Journal of Al-Qadisiyah for Computer Science and Mathematics Vol. 15(1) 2023 , pp Math. 127–149

JQCM
University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Third-Order Differential Subordination and Superordination Results for Analytic Univalent Functions Using Hadamard Product Operator

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ARTICLE INFO

Article history:
Received: 13/01/2023
Revised form: 25/02/2023
Accepted : 27/02/2023
Available online: 31/03/2023

Keywords:

Analytic function, Differential Subordination, Sandwich results, Third-Order, Hadamard product Operator.

ABSTRACT

In this paper, we aim to obtain some results of third-order of differential subordination and superordination with sandwich theorems for analytic univalent functions using the operator $(S_{b,\alpha,\gamma}^{\mu,\delta})$. Some new results has been introduced.

MSC: 30C45

https://doi.org/ 10.29304/jqcm.2023.15.1.1189

1. Introduction

Assume that $H = H(U)$ be a class of functions which are analytic in the open unit disk $U = \{z: z \in \mathbb{C}\ and\ |z| < 1\}$.

Let $H[a, n]$ $(n \in \mathbb{N} = \{1, 2, 3, \cdots\}\ and\ a \in \mathbb{C})$, be the subclass of $H(U)$ and $H[a, n] = \{f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$. We denote by $A \subset H(U)$ the subclass of H which are analytic functions in U, and have normalized Taylor-Maclaurin series of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U). \tag{1.1}$$

Let f and g are analytic functions in the class $H(U)$, f is said to be subordinate to g, written as

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$$f \prec g \text{ in } U \text{ or } f(z) \prec g(z), \qquad (z \in U),$$

if there exists a Schwarz function $\omega \in H$, which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$, such that $f(z) = g(\omega(z))$, $(z \in U)$.

Additionally, if the function g is univalent in U, we get that (like[19]).

$$g(z) \prec f(z) \leftrightarrow g(0) = f(0) \text{ and } g(U) \subset f(U).$$

Abd [28] introduced the following operator:

$$S_{b,\alpha,\gamma}^{\mu,\delta} f(z) = z + \sum_{n=2}^{\infty} \beta_{n,\mu} a_n z^n, \tag{1.2}$$

where

$$\beta_{n,\mu} = \left(\frac{1+b}{n+b}\right)^{\delta} \left(\frac{\alpha + n\gamma}{\alpha + \gamma}\right)^{\mu}, \qquad (b \in \mathbb{C} \setminus \{0, -1, -2, \dots\}, \delta, \mu \in \mathbb{C}, z \in U, f \in A).$$

It is easily verified from (1.2) the identity:

$$z\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z)\right)' = (1+b) S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z) - b S_{b,\alpha,\gamma}^{\mu,\delta} f(z). \tag{1.3}$$

The recent work by Ponnusamy and Juneja [20] introduced the concept of third-order differential subordination. The recent work on differential subordination by some authors [4,8,12,16,15,17,19,21,23,24,25,26,27] drew attention from many experts in this area. see ([1,2,3,5,6,7,9,10,11,13,14,18,22]).

In this work, we investigate suitable classes of admissible function associated with an operator ($S_{b,\alpha,\gamma}^{\mu,\delta}$), with certain corollaries, several new findings on differential subordinations are made.

2. Preliminaries

The following lemmas and definition are needing in the proofs of our results.

***Definition* (2.1) [4]**: *Let $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ and suppose the function $h(z)$ is univalent in U. If the function $p(z)$ is analytic in U and satisfies the following third-order differential subordination:*

$$\phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \prec h(z), \tag{2.1}$$

then $p(z)$ is called a solution of the differential subordination (2.1). A univalent function $q(z)$ is called a dominant of the solutions of (2.1), if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2.1). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (2.1) is said to be the best dominant.

***Definition* (2.2) [26]**: *Let $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ and the function $h(z)$ be analytic in U. If the function $p(z)$ and $\phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z)$, are univalent in U and satisfies the following third-order differential superordination:*

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z), \tag{2.2}$$

then $p(z)$ is called a solution of the differential superordination (2.2). An analytic function $q(z)$ is called a subordinant of the solutions of (2.2), if $q(z) \prec p(z)$ for all $p(z)$ satisfying (2.2). A univalent subordinant $\tilde{q}(z)$ that satisfies the condition $q(z) \prec \tilde{q}(z)$ for all subordinant $q(z)$ of (2.2) is said to be the best subordinant.

***Definition* (2.3) [4]**: *Let Q be the set of all functions q that are analytic and univalent on the set $\overline{U} \setminus E(q)$, where*

$$E(q) = \{\xi : \xi \in \partial U : \lim_{z \to \xi} q(z) = \infty\},$$

and $\min|q'(\xi)| = p > 0$ *for* $\xi \in \partial U \backslash E(q)$. *Further, let the subclass of* $\mathbb{Q}$ *for which* $q(0) = \mathrm{a}$, *be denoted by* $\mathbb{Q}(\mathrm{a})$, *with*

$$\mathbb{Q}(0) = \mathbb{Q}_0 \text{ and } \mathbb{Q}(1) = \mathbb{Q}_1 = \{q \in \mathbb{Q}: q(0) = 1\}.$$

The subordination methodology is applied to an appropriate classes of admissible functions.

The following class of admissible functions is given by Antonino and Miller [4].

Definition (2.4) [4]: *Let* Ω *be a set in* $\mathbb{C}$ *and* $q \in \mathbb{Q}$ *and* $n \in N\backslash\{1\}$. *The class of admissible functions* $\Psi_n[\Omega, q]$ *consists of those functions* $\phi: \mathbb{C}^4 \times U \rightarrow \mathbb{C}$, *which satisfy the next admissibility conditions:*

$$\phi(r, s, t, e;\ z) \notin \Omega$$

whenever

$$r = q(\xi)\,, \qquad s = k\xi q'(\xi), \qquad \mathcal{R}e\left(\frac{t}{s} + 1\right) \geq k\, \mathcal{R}e\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)$$

and

$$\mathcal{R}e\left(\frac{e}{s}\right) \geq k^2 \mathcal{R}e\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U, \xi \in \partial U \backslash E(q)$, *and* $k \geq n$.

Lemma (2.1) [4]: *Let* $p \in H[a, n]$, *with* $n \geq 2$, *and* $q \in \mathbb{Q}(\mathrm{a})$ *satisfy the next conditions:*

$$\mathcal{R}e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0 \quad , \text{and} \quad \left|\frac{zp'(z)}{q'(\xi)}\right| \leq k,$$

where $z \in U\,, \xi \in \partial U \setminus E(q)$, *and* $k \geq n$ *.If* Ω *is a set in* $\mathbb{C}$, $\phi \in \Psi_n[\Omega, q]$ *and*

$$\phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \in \Omega,$$

then

$$p(z) \prec q(z), \qquad (z \in U).$$

Definition (2.5) [26]: *Let* Ω *be a set in* $\mathbb{C}$, $q \in H[\mathrm{a}, n]$ *and* $q'(z) \neq 0$ *and* $n \in N\backslash\{1\}$. *The class of admissible functions* $\Psi'_n[\Omega, q]$ *consists of those functions* $\phi : \mathbb{C}^4 \times \overline{U} \rightarrow \mathbb{C}$ *that satisfy the following admissibility conditions:*

$$\phi(r, s, t, e;\ \xi) \in \Omega$$

whenever

$$r = q(z) \quad , \quad s = \frac{zq'(z)}{m} \quad , \quad \mathcal{R}e\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \mathcal{R}e\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

and

$$\mathcal{R}e\left(\frac{e}{s}\right) \leq \frac{1}{m^2} \mathcal{R}e\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

where $z \in U, \xi \in \partial U$, *and* $m \geq n \geq 2$.

