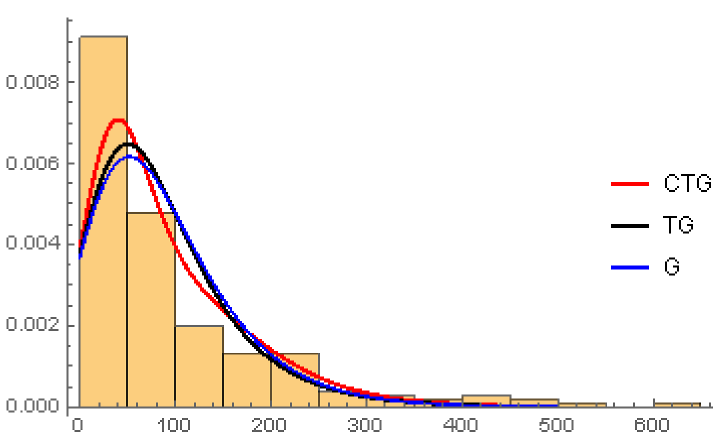
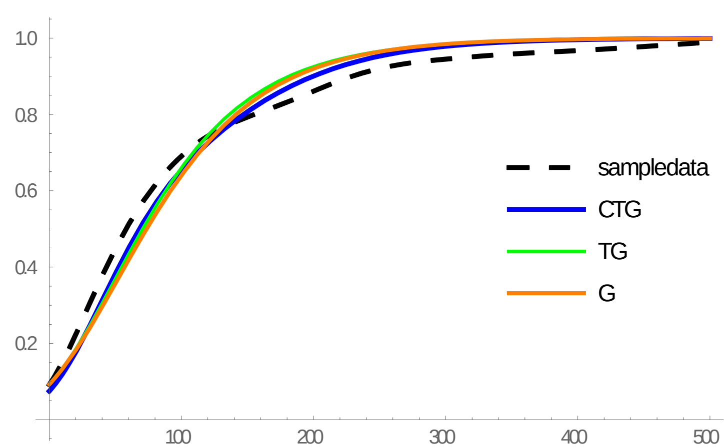


Fig.5: (a) Fitted pdf for data set 1



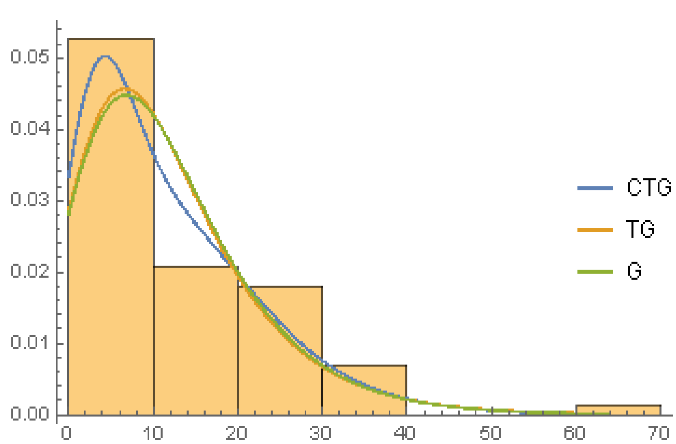
(b) Empirical cdf and theoretical cdf for data set 1.

Table 7: MLE's of the parameters and respective SE's for various distributions for Data Set 2

Distribution	Parameter	Estimate	SE
CTGD	μ	7.93564	1.20477
	σ	7.46740	1.10877
	λ_1	0.74310	0.43550
	λ_2	-1	0.76646
TGD	μ	7.75273	1.67731
	σ	8.57673	1.08341
	λ	0.20949	0.29311
GD	μ	6.82756	1.01171
	σ	8.21074	0.82291

Table 8: Goodness-of fit statistics using the selection criteria values for Data Set 2

Model	$-2\ell(\theta)$	AIC	BIC	W*	A*	K-S	P-value
CTGD	**530.568**	**538.568**	**547.674**	**0.24496**	**1.67107**	**0.15271**	**0.06959**
TGD	539.017	545.017	551.847	0.30932	2.06195	0.16339	0.0428
GD	539.506	543.506	548.059	0.31188	2.07660	0.1658	0.03818



