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Statistically Limit and Statistically Continuous Functions

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ABSTRACT

In this paper, statistical convergence will be investigated. Then it sheds light on the study of some important concepts, characteristics and results of the previous study. In addition, we introduced a definition in limit and continuity which is the statistical limit and statistical continuous functions.

MSC..

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1- Introduction:

In general, it is known that the idea of statistical convergence for sequences of real numbers was introduced by H. Fast [2] and H. Steinhaus [11] and based on the asymptotic density of the AN group and the idea was presented independently in the same year 1951. The generalizations and general applications of this idea were investigated by various authors. Within this generalizations the convergence was generalized statistic on sequences in metric spaces (see, for example,[11]). Kosinak introduced and studied statistical convergence in Unitary Spaces Some applications were made to the theory of selection principles, and function spaces.excess spaces. After introducing Fast [2], it was a very fast investigation of [6,5], the concept of statistics convergence has been studied in probabilistic standard space and in intuitive fuzziness Normative areas, respectively, and also Maddox [8] presented the statistical convergence in locally convex areas.

The natural density of a set **K** of positive integers is defined by [2]:

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$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k\epsilon K\}|,$$

where $|\{k \le n : k\epsilon K\}|$ denotes the number of elements of K not exceeding n. It is clear that for a finite set K, we have $\delta(K) = 0$

If $\{a_k\}$ is a sequence such that a_k Satisfies property $\mathcal{P}$ for all k except a set of natural density zero, then we say that a_k satisfies $\mathcal{P}$ for "almost all k ", and we abbreviate this by "a.a. k."

If δ is any density, we define $\bar{\delta}$, the upper density associated with δ, by:

$$\bar{\delta}(\mathcal{A}) = 1 - \delta(\mathbb{N} \setminus \mathcal{A}).$$

For any set of natural numbers $\mathcal{A}$ [3]

2-Preliminaries

We will recall some definitions, theorems, and properties from previous studies in statistical convergence.

2.1 Definition [2]: The sequence $\{a_k\}$ **of real number** is statistically convergent to the number j provided that for each $\mathcal{E} > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |a_k - j| \ge \mathcal{E}\}| = 0,$$

(i.e.) $|a_k - j| < \epsilon$ a. a. k.

In this case, we write $a_k \overset{s.c.}{\rightarrow} j$

2.2 Theorem [10]: If $a_k \overset{s.c.}{\rightarrow} j_1$, $\mu_k \overset{s.c.}{\rightarrow} j_2$, then

(i) $\{a_k + \mu_k\} \overset{s.c.}{\rightarrow} j_1 + j_2$.

(ii) $\{\mu_k \cdot a_k\} \overset{s.c.}{\rightarrow} j_1 j_2$.

2.3 Theorem [10]: $a_k \overset{s.c.}{\rightarrow} j$ if and only if there exists a set

$$K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subset \mathbb{N},$$

such that, $\delta(K) = 1$ and $a_{k_n} \rightarrow j$ as $n \rightarrow \infty$.

2.4 Lemma: If $a_n \overset{s.c.}{\rightarrow} a_0, a_n \neq 0\ \forall n, a_0 \neq 0$ then $\frac{1}{a_n} \overset{s.c.}{\rightarrow} \frac{1}{a_0}$

Proof :Since $a_n \overset{s.c.}{\rightarrow} a_0 \Rightarrow \exists K \subseteq \mathbb{N}\ \ \delta(K) = 1$ and $a_{k_n} \rightarrow a_0$ as $n \rightarrow \infty$

Sothat $\left\{\frac{1}{a_{k_n}}\right\} \rightarrow \frac{1}{a_0}$ as $n \rightarrow \infty$

Therfore $\left\{\frac{1}{a_{k_n}}\right\}_{n=1}^{\infty}$ is sub sequence of $\left\{\frac{1}{a_n}\right\}$ and $\left\{\frac{1}{a_{k_n}}\right\} \rightarrow \frac{1}{a_0}$ **Thus** $\left\{\frac{1}{a_n}\right\} \overset{s.c.}{\rightarrow} \frac{1}{a_0}$ by (theorem(**2.5**))

2.5 Lemma: I f $a_n \overset{s.c.}{\rightarrow} a_o$, $i_n \overset{s.c.}{\rightarrow} i_o$, $i_n \neq 0\ \forall n, i_o \neq 0$, then $\frac{a_n}{i_n} \overset{s.c.}{\rightarrow} \frac{a_o}{i_o}$.