Lemma (2.2) [26]: *Let* $q \in H[\mathrm{a}, n]$ *with* $\phi \in \Psi'_n[\Omega, q]$. *If*

$$\phi(p(z), zp'(z), z^2p''(z), z^3p'''(z);\ z)$$

is univalent in U and $p \in \mathcal{Q}(a)$ satisfying the following conditions:

$$\mathcal{R}e\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{zp'(z)}{q'(z)}\right| \leq m,$$

where $z \in U, \xi \in \partial U$, and $m \geq n \geq 2$, then

$$\Omega \subset \{\phi(p(z), zp'(z), z^2p''(z), z^3p'''(z);\ z) : z \in U\}$$

implies that

$$q(z) \prec p(z), \qquad (z \in U).$$

The present paper utitizes the techniques on the third-order differential subordination and superordination outcomes of Antonino and Miller [4] and others [8,12,15,16,17,21,23,25,27] *and different conditions* (*see*[1,2,3,5,6,7,9,10,11,13,14,18,22]). *Certain classes of admissible functions are investigated in this idea, some properties of the third-order differential subordination and superordination for analytic functions in U related to the operator $\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z)\right)$ are also mentioned.*

3. Third-Order Differential Subordination Results:

In this part, we starting with a given set Ω and function q, and we create asset of acceptable function so that (1.2) *is true, to achieve this, we create the following new class of admissible functions, which required to establish the crucial third-order differential subordination theorems for the operator $(S_{b,\alpha,\gamma}^{\mu,\delta})$ defined by* (1.2).

Definition (3.1): *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap H_0$. The class $\ell_j[\Omega, q]$ of admissible functions consists of those functions $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the next admissibility conditions:*

$$\phi(u, v, x, y; z) \notin \Omega,$$

whenever

$$u = q(\xi), \qquad v = \frac{\xi k q'(\xi) + bq(\xi)}{(1+b)},$$

$$\mathcal{R}e\left(\frac{(1+b)[x(1+b) - 2bv] + b^2u}{v(1+b) - bu}\right) \geq k\ \mathcal{R}e\left(\frac{\xi q''(z)}{q'(z)} + 1\right)$$

and

$$\mathcal{R}e\left(\frac{(y + 3x)(1+b)^3 + ub^2(3+2b) + [3b^2 + 2(3b+1)][b(v-u) + v}{v(1+b) - bu}\right)$$

$$\geq k^2\mathcal{R}e\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U, \xi \in \partial U \backslash E(q)$ and $k \geq 2$.

Theorem (3.1): *Let $\phi \in \ell_j[\Omega, q]$. If the functions $f \in \mathbb{A}$ and $q \in \mathcal{Q}_0$ satisfies the next condition:*

$$\mathcal{R}e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \qquad \left|\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{q'(\xi)}\right| \leq k \tag{3.1}$$

and

$$\left\{\phi\left(S_{b,\alpha,\gamma}^{\mu,\delta}f(z),S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z),S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z),S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z);z\right):z\in U\right\}\subset\Omega, \tag{3.2}$$

then

$$S_{b,\alpha,\gamma}^{\mu,\delta}(z)\prec q(z)\qquad(z\in U).$$

Proof. *Let if, we put*

$$p(z)=S_{b,\alpha,\gamma}^{\mu,\delta}f(z). \tag{3.3}$$

Then from equation (1.3) *and* (3.3), *we have*

$$S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)=\frac{zp'(z)+bp(z)}{(1+b)}. \tag{3.4}$$

By similar argument, we get

$$S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)=\frac{z^2p''(z)+(1+2b)zp'(z)+b^2p(z)}{(1+b)^2}. \tag{3.5}$$

and

$$S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)=\frac{z^3p'''(z)+3(b+1)z^2p''(z)+(3b^2+3b+1)zp'(z)+b^3p(z)}{(1+b)^3}. \tag{3.6}$$

we will now build a transformation from $\mathbb{C}^4$ *to* $\mathbb{C}$ *by*

$$u(r,s,t,e)=r\quad,\qquad v(r,s,t,e)=\frac{s+br}{(1+b)},$$

$$x(r,s,t,e)=\frac{t+(1+2b)s+b^2r}{(1+b)^2}, \tag{3.7}$$

and

$$y(r,s,t,e)=\frac{e+3(1+b)t+(3b^2+3b+1)s+b^3r}{(1+b)^3}. \tag{3.8}$$

Let

$$\varphi(r,s,t,e)=\phi(u,v,x,y)=\phi\left(r,\frac{s+br}{(1+b)},\frac{t+(1+2b)s+b^2r}{(1+b)^2},\frac{e+3(1+b)t+(3b^2+3b+1)s+b^3r}{(1+b)^3}\right) \tag{3.9}$$

by applying Lemma (2.1), *Using equations* (3.3) *to* (3.8), *and from* (3.9), *we get*

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)=\phi\left(S_{b,\alpha,\gamma}^{\mu,\delta}f(z),S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z),S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z),S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z);z\right) \tag{3.10}$$

Hence, (3.2) *leads to*

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)\in\Omega.$$

we note that

$$\frac{t}{s}+1=\frac{(1+b)[x(1+b)-2bv]+b^2u}{v(1+b)-bu}.$$

and

$$\frac{e}{s}=\frac{(y+3x)(1+b)^3+ub^2(3+2b)+[3b^2+2(3b+1)][b(v-u)+v]}{v(1+b)-bu}.$$

Thus, we see that the admissibility condition in definition (3.1) *for* $\phi\in\ell_j[\Omega,q]$ *is equivalent to the admissibility condition in definition* (2.4) *for* $\varphi\in\Psi_2[\Omega,q]$ *as given with* $n=2$. *Therefore, by using* (3.1) *and applying Lemma* (2.1), *we get*

$$S^{\mu,\delta}_{b,\alpha,\gamma}(z)\prec q(z).$$

This completes the proof of theorem(3.1).□

The following outcome is an extension of Theorem (3.1), *when the behavior of* $q(z)$ *on* ∂U *is not known.*

Corollary (3.1): *Let* $\Omega\subset\mathbb{C}$ *and the function* q *be univalent in* U *with* $q(0)=1$. *Let* $\phi\in\ell_j[\Omega,q_\rho]$ *for some* $\rho\in(0,1)$, *where* $q_\rho(z)=q(\rho z)$. *If* $f\in\mathbb{A}$ *and* q_ρ *satisfy the next conditions:*

$$\mathcal{R}e\left(\frac{\xi q''_\rho(\xi)}{q'_\rho(\xi)}\right)\geq 0\quad,\quad\left|\frac{S^{\mu,\delta-1}_{b,\alpha,\gamma}f(z)}{q'_\rho(\xi)}\right|\leq k,\qquad \left(z\in U,k\geq 2,\xi\in\partial U\setminus E(q_\rho)\right)$$

and

$$\phi\left(S^{\mu,\delta}_{b,\alpha,\gamma}f(z),S^{\mu,\delta-1}_{b,\alpha,\gamma}f(z),S^{\mu,\delta-2}_{b,\alpha,\gamma}f(z),S^{\mu,\delta-3}_{b,\alpha,\gamma}f(z);z\right)\in\Omega,$$

then

$$S^{\mu,\delta}_{b,\alpha,\gamma}f(z)\prec q(z)\qquad(z\in U).$$

Proof. *Using Theorem* (3.1), *we can obtain*

$$S^{\mu,\delta}_{b,\alpha,\gamma}f(z)\prec q_\rho(z)\qquad(z\in U).$$

The following subordination property makes this result obvious

$$q_\rho(z)\prec q(z)\qquad(z\in U).$$

This completes the proof of corollary(3.1).□

In particular case, If $h(z)$ *conformal mapping of* U *onto* Ω, *such that* $\Omega\neq\mathbb{C}$ *is a simply connected domain, then* $\Omega=h(U)$, *and we define the class* $\ell_j[h(U),q]$ *is by* $\ell_j[h,q]$.