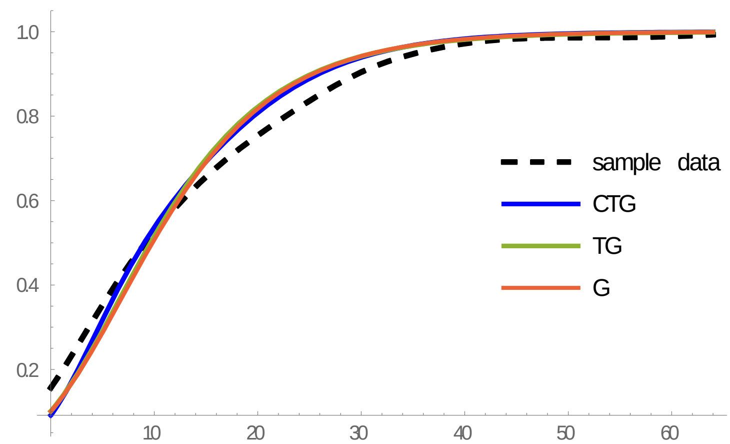


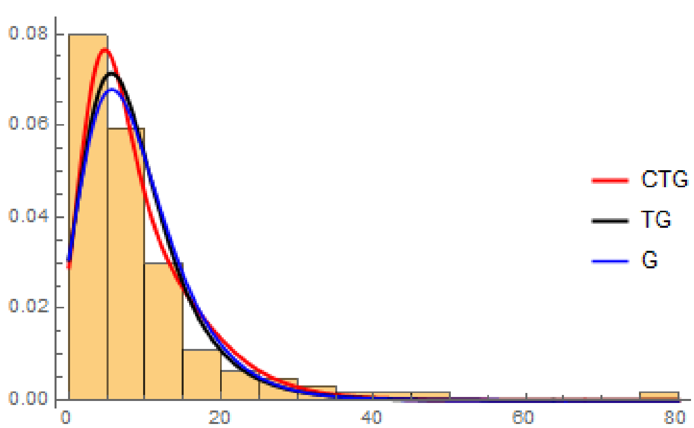
Fig.6: (a) Fitted pdf for Data Set 2 **(b) Empirical cdf and theoretical cdf for Data Set 2.**

Table 9: MLE's of the parameters and respective SE's for various distributions for Data Set 3

Distribution	Parameter	Estimate	SE
CTGD	μ	7.54780	0.92320
	σ	5.68244	0.58843
	λ_1	1	0.46787
	λ_2	-0.90670	0.56144
TGD	μ	7.13527	0.77758
	σ	6.10075	0.583667
	λ	0.49076	0.18292
GD	μ	5.66632	0.49852
	σ	5.44427	0.41357

Table 10: Goodness-of fit statistics using the selection criteria values for Data Set 3

Model	$-2\mathfrak{L}(\theta)$	AIC	BIC	W*	A*	K-S	P-value
CTGD	**851.286**	**859.286**	**870.694**	**0.22939**	**1.56577**	**0.088798**	**0.2651**
TGD	860.348	866.348	874.904	0.37397	2.33251	0.10209	0.1387
GD	865.204	869.204	874.908	0.43302	2.69157	0.11354	0.07373



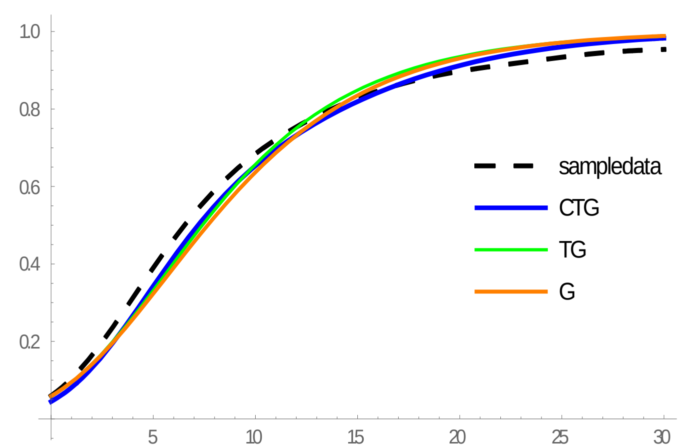


Fig.7: (a) Fitted pdf for Data Set 3 **(b) Empirical cdf and theoretical cdf for Data Set 3.**

8. Concluding Remarks

This article examines the cubic rank transmuted Gumbel (CTG) distribution, a novel generalized distribution. The distribution's hazard function, quantile function, moments, distribution of the order statistics, and entropy are among the structural aspects that are examined. The model parameters are estimated using a technique called maximum likelihood estimation. The importance and potential of the CTG distribution is demonstrated by examples from three real life datasets.

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