Proof : **Since** $a_n \overset{s.c.}{\rightarrow} a_0$ **then** $\frac{1}{a_n} \overset{s.c.}{\rightarrow} \frac{1}{a_0}$ by Lemma(**2. 4**)

Thus $i_n \cdot \frac{1}{a_n} = \frac{i_n}{a_n} \overset{s.c.}{\rightarrow} i_0 \cdot \frac{1}{a_0} = \frac{i_0}{a_0}$.

2.6 Lemma [4]: If $\{a_k\}$ is a sequence such that $a_k \overset{s.c.}{\rightarrow} j$, then a has a subsequence $\{a_{k_n}\}$ such that $\{a_{k_n}\} \longrightarrow j$.

2.7 Theorem [12]: If a sequence $\{a_k\}$ is statistically convergent, then the convergence limit point is unique.

2.8 Theorem [12]: If $a_k \rightarrow j$ then $a_k \overset{s.c.}{\rightarrow} j$. The converse need not be true in general.

2.9 Theorem [9]: Let $(a_n), (i_n)$ and (s_n) be real sequences such that $a_n \leqslant i_n \leqslant s_n$ for all $n \in \mathcal{K} \subseteq \mathbb{N}$, with $\delta(\mathcal{K}) = 1$ and $a_n \overset{s.c.}{\rightarrow} j, s_n \overset{s.c.}{\rightarrow} j$Then $i_n. \overset{s.c.}{\rightarrow} j$.

3- Main Results

We introduced the definitions of statistical limit and continuous statistical functions And we generalized and proved some theorems and properties in our new definition, and we added some new theorems, properties and terms within this new definition.

3.1 Theorem [1]: Let $\mathcal{F}:\mathcal{X}\to\mathcal{Y}$, $\mathcal{U}_0\in\mathcal{X}$ Then $\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U})=\jmath$ if and only if for every sequence $\{\mathcal{U}_n\}\to\mathcal{U}_0$ the sequence $\{\mathcal{F}(\mathcal{U}_n)\}\to\jmath$.

In the following we define the statistically limit of a function $\mathcal{F}$.

3.2 Definition:Let $\mathcal{F}:\mathcal{X}\to\mathcal{Y}$, $\mathcal{U}_0\in\mathcal{X}$, then the statistically limit of $\mathcal{F}$ at $\mathcal{U}_o$ is equal to $\jmath$ if For every sequence $\{\mathcal{U}_n\}$ in $\mathcal{X}$ converge statistically to $\mathcal{U}_o$, the sequence $\{\mathcal{F}(\mathcal{U}_n)\}$ converge statistically to $\jmath$. In this case, we write $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U})=\jmath$.

3.3 Example: Let $S.\lim_{\mathcal{U}\to 2}\frac{1}{1-\mathcal{U}}=-1$.

Proof: Suppose $\mathcal{U}_n\xrightarrow{s.c.}2$ then $\{1-\mathcal{U}_n\}$ is statistically convergent to (-1)

Hence $\frac{1}{1-\mathcal{U}_n}\xrightarrow{s.c}-1$, by theorem(2.5).

3.4 Theorem: If $\mathcal{F},\mathcal{G}:\mathcal{X}\to\Re$ are functions such that $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U})=\jmath_1$ and $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{G}(\mathcal{U})=\jmath_2$, then

1. $S.\lim_{\mathcal{U}\to\mathcal{U}_0}(\mathcal{F}\pm\mathcal{G})(\mathcal{U})=\jmath_1\pm\jmath_2$.
2 $S.\lim_{\mathcal{U}\to\mathcal{U}_0}(\mathcal{F}.\mathcal{G})(\mathcal{U})=\jmath_1.\jmath_2$.
3 If $\mathcal{G}(\mathcal{U})\neq 0\ \forall\mathcal{U}$ and $\jmath_2\neq 0$, then $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\frac{\mathcal{F}(\mathcal{U})}{\mathcal{G}(\mathcal{U})}=\frac{\jmath_1}{\jmath_2}$.

Proof : (1)We prove $S.\lim_{\mathcal{U}\to\mathcal{U}_0}(\mathcal{F}+\mathcal{G})(\mathcal{U})=\jmath_1+\jmath_2$ and similarly we can prove $S.\lim_{\mathcal{U}\to\mathcal{U}_0}(\mathcal{F}-\mathcal{G})(\mathcal{U})=\jmath_1-\jmath_2$

Let $\mathcal{U}_n\xrightarrow{s.c.}\mathcal{U}$.

Since $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U})=\jmath_1$ and $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{G}(\mathcal{U})=\jmath_2$. Then $\{\mathcal{F}(\mathcal{U}_n)\}\xrightarrow{s.c.}\jmath_1$ and $\{\mathcal{G}(\mathcal{U}_n)\}\xrightarrow{s.c.}\jmath_2$.