The follows outcome are immediate consequence of Theorem (3.1), *and corollary* (3.1), *respectively.*

Theorem (3.2): *Let* $\phi\in\ell_j[h,q]$. *If* $f\in\mathbb{A}$ *and* $q\in\mathcal{Q}_0$ *satisfy the next conditions:*

$$\mathcal{R}e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right)\geq 0\quad,\quad\left|\frac{S^{\mu,\delta-1}_{b,\alpha,\gamma}f(z)}{q'(\xi)}\right|\leq k,\tag{3.11}$$

and

$$\phi\left(S^{\mu,\delta}_{b,\alpha,\gamma}f(z),S^{\mu,\delta-1}_{b,\alpha,\gamma}f(z),S^{\mu,\delta-2}_{b,\alpha,\gamma}f(z),S^{\mu,\delta-3}_{b,\alpha,\gamma}f(z);z\right)\prec h(z),\tag{3.12}$$

then

$$S_{b,\alpha,\gamma}^{\mu,\delta} f(z) \prec q(z) \qquad (z \in U).$$

Corollary (3.2): *Let $\Omega \subset \mathbb{C}$ and the function q be univalent in U with $q(0) = 1$. Let $\phi \in \ell_j\,[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathbb{A}$ and q_ρ satisfy the next conditions:*

$$\mathcal{R}e\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geq 0 \quad , \quad \left|\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{q_\rho'(\xi)}\right| \leq k,$$

$$(z \in U, k \geq 2, \xi \in \partial U \backslash E(q_\rho)),$$

and

$$\phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right) \prec h(z),$$

then

$$S_{b,\alpha,\gamma}^{\mu,\delta} f(z) \prec q(z) \qquad (z \in U).$$

The best dominant of the differential subordination (3.12), *is given by the following result.*

Theorem (3.3): *Let the function h be univalent in U and let $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ and φ defined by* (3.9). *Assume that the differential equation:*

$$\varphi(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z) = h(z) \tag{3.13}$$

has a solution $q(z)$ with $q(0) = 1$, which satisfy condition (3.1). *If $f \in \mathbb{A}$ satisfies the condition* (3.12) *and if*

$$\phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right),$$

is analytic in U, then

$$S_{b,\alpha,\gamma}^{\mu,\delta} f(z) \prec q(z) \qquad (z \in U)$$

and $q(z)$ is the best dominant.

Proof. *By using Theorem* (3.1), *we deuced q is a dominant of* (3.12). *Since q satisfies* (3.13), *it is also a solution of* (3.12) *and therefore, q will be dominated by all dominants. Hence q is the best dominant. This completes the proof of Theorem*(3.3).□

With respect to Definition (3.1), *and in the particular case $q(z) = Mz$, $M > 0$, the class of admissible functions $\ell_j[\Omega, q]$ represented by $\ell_j[\Omega, M]$, is written as follows.*

Definition (3.2): *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\ell_j\,[\Omega, M]$ consists of those functions $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ such that*

$$\phi\left(\begin{array}{c} Me^{i\theta}, \left(\dfrac{k+b}{(1+b)}\right) Me^{i\theta}, \dfrac{L + [(2b+1)k + b^2]Me^{i\theta}}{(1+b)^2}, \\ \dfrac{N + 3(1+b)L + [(3b^2 + (3b+1) + b^3]Me^{i\theta}}{(1+b)^3}; z \end{array}\right) \notin \Omega, \tag{3.14}$$

whenever $z \in U$,

$$\mathcal{R}e(Le^{-i\theta}) \geq (k-1)kM,$$

and

$$\mathcal{R}e\left(Ne^{-i\theta}\right) \geq 0 \quad \forall\theta \in \mathcal{R};\ k \geq 2.$$

Corollary (3.3): *Let* $\phi \in \ell_j[\Omega, M]$. *If* $f \in \mathbb{A}$ *satisfies the next conditions:*

$$\left|S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)\right| \leq kM, \quad (z \in U;\ k \geq 2;\ M > 0),$$

and

$$\phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right) \in \Omega,$$

then

$$|S_{b,\alpha,\gamma}^{\mu,\delta} f(z)| < M.$$

In this special case, if $\Omega = q(U) = \{w : |w| < M\}$, *then the class* $\ell_j[\Omega, M]$ *is represented by* $\ell_j\ [M]$.

Corollary (3.4): *Let* $\phi \in \ell_j[M]$. *If* $f \in \mathbb{A}$ *satisfies the next conditions:*

$$\left|S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)\right| \leq kM, \quad (z \in U; k \geq 2; M > 0),$$

and

$$\left|\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right)\right| < M,$$

then

$$|S_{b,\alpha,\gamma}^{\mu,\delta} f(z)| < M.$$

Corollary (3.5): *Let* $k \geq 2$, *and* $M > 0$. *If* $f \in \mathbb{A}$ *satisfies the next conditions:*

$$|S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)| \leq kM,$$

and

$$\left|S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z) - S_{b,\alpha,\gamma}^{\mu,\delta} f(z)\right| < \frac{M}{b+1},$$

then

$$|S_{b,\alpha,\gamma}^{\mu,\delta} f(z)| < M.$$

Proof. *Let* $\phi(u, v, x, y; z) = v - u$ *and* $\Omega = h(U)$ *where*

$$h(z) = \frac{Mz}{|b+1|}, \quad (M > 0),$$

According to Corollary (3.3), *we shall present that* $\phi \in \ell_j[\Omega, M]$, *that is, the admissibility condition* (3.14) *is satisfied. This follows readily since it is seen that*

$$|\phi(u, v, x, y; z)| = \left|\frac{(k-1)Me^{i\theta}}{(1+b)}\right| \geq \frac{M}{|1+b|}$$

whenever $z \in U, \theta \in \mathcal{R}$ *and* $k \geq 2$. *The required result follows from corollary* (3.5) *proof is complete.*□

Definition (3.3): *Let* Ω *be a set in* $\mathbb{C}$, $q \in Q_1 \cap H_1$. *The class of admissible functions* $\ell_{j,1}[\Omega, q]$ *consists of those functions* $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ *that satisfy the next admissibility conditions:*

$$\phi(u, v, x, y; z) \notin \Omega,$$

whenever

$$u = q(\xi) \ , \quad v = \frac{k\xi q'(\xi) + (1+b)q(\xi)}{(1+b)},$$

$$\mathcal{R}e\left(\frac{(1+b)[x - 2v + u]}{(v - u)}\right) \geq k\ \mathcal{R}e(\frac{\xi q''(\xi)}{q'(\xi)} + 1)$$

and

$$\mathcal{R}e\left(\frac{(1+b)(y-u) + 3(1+b)(2+b)(u-x) + (3b^2+12b+11)(v-u)}{(v-u)}\right) \geq k^2 \mathcal{R}e\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U, \xi \in \partial U / E(q)$, *and* $k \geq 2$.

Theorem (3.4): *Let* $\phi \in \ell_{j,1}[\Omega, q]$. *If the functions* $f \in A$ *and* $q \in Q_1$ *satisfy the next conditions:*

$$\mathcal{R}e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0 \quad , \quad \left|\frac{S^{\mu,\delta-1}_{b,\alpha,\gamma} f(z)}{zq'(\xi)}\right| \leq k, \tag{3.15}$$

and

$$\left\{\phi\left(\frac{S^{\mu,\delta}_{b,\alpha,\gamma} f(z)}{z}, \frac{S^{\mu,\delta-1}_{b,\alpha,\gamma} f(z)}{z}, \frac{S^{\mu,\delta-2}_{b,\alpha,\gamma} f(z)}{z}, \frac{S^{\mu,\delta-3}_{b,\alpha,\gamma} f(z)}{z}; z\right); z \in U\right\} \subset \Omega, \tag{3.16}$$

then

$$\frac{S^{\mu,\delta}_{b,\alpha,\gamma} f(z)}{z} \prec q(z) \qquad (z \in U).$$

Proof. *Let if, we put*

$$p(z) = \frac{S^{\mu,\delta}_{b,\alpha,\gamma} f(z)}{z}. \tag{3.17}$$

Then from equation (1.3) *and* (3.17), *we have*

$$\frac{S^{\mu,\delta-1}_{b,\alpha,\gamma} f(z)}{z} = \frac{zp'(z) + 2p(z)}{(1+b)}. \tag{3.18}$$

By similar argument, we get

$$\frac{S^{\mu,\delta-2}_{b,\alpha,\gamma} f(z)}{z} = \frac{z^2 p''(z) + (2b+3)zp'(z) + (1+b)^2 p(z)}{(1+b)^2} \tag{3.19}$$

and

$$\frac{S^{\mu,\delta-3}_{b,\alpha,\gamma} f(z)}{z} = \frac{z^3 p'''(z) + 3(b+2)z^2 p''(z) + (3b^2+9b+7)zp'(z) + (1+b)^3 p(z)}{(1+b)^3}. \tag{3.20}$$

We will now build a transformation from $\mathbb{C}^4$ *to* $\mathbb{C}$ *by*

$$u(r,s,t,e) = r \ , \qquad v(r,s,t,e) = \frac{s+(1+b)r}{(1+b)},$$

$$x(r,s,t,e) = \frac{t+(2b+3)s+(1+b)^2 r}{(1+b)^2}, \tag{3.21}$$

and

$$y(r,s,t,e) = \frac{e+3(b+2)t+(3b^2+9b+7)s+(1+b)^3 r}{(1+b)^3}. \tag{3.22}$$

Let

$$\varphi(r,s,t,e) = \phi(u,v,x,y;z) = \phi\left(\begin{array}{c} r, \dfrac{s+(1+b)r}{(1+b)}, \dfrac{t+(2b+3)s+(1+b)^2 r}{(1+b)^2}, \\ \dfrac{e+3(b+2)t+(3b^2+9b+7)s+(1+b)^3 r}{(1+b)^3}; z \end{array} \right). \tag{3.23}$$

By applying Lemma (2.1), *Using equations* (3.17) *to* (3.20), *and from* (3.23), *we get*

$$\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}{z}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{z}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}{z}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}{z}; z \right). \tag{3.24}$$

Hence, clearly (3.16) becomes

$$\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega.$$

Note that

$$\frac{t}{s} + 1 = \frac{(1+b)[x - 2v + u]}{(v-u)}$$

and

$$\frac{e}{s} = \frac{(1+b)(y-u) + 3(1+b)(2+b)(u-x) + (3b^2+12b+11)(v-u)}{(v-u)}.$$

Thus, we see that the admissibility condition in definition (3.3) *for* $\phi \in \ell_{j,1}[\Omega, q]$ *is equivalent to the admissibility condition in definition* (2.4) *for* $\varphi \in \Psi_2[\Omega, q]$ *as given with* $n = 2$. *Therefore, by using* (3.15) *and applying Lemma* (2.1), *we get*

$$\frac{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}{z} \prec q(z).$$

This completes the proof of theorem(3.4).□

In particular case, If $h(z)$ *conformal mapping of* U *onto* Ω, *such that* $\Omega \neq \mathbb{C}$ *is a simply connected domain, then* $\Omega = h(U)$, *and we define the class* $\ell_{j,1}\,[h(U), q]$ *is by* $\ell_{j,1}\,[h, q]$.

The follows outcome are immediate consequence of Theorem (3.4).

Theorem (3.5): *Let* $\phi \in \ell_{j,1}[\Omega, q]$. *If* $f \in A$ *and* $q \in Q_1$ *satisfy the next conditions:*

$$\mathcal{R}e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \qquad \left|\frac{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}{zq'(\xi)}\right| \leq k \tag{3.25}$$

and

$$\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{z};z\right)\prec h(z) \tag{3.26}$$

then

$$\frac{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}{z}\prec q(z) \qquad (z\in U).$$

with respect to Definition (3.3), *and in the particular case* $q(z)=Mz,\ M>0$, *the class of admissible functions* $\ell_{j,1}[\Omega,q]$ *represented by* $\ell_{j,1}[\Omega,M]$, *is written as follows.*

Definition (3.4): *Let* Ω *be a set in* $\mathbb{C}$ *and* $M>0$. *The class of admissible functions* $\ell_{j,1}[\Omega,M]$ *consists of those functions* $\phi:\mathbb{C}^4\times U\to\mathbb{C}$ *such that*

$$\phi\left(\begin{array}{c}Me^{i\theta},\dfrac{K+(1+b)Me^{i\theta}}{(1+b)},\dfrac{L+\left((2b+3)K+(1+b)^2\right)Me^{i\theta}}{(1+b)^2},\\ \dfrac{N+3(b+2)L+\left((3b^2+9b+7)K+(1+b)^3\right)Me^{i\theta}}{(1+b)^3};z\end{array}\right)\notin\Omega, \tag{3.27}$$

whenever

$$z\in U,\qquad \mathcal{R}e(Le^{-i\theta})\ \geq\ (k-1)kM,$$

and

$$\mathcal{R}e\left(Ne^{-i\theta}\right)\geq 0 \qquad ,\forall\theta\in\mathcal{R};\ k\geq 2.$$

Corollary (3.6): *Let* $\phi\in\ \ell_{j,1}[\Omega,M]$. *If the function* $f\in\mathbb{A}$ *satisfies the next conditions:*

$$\left|\frac{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}{z}\right|\leq kM, \qquad (z\in U;k\geq 2:M>0)$$

and

$$\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{z};z\right)\in\Omega,$$

then

$$\left|\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{z}\right|<M.$$

In this special case, if $\Omega=q(U)=\{w:\ |w|<M\}$, *then the class* $\ell_{j,1}[\Omega,M]$ *is represented by* $\ell_j[M]$.