So that $\{\mathcal{F}(\mathcal{U}_n)+\mathcal{G}(\mathcal{U}_n)\}\xrightarrow{s.t}\jmath_1+\jmath_2$, by theorem(2.2),part(i).

Thus $S.\lim_{\mathcal{U}\to\mathcal{U}_0}[\mathcal{F}(\mathcal{U})+\mathcal{G}(\mathcal{U})]=\jmath_1+\jmath_2$.

(2) We prove $S.\lim_{\mathcal{U}\to\mathcal{U}_0}(\mathcal{F}.\mathcal{G})(\mathcal{U})=\jmath_1.\jmath_2$.

Let $\mathcal{U}_n\xrightarrow{s.t}\mathcal{U}$. Since $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U})=\jmath_1$ and $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{G}(\mathcal{U})=\jmath_2$. Then $\{\mathcal{F}(\mathcal{U}_n)\}\xrightarrow{s.c.}\jmath_1$ and $\{\mathcal{G}(\mathcal{U}_n)\}\xrightarrow{s.c.}\jmath_2$.

So that $\{\mathcal{F}(\mathcal{U}_n).\mathcal{G}(\mathcal{U}_n)\}\xrightarrow{s.c.}\jmath_1.\jmath_2$, by theorem(2.2),part(ii). Thus $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\{\mathcal{F}(\mathcal{U}).\mathcal{G}(\mathcal{U})\}=\jmath_1.\jmath_2$.

(3) Let $\mathcal{U}_n\xrightarrow{s.c.}\mathcal{U}$, so that $\mathcal{F}(\mathcal{U}_n)\xrightarrow{s.c.}\jmath_1$ and $\mathcal{G}(\mathcal{U}_n)\xrightarrow{s.c.}\jmath_2$,

$\left(\frac{\mathcal{F}}{\mathcal{G}}\right)(\mathcal{U}_n)=\frac{\mathcal{F}(\mathcal{U}_n)}{\mathcal{G}(\mathcal{U}_n)}\xrightarrow{s.c.}\frac{\jmath_1}{\jmath_2}$, by Lemma (2.5).

$S.\lim_{\mathcal{U}\to\mathcal{U}_0}\frac{\mathcal{F}(\mathcal{U}_n)}{\mathcal{G}(\mathcal{U}_n)}=\frac{\jmath_1}{\jmath_2}$.

3.5Theorem: If $\mathcal{F}:\mathcal{X}\to\mathcal{Y}$, $\mathcal{G}:\mathcal{Y}\to\mathcal{Z}$ such that $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U})=\jmath$ and $\mathcal{G}$ continuous at $\jmath$ then $S.\lim_{\mathcal{U}\to\mathcal{U}_0}(\mathcal{G}\circ\mathcal{F})(\mathcal{U})=\mathcal{G}(\jmath)$

Proof: Let $\mathcal{U}_n\xrightarrow{s.c.}\mathcal{U}_0$

Since $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U})=\jmath$ then $\{\mathcal{F}(\mathcal{U}_n)\}\xrightarrow{s.c.}\jmath$

There exist $K=\{k_1<k_2<\cdots<k_n<\cdots\}$ such that $\delta(K)=1$ and $\{\mathcal{F}(\mathcal{U}_{k_n})\}\to\jmath$ as $n\to\infty$ by (theorem(2.3))

Since $\mathcal{G}$ continuous at $\jmath$ then $\mathcal{G}(\mathcal{F}(\mathcal{U}_{k_n}))\to\mathcal{G}(\jmath)$

Therefore $\mathcal{G}\ (\mathcal{F}(\mathcal{U}_n)) \xrightarrow{\text{s.c.}} \mathcal{G}(j)$ as $n \to \infty$, by theorem(2.3).

Thus $S.\lim_{\mathcal{U}\to\mathcal{U}_0} \mathcal{G}\circ\mathcal{F}(\mathcal{U}) = \mathcal{G}(j)$.

3.6 Theorem: If $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U})$ exist then it is unique

Proof: Assume $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U}) = j_1$, $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U}) = j_2$

If $\mathcal{U}_n \xrightarrow{\text{s.c.}} \mathcal{U}_0$

Then $\{\mathcal{F}(\mathcal{U}_n)\} \xrightarrow{\text{s.c.}} j_1$ and $\{\mathcal{F}(\mathcal{U}_n)\} \xrightarrow{\text{s.c.}} j_2$

Since the convergent point of statistically convergent sequence is unique. Then $j_1 = j_2$.