Corollary (3.7) : *Let* $\phi\in\ \ell_{j,1}[\Omega,M]$. *If the function* $f\in\mathbb{A}$ *satisfies the next conditions:*

$$\left|\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{z}\right|\leq kM \qquad (z\in U;k\geq 2:M>0)$$

and

$$\left|\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{z};z\right)\right|<M,$$

then

$$\left|\frac{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}{z}\right|<M.$$

Definition (3.5): *Let* Ω *be a set in* $\mathbb{C}$ *and* $q\in\mathcal{Q}_1\cap H_1$. *The class* $\ell_{j,2}[\Omega,q]$ *of admissible functions consists of those functions* $\phi:\mathbb{C}^4\times U\to\mathbb{C}$, *which satisfy the next admissibility conditions:*

$$\phi(u,v,x,y;z)\notin\Omega,$$

whenever

$$u=q(\xi),\qquad v=\frac{1}{(1+b)}\left[\frac{k\xi q'(\xi)+(1+b)\big(q(\xi)\big)^2}{q(\xi)}\right],$$

$$\mathcal{R}e\left(\frac{(1+b)[vx+2u^2-3uv]}{v-u}\right)\geq k\,\mathcal{R}e\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right),$$

and

$$\mathcal{R}e\big[vx(y-x)(1+b)^2-v(1+b)(x-v)(1-v-x+3u)-3v(b+1)(x-v)(v-u)+2(v-u)$$
$$+3u(1+b)(v-u)+(v-u)^2(1+b)\big((1+b)(v-5u)-3\big)+u^2(v-u)(1+b)^2\big]\,(v-u)^{-1}$$
$$\geq k^2\ \mathcal{R}e\left(\frac{\xi^2q'''(\xi)}{q'(\xi)}\right),$$

where $z\in U$, $\xi\in\partial U\backslash E(q)$ *and* $k\geq 2$.

Theorem (3.6): *Let* $\phi\in\ell_{j,2}[\Omega,q]$. *If the functions* $f\in\mathbb{A}$ *and* $q\in\mathcal{Q}_1$ *satisfy the next conditions:*

$$\mathcal{R}e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right)\geq 0,\qquad\left|\frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)q'(\xi)}\right|\leq k,\tag{3.28}$$

and

$$\left\{\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-4}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)};z\right):z\in U\right\}\subset\Omega\tag{3.29}$$

then

$$\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}\prec q(z),\qquad(z\in U).$$

Proof. *Let if, we put*

$$p(z)=\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}.\tag{3.30}$$

From equation (1.3) *and* (3.30), *we have*

$$\frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}=\frac{1}{(1+b)}\left[\frac{zp'(z)+(1+b)p^2(z)}{p(z)}\right]=\frac{A}{1+b},\tag{3.31}$$

By a similar argument, we get

$$\frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}=\frac{B}{1+b},\tag{3.32}$$

and

$$\frac{S_{b,\alpha,\gamma}^{\mu,\delta-4}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}=\frac{1}{1+b}[B+B^{-1}(C+A^{-1}D-A^{-2}C^2)],\tag{3.33}$$

where

$$B=\frac{zp'(z)}{p(z)}+(1+b)p(z)+\frac{\frac{z^2p''(z)+zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+(1+b)zp'(z)}{\frac{zp'(z)}{p(z)}+(1+b)p(z)}$$

$$C=\frac{z^2p''(z)+zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+(1+b)zp'(z)$$

and

$$D=\frac{z^3p'''(z)+3z^2p''(z)+zp'(z)}{p(z)}-\frac{3z^2\left(p'(z)\right)^2+3z^3p''(z)p'(z)}{\left(p(z)\right)^2}+2\left(\frac{zp'(z)}{p(z)}\right)^3+(1+b)z^2p''(z)+(1+b)zp'(z).$$

we will now build a transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$u(r,s,t,e)=r,\qquad v(r,s,t,e)=\frac{1}{1+b}\left[\frac{s}{r}+(1+b)r\right]=\frac{E}{1+b},$$

$$x(r,s,t,e)=\frac{1}{1+b}\left[\frac{s}{r}+(1+b)r+\frac{\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+(1+b)s}{\frac{s}{r}+(1+b)r}\right]=\frac{F}{1+b},\tag{3.34}$$

and

$$y(r,s,t,e)=\frac{1}{1+b}[F+F^{-1}(L+E^{-1}H-E^{-2}L^2)],\tag{3.35}$$

where

$$L=\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+(1+b)s$$

and

$$H=\frac{e}{r}+\frac{3t}{r}+\frac{s}{r}-3\left(\frac{s}{r}\right)^2-3\frac{st}{r^2}+2\left(\frac{s}{r}\right)^3+(1+b)(s+t).$$

Let

$$\varphi(r,s,t,e)=\phi(u,v,x,y)=$$

$$\phi\left(r, \frac{E}{1+b}, \frac{F}{1+b}, \frac{1}{1+b}\left[F + F^{-1}(L + E^{-1}H - E^{-2}L^2)\right]\right). \tag{3.36}$$

by applying Lemma (2.1), *Using equations* (3.30) *to* (3.33), *and from* (3.36), *we get*

$\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) =$

$$\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-4}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}; z\right). \tag{3.37}$$

Hence, clearly(3.29) *becomes*

$$\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega.$$

we note that

$$\frac{t}{s} + 1 = \frac{(1+b)[xv + 2u^2 - 3vu]}{(v-u)}$$

and

$$\frac{e}{s} = \Big[vx(y-x)(1+b)^2 - v(1+b)(x-v)(1-v-x+3u) - 3v(b+1)(x-v)(v-u) + 2(v-u) + 3u(1+b)(v-u) + (v-u)^2(11+b)\big((1+b)(v-5u) - 3\big) + u^2(v-u)(1+b)^2\Big](v-u)^{-1}.$$

Thus, we see that the admissibility condition in definition (3.5) *for* $\phi \in \ell_{j,2}[\Omega, q]$ *is equivalent to the admissibility condition in definition* (2.4) *for* $\varphi \in \Psi_2[\Omega, q]$ *as given with* $n = 2$. *Therefore, by using* (3.30) *and applying Lemma* (2.1), *we get*

$$\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)} \prec q(z).$$

This completes the proof of theorem(3.6).□

In particular case, If $h(z)$ *conformal mapping of* U *onto* Ω, *such that* $\Omega \neq \mathbb{C}$ *is a simply connected domain, then* $\Omega = h(U)$, *and we define the class* $\ell_{j,2}[h(U), q]$ *is by* $\ell_{j,2}[h, q]$.

The follows outcome are immediate consequence of Theorem (3.6).

Theorem **(3.7)**: *Let* $\phi \in \ell_{j,2}[\Omega, q]$. *If* $f \in \mathbb{A}$ *and* $q \in \mathbb{Q}_1$ *satisfy the conditions* (3.29) *and*

$$\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-4}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}; z\right) \prec h(z),$$

then

$$\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)} \prec q(z), \qquad (z \in U).$$

4. Third-Order Differential Superordination Results

This part analyzes the third-order differential superordination properties.

Definition (4.1): *Let* Ω *be a set in* $\mathbb{C}$ *and* $q \in \mathcal{Q}_0 \cap H_0$ *with* $q'(z) \neq 0$ *. The class of admissible functions* $\ell_j'[\Omega, q]$ *consists of those functions* $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$, *that satisfy the following admissibility conditions:*

$$\phi(u, v, x, y; \xi) \in \Omega,$$

whenever

$$u = q(z), \qquad v = \frac{zq'(z) + mbq(z)}{m(1+b)},$$

$$\mathcal{R}e\left(\frac{(1+b)[x(1+b) - 2bv] + b^2 u}{v(1+b) - bu}\right) \geq \left(\frac{1}{m}\right) \mathcal{R}e\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

and

$$\mathcal{R}e\left(\frac{(y+3x)(1+b)^3 + ub^2(3+2b) + [3b^2 + 2(3b+1)][b(v-u)+v]}{v(1+b) - bu}\right)$$

$$\geq \left(\frac{1}{m}\right)^2 \mathcal{R}e\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

where $z \in U, \xi \in \partial U$ *and* $m \geq 2$.

Theorem **(4.1)**: *Let* $\phi \in \ell_j'[\Omega, q]$. *If* $f \in \mathbb{A}$ *with* $S_{b,\alpha,\gamma}^{\mu,\delta} f(z) \in \mathcal{Q}_0$ *and if* $q \in H_0$ *with* $q'(z) \neq 0$, *satisfying the following conditions:*

$$\mathcal{R}e\left(\frac{\xi q''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{q'(z)}\right| \leq m \tag{4.1}$$

and the function

$$\phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right)$$

is univalent in U, *then*

$$\Omega \subset \left\{\phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right) : z \in U\right\}, \tag{4.2}$$

implies that

$$q(z) \prec S_{b,\alpha,\gamma}^{\mu,\delta} f(z), \qquad (z \in U).$$

Proof. *Let the function* $p(z)$ *be defined by* (3.3) *and* φ *given by* (3.9). *Since* $\phi \in \ell_j'[\Omega, q]$. *From* (3.10) *and* (4.2), *we have*

$$\Omega \subset \{\phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) : z \in U\}.$$

From (3.9), *we note this the admissibility condition in Definition* (4.1) *for* $\phi \in \ell_j'[\Omega, q]$ *is equivalent to the admissibility in Definition* (2.5) *for* $\varphi \in \Psi_n'[\Omega, q]$ *with* $n = 2$. *Hence* $\varphi \in \Psi_2'[\Omega, q]$ *and by using* (4.2) *and applying Lemma* (2.2), *we get*

$$q(z) \prec S_{b,\alpha,\gamma}^{\mu,\delta} f(z) \qquad (z \in U).$$

This completes the proof of theorem (4.1).□

In particular case, If $\mathrm{h}(z)$ *conformal mapping of* U *onto* Ω, *such that* $\Omega \neq \mathbb{C}$ *is a simply connected domain, then* $\Omega = \mathrm{h}(U)$, *and we define the class* $\ell_j[\mathrm{h}(U), q]$ *is by* $\ell_j\ [\mathrm{h}, q]$.

The follows outcome are immediate consequence of Theorem (4.1).

Theorem (4.2): *Let* $\phi \in \ell'_j[\mathrm{h}, q]$ *and let* h *analytic in* U. *If* $f \in \mathbb{A}$, *and* $S_{b,\alpha,\gamma}^{\mu,\delta} \in \mathbb{Q}_0$, *and if* $q \in H_0$ *with* $q'(z) \neq 0$, *satisfying the conditions* (4.1) *and the function*

$$\phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right)$$

is univalent in U, *then*

$$\mathrm{h}(z) \prec \phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right) \qquad (4.3)$$

implies that

$$q(z) \prec S_{b,\alpha,\gamma}^{\mu,\delta} f(z), \qquad (z \in U).$$

Theorem (4.1) *and* (4.2) *can only be used to get subordinant for the third-order differential superordination of the form* (4.2) *or* (4.3). *The next theorem gives the existence of the best subordinant of* (4.3) *for suitable* ϕ.

Theorem (4.3): *Let the function* h *univalent in* U, *and let* $\phi: \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ *and* φ *be defined by* (3.9). *Assume that the following differential equation:*

$$\varphi(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z) = \mathrm{h}(z) \qquad (4.4)$$

has a solution $q(z) \in \mathbb{Q}_0$. *If the functions* $f \in \mathbb{A}$, *and* $S_{b,\alpha,\gamma}^{\mu,\delta} \in \mathbb{Q}_0$ *and if* $q \in H_0$ *with* $q'(z) \neq 0$, *which satisfy the following conditions* (4.1) *and the function*

$$\phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right)$$

is analytic in U, *then*

$$h(z) \prec \phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right)$$

implies that

$$q(z) \prec S_{b,\alpha,\gamma}^{\mu,\delta} f(z), \qquad (z \in U).$$

and $q(z)$ *is the best subordinant.*

Proof. *From Theorem* (4.1) *and* (4.2), *we see that* q *is a subordinant of* (4.3). *Since* q *satisfies* (4.4), *it is also a solution of* (4.3) *and therefore,* q *will be subordinant by all subordinants. Hence* q *is the best subordinant.*

theorem proof is complete.□

Definition (4.2): *Let* Ω *be a set in* $\mathbb{C}$ *and* $q \in H_1$ *with* $q'(z) \neq 0$. *The class of admissible function* $\ell'_{j,1}[\Omega, q]$ *consists of those functions* $\phi: \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$, *that satisfy the next admissibility conditions:*

$$\phi(u, v, x, y; \xi) \in \Omega,$$

whenever

$$u = q(z), \qquad v = \frac{zq'(\xi) + m(1+b)q(z)}{m(1+b)},$$

$$\mathcal{R}e\left(\frac{(1+b)[x-2v+u]}{(v-u)}\right)\geq\frac{1}{m}\mathcal{R}e\left(\frac{zq''(z)}{q'(z)}+1\right)$$

and

$$\mathcal{R}e\left(\frac{(1+b)(y-u)+3(1+b)(2+b)(u-x)+(3b^2+12b+11)(v-u)}{(v-u)}\right)\geq\left(\frac{1}{m}\right)^2\mathcal{R}e\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z\in U,\xi\in\partial U$ *, and* $m\geq 2$.

Theorem (4.4): *Let* $\phi\in\ell'_{j,1}[\Omega,q]$. *If the function* $f\in A$ *and* $\frac{S^{\mu,\delta-1}_{b,\alpha,\gamma}f(z)}{z}\in Q_1$ *and if* $q\in H_1$ *with* $q'(z)\neq 0$, *satisfying the following conditions:*

$$\mathcal{R}e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right)\geq 0,\qquad\left|\frac{S^{\mu,\delta-1}_{b,\alpha,\gamma}f(z)}{zq'(z)}\right|\leq m \tag{4.5}$$

and the function

$$\phi\left(\frac{S^{\mu,\delta}_{b,\alpha,\gamma}f(z)}{z},\frac{S^{\mu,\delta-1}_{b,\alpha,\gamma}f(z)}{z},\frac{S^{\mu,\delta-2}_{b,\alpha,\gamma}f(z)}{z},\frac{S^{\mu,\delta-3}_{b,\alpha,\gamma}f(z)}{z};z\right),$$

is univalent in U, *then*

$$\Omega\subset\left\{\phi\left(\frac{S^{\mu,\delta}_{b,\alpha,\gamma}f(z)}{z},\frac{S^{\mu,\delta-1}_{b,\alpha,\gamma}f(z)}{z},\frac{S^{\mu,\delta-2}_{b,\alpha,\gamma}f(z)}{z},\frac{S^{\mu,\delta-3}_{b,\alpha,\gamma}f(z)}{z};z\right);z\in U\right\}, \tag{4.6}$$

implies that

$$q(z)\prec\frac{S^{\mu,\delta}_{b,\alpha,\gamma}f(z)}{z},\qquad(z\in U).$$

Proof. *Let* $p(z)$ *given by* (3.17) *and* φ *given by* (3.23). *Since* $\phi\in\ell'_{j,1}[\Omega,q]$, *from* (3.24) *and* (4.6), *we have*

$$\Omega\subset\{\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z):z\in U\}$$

From (3.21) *and* (3.22), *we note this the admissibility condition in definition* (4.1) *for* $\phi\in\ell'_{j,1}[\Omega,q]$ *is equivalent to the admissibility in definition* (2.4) *for* $\varphi\in\Psi'_n[\Omega,q]$ *with* $n=2$. *Hence* $\varphi\in\Psi'_2[\Omega,q]$ *and by using* (4.5) *and applying Lemma* (2, 2), *we get*

$$q(z)\prec\frac{S^{\mu,\delta}_{b,\alpha,\gamma}f(z)}{z},\qquad(z\in U).$$

This completes the proof of theorem (4.4).□

In particular case, If $h(z)$ *conformal mapping of* U *onto* Ω, *such that* $\Omega\neq\mathbb{C}$ *is a simply connected domain, then* $\Omega=h(U)$, *and we define the class* $\ell'_{j,1}[h(U),q]$ *is by* $\ell'_{j,1}[h,q]$.

The follows outcome are immediate consequence of Theorem (4.4).

Theorem (4.5): *Let* $\phi\in\ell'_{j,1}[h,q]$ *and* h *be an analytic function in* U. *If the functions* $f\in A$, *with* $q\in H_1$ *and* $q'(z)\neq 0$, *satisfying the next conditions* (4.5) *and the function*

$$\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{z};z\right)$$

is univalent in U, then

$$h(z)\prec\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{z},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{z};z\right),$$

implies that

$$q(z)\prec\frac{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}{z},\qquad (z\in U).$$

Definition (4.3): *Let Ω be a set in $\mathbb{C}$ and $q\in H_1$ with $q'(z)\neq 0$. The class of admissible functions $\ell'_{j,2}[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^4\times\overline{U}\to\mathbb{C}$, that satisfy the next admissibility conditions:*

$$\phi(u,v,x,y;\xi)\in\Omega,$$

whenever

$$u=q(z),\qquad v=\frac{1}{(1+b)}\left[\frac{zq'(z)+m(1+b)\big(q(z)\big)^2}{mq(z)}\right],$$

$$\mathcal{R}e\left(\frac{(1+b)[vx+2u^2-3uv]}{v-u}\right)\geq\frac{1}{m}\,\mathcal{R}e\left(\frac{zq''(z)}{q'(z)}+1\right),$$

and

$$\begin{aligned}\mathcal{R}e\,\big[&vx(y-x)(1+b)^2-v(1+b)(x-v)(1-v-x+3u)-3v(1+b)(x-v)(v-u)+2(v-u)\\&+3u(1+b)(v-u)+(v-u)^2(1+b)\big((1+b)(v-5u)-3\big)+u^2(v-u)(1+b)^2\big]\,(v-u)^{-1}\\&\geq\left(\frac{1}{m}\right)^2\mathcal{R}e\left(\frac{z^2q'''(z)}{q'(z)}\right)\end{aligned}$$

where $z\in U$, $\xi\in\partial U$ and $m\geq 2$.

Theorem (4.6): *Let $\phi\in\ell'_{j,2}[\Omega,q]$. If $f\in A$ and $\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)}\in Q_1$, and if $q\in H_1$ with $q'(z)\neq 0$, satisfying the next conditions:*

$$\mathcal{R}e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right)\geq 0,\qquad \left|\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)q'(z)}\right|\leq m \tag{4.7}$$

and the function

$$\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-4}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)};z\right)$$

is univalent in U, then

$$\Omega\subset\left\{\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta}f(z)},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1}f(z)},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-2}f(z)},\frac{S_{b,\alpha,\gamma}^{\mu,\delta-4}f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-3}f(z)};z\right):\in U\right\} \tag{4.8}$$

implies that

$$q(z) \prec \frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}, \qquad (z \in U).$$

Proof. *Let* $p(z)$ *given by* (3.30) *and* φ *given by* (3.36). *Since* $\phi \in \ell'_{j,2}[\Omega, q]$, *from* (3.37) *and* (4.8) *we have*

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z): z \in U\}.$$

From (3.34) *and* (3.35), *we note this the admissibility condition in Definition* (4.3) *for* $\phi \in \ell'_{j,2}[\Omega, q]$ *is equivalent to the admissibility in Definition* (2.4) *for* $\varphi \in \Psi'_n[\Omega, q]$ *with* $n = 2$. *Hence* $\varphi \in \Psi'_2[\Omega, q]$, *and by using* (4.7) *and applying Lemma* (2.2), *we get*

$$q(z) \prec \frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}, \qquad (z \in U).$$

This completes the proof of theorem(4.6).□

Theorem (4.7): *Let* $\phi \in \ell'_{j,2}[\Omega, q]$. *If the function* $f \in \mathbb{A}$ *and* $\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)} \in \mathbb{Q}_1$, *and if* $q \in H_1$ *with* $q'(z) \neq 0$, *satisfying the next conditions* (4.7) *and the function*

$$\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-4} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}; z\right),$$

is univalent in U, *then*

$$h(z) \prec \phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-4} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}; z\right)$$

implies that

$$q(z) \prec \frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}, \qquad (z \in U).$$

5. Sandwich Results

we arrive at the next sandwich Theorem by combining Theorems (3.2) *and* (4.2).

Theorem (5.1): *Let* h_1 *and* q_1 *are analytic functions in* U, *and let* h_2 *be an univalent in* U, *and* $q_2 \in \mathbb{Q}_0$ *with* $q_1(0) = q_2(0) = 1$ *and* $\phi \in \ell_j[h_2, q_2] \cap \ell'_j[h_1, q_1]$. *If the function* $f \in \mathbb{A}$ *with* $S_{b,\alpha,\gamma}^{\mu,\delta} f(z) \in \mathbb{Q}_0 \cap H_0$ *and the function*

$$\phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right),$$

is univalent in U, *and if the conditions* (3.1) *and* (4.1) *are satisfied, then*

$$h_1(z) \prec \phi\left(S_{b,\alpha,\gamma}^{\mu,\delta} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z), S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z); z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec S_{b,\alpha,\gamma}^{\mu,\delta} f(z) \prec q_2(z), \qquad (z \in U). \tag{5.1}$$

If, on the other hand, we arrive at the next sandwich theorem by combining Theorems (3.5) *and* (4.5).

Theorem **(5.2)**: *Let* h_1 *and* q_1 *are analytic functions in* U, *and let* h_2 *be an univalent in* U, *and* $q_2 \in Q_1$ *with* $q_1(0) = q_2(0) = 1$ *and* $\phi \in \ell_{j,1}[h_2, q_2] \cap \ell'_{j,1}[h_1, q_1]$. *If the function* $f \in A$ *with* $\frac{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}{z} \in Q_1 \cap H_1$ *and the function*

$$\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}{z}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{z}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}{z}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}{z}; z\right),$$

is univalent in U, *and if the conditions* (3.15) *and* (4.5) *are satisfied, then*

$$h_1(z) \prec \phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}{z}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{z}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}{z}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}{z}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}{z} \prec q_2(z), \qquad (z \in U). \tag{5.2}$$

we arrive at the next sandwich Theorem by combining Theorems (3.6) *and* (4.6).

Theorem **(5.3)**: *Let* h_1 *and* q_1 *are analytic functions in* U, *and let* h_2 *be an univalent in* U, *and* $q_2 \in Q_1$ *with* $q_1(0) = q_2(0) = 1$ *and* $\phi \in \ell_{j,2}[h_2, q_2] \cap \ell'_{j,2}[h_1, q_1]$. *If the function* $f \in A$ *with* $\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)} \in Q_1 \cap H_1$ *and the function*

$$\phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-4} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}; z\right),$$

is univalent in U, *and if the conditions* (3.28) *and* (4.7) *are satisfied, then*

$$h_1(z) \prec \phi\left(\frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-2} f(z)}, \frac{S_{b,\alpha,\gamma}^{\mu,\delta-4} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta-3} f(z)}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{S_{b,\alpha,\gamma}^{\mu,\delta-1} f(z)}{S_{b,\alpha,\gamma}^{\mu,\delta} f(z)} \prec q_2(z), \qquad (z \in U). \tag{5.3}$$

References

[1] R. Abd Al-Sajjad and W. G. Atshan, Certain analytic function sandwich theorems involving operator defined by Mittag-Leffler function, AIP Conference Proceedings, 2398, (2022), 060065, pp.1-8.

[2] S. A. Al-Ameedee, W. G. Atshan and F. A. Al-Maamori, On sandwich results of univalent functions defined by a linear operator, Journal of Interdisciplinary Mathematics, 23(4) (2020), 803-809.

[3] S. A. Al-Ameedee, W. G. Atshan and F. A. Al-Maamori, Some new results of differential subordinations for Higher-order derivatives of multivalent functions, Journal of Physics: Conference Series, 1804 (2021) 012111, 1-11.

[4] J. A. Antonino, S. S. Miller, Third-Order differential inequalities and subordination in the complex plane, Complex Var. Elliptic Equ., 56(2011), 439-454.

[5] W. G. Atshan and A. A. R. Ali, On some sandwich theorems of analytic functions involving Noor-Sălăgean operator, Advances in Mathematics: Scientific Journal, 9 (10) (2020), 8455-8467.

[6] W. G. Atshan and A. A. R. Ali, On sandwich theorems results for certain univalent functions defined by generalized operators, Iraqi Journal of Science, 62(7) (2021), pp: 2376-2383.

[7] W. G. Atshan and E. I. Badawi, On sandwich theorems for certain univalent functions defined by a new operator, Journal of Al-Qadisiyah for Computer Science and Mathematics, 11(2) (2019) ,72–80.

[8] W. G. Atshan, A. H. Battor and A. F. Abaas, On third-order differential subordination results for univalent analytic functions involving an operator, Journal of Physics: Conference Series, 1664 (2020) 012044, 1-19.
[9] W. G. Atshan, A. H. Battor and A. F. Abaas, Some sandwich theorems for meromorphic univalent functions defined by new integral operator, Journal of Interdisciplinary Mathematics, 24(3) (2021), 579-591.
[10] W. G. Atshan and R. A. Hadi, Some differential subordination and superordination results of p-valent functions defined by differential operator, Journal of Physics: Conference Series, 1664 (2020) 012043, 1-15.
[11] W. G. Atshan and S. A. Jawad, On differential sandwich results for analytic functions, Journal of Al-Qadisiyah for Computer Science and Mathematics, 11(1) (2019), 96-101.
[12] W. G. Atshan, R. A. Hiress and S. Altinkaya, On third-order differential subordination and superordination properties of analytic functions defined by a generalized operator, Symmetry, 14 (2) (2022), 418, 1-17.
[13] W. G. Atshan and S. R. Kulkarni, On application of differential subordination for certain subclass of meromorphically p-valent functions with positive coefficients defined by linear operator, Journal of Inequalities in Pure and Applied Mathematics, 10(2) (2009), Article 53, 11 pp.
[14] N. E. Cho, T. Bulboacᾰ and H. M. Srivastava, A general family of integral and associated subordination and superordination properties of some special analytic function classes, Apple. Math. Comput., 219(2012), 2278-2288.
[15] A. M. Darweesh, W. G. Atshan, A. H. Battor and A. A. Lupas, Third-order differential subordination results for analytic functions associated with a certain differential operator, Symmetry, 14(1) (2022), 99, 1-15.
[16] H. A. Farzana, B. A. Stephen and M. P. Jeyaramam, Third-order differential subordination of analytic functions defined by functional derivative operator, An Stiint. Univ. Al. I. Cuza Iasi Mat. (New Ser)., 62(2016), 105-120
[17] M. P. Jeyaraman and T. K. Suresh, Third-order differential subordination of analytic functions, Acta Univ. Apulensis Math. Inform., 35 (2013), 187–202.
[18] I. A. Kadum, W. G. Atshan and A. T. Hameed, Sandwich theorems for a new class of complete Homogeneous symmetric functions by using cyclic operator, Symmetry, 14(10) (2022),2223, 1-16.
[19] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, (2000).
[20] S. Ponnusamy and O. P. Juneja, Third-order differential inequalities in the complex plane, in Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, (1992).
[21] D. Rᾰducanu, Third order differential subordinations for analytic functions associated with generalized Mittag-Leffler functions, Mediterr. J. Math., 14 (4) (2017), Article ID 167, 1–18.
[22] M. A. Sabri, W. G. Atshan and E. El-Seidy, On sandwich-type results for a subclass of certain univalent functions using a new Hadamard product operator, Symmetry, 14(5) (2022), 931, pp.1-11.
[23] A. H. Saeed and W. G. Atshan, Third-order sandwich results for analytic functions defined by generalized operator, AIP Conference Proceedings, 2398, (2022), 060055, pp.1-14.
[24] H. Tang and E. Deniz, Third-order differential subordination results for analytic functions involving the generalized Bessel functions, Acta Math. Sci. Ser. B Engl. Ed.,34 (2014),1707-1719.
[25] H. Tang, H. M. Srivastava, E. Deniz and S. Li, Third-order differential superordination involving the generalized Bessel functions, Bull. Malays. Math. Sci. Soc., 38 (2015), 1669–1688.
[26] H. Tang, H. M. Srivastava, S. Li and L. Ma, Third order differential subordination and superordination results for meromorphically multivalent functions associated with the Liu-Srivastava operator, Abstr. App. Anal., 2014 (2014), Article ID 792175, 1-11.
[27] S. D. Theyab, W. G. Atshan and H. K. Abdullah, On some sandwich results of univalent functions related by differential operator, Iraqi Journal of Science, Vol. 63, no.11, (2022), pp. 4928-4936.
[28] S. J. Abd, New studies on Differential Subordination and Quasi-Subordination Theory, M.SC. Thesis, University of Al-Qadisiyah, Diwaniyah, Iraq, (2022).