3.7 Theorem: let $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ are functions from $\mathcal{A} \subseteq \Re$ into $\Re$ such that $\mathcal{F}(\mathcal{U}) \le \mathcal{G}(\mathcal{U}) \le \mathcal{H}(\mathcal{U})$ for all $\mathcal{U} \in \mathcal{A}$ and $\mathcal{U}_0 \in \mathcal{A}$

If $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U}) = S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{H}(\mathcal{U}) = j$ Then $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{G}(\mathcal{U})$ exists and equal to j

Proof: Let $\mathcal{U}_n$ be a sequence in $\mathcal{A}$ such that $\mathcal{U}_n \xrightarrow{\text{s.c.}} \mathcal{U}_0$

Since $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{F}(\mathcal{U}) = j$

And $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{H}(\mathcal{U}) = j$ then $\mathcal{F}(\mathcal{U}_n) \xrightarrow{\text{s.c.}} j$ and $\mathcal{H}(\mathcal{U}_n) \xrightarrow{\text{s.c.}} j$

Since $\mathcal{F}(\mathcal{U}_n) \le \mathcal{G}(\mathcal{U}_n) \le \mathcal{H}(\mathcal{U}_n)$ $\forall$ n Then $\mathcal{G}(\mathcal{U}_n) \xrightarrow{\text{s.c.}} j$

So that $S.\lim_{\mathcal{U}\to\mathcal{U}_0}\mathcal{G}(\mathcal{U}) = j$.

3.8 Definition [1]: Let $\mathcal{F}: \mathcal{X} \to \mathcal{Y}$, $\mathcal{U}_0 \in \mathcal{X}$. The function $\mathcal{F}$ is continuous $\mathcal{U}_0$ at if for every sequence $\{\mathcal{U}_n\}$ converg to $\mathcal{U}_0$ then$\{\mathcal{F}(\mathcal{U}_n)\} \to \mathcal{F}(\mathcal{U}_o)$. We say that $\mathcal{F}$ is continuous if is continuous at every point.

In this case, we write $C(\Re) = \{\mathcal{F}: \Re \to \Re \backslash \mathcal{F}$is continuous $\}$.

In the following we define the statistically continuous of a function $\mathcal{F}$.

3.9 Definition: Let $\mathcal{F}: \Re \to \Re$, be a function and $\mathcal{U}_0 \in \Re$. Then $\mathcal{F}$ is statistically continuous at $\mathcal{U}_0$ if $\forall \{\mathcal{U}_n\} \xrightarrow{\text{s.c.}} \mathcal{U}_0$ then $\{\mathcal{F}(\mathcal{U}_n)\} \xrightarrow{\text{s.c.}} \mathcal{F}(\mathcal{U}_o)$. We say that $\mathcal{F}$ is statistically continuous if is statistically continuous at every point in $\Re$.

3.10 Example: Let $\mathcal{F}: \Re \to \Re$ defined by $\mathcal{F}(\mathcal{U}) = \mathcal{U}^2$ $\forall\, \mathcal{U} \in \Re$ then $\mathcal{F}$ is statistically continuous

Proof: Let $\mathcal{U}_0 \in \Re$ To proof $\mathcal{F}$ is statistically continuous at $\mathcal{U}_o$.

Let $\mathcal{U}_n$ be a sequence in $\Re$ such that $\mathcal{U}_n \xrightarrow{\text{s.c.}} \mathcal{U}_0$ then ${\mathcal{U}_n}^2 \xrightarrow{\text{s.c.}} {\mathcal{U}_0}^2$

Thus $\mathcal{F}(\mathcal{U}_n) \xrightarrow{\text{s.c.}} \mathcal{F}(\mathcal{U}_o)$.

$\therefore$ $\mathcal{F}$ Is statistically continuous at $\mathcal{U}_o$.

3.11 ***Example*** Let $\mathcal{F}: \Re \to \Re$, be a function defin as

$$\mathcal{F}(\mathcal{U}) = \begin{cases} 1 & \text{if } \mathcal{U} \in \mathbb{Q} \\ 5 & \text{if } \mathcal{U} \in \mathbb{Q}' \end{cases}$$

Then $\mathcal{F}$ is is not statistically continuous at every point .

Proof: Let $\mathcal{U}_{o.} \in \Re$

1- If $\mathcal{U}_0 \in \mathbb{Q}$ then $\mathcal{F}(\mathcal{U}_0) = 1$, So there exist a $\mathcal{U}_n \in \mathbb{Q}'$such that$\{\mathcal{U}_n\} \to \mathcal{U}_0$

$\therefore \{\mathcal{U}_n\} \xrightarrow{\text{s.c.}} \mathcal{U}_0$

Since $\{\mathcal{F}(\mathcal{U}_n)\} = \{5\} \overset{\text{s.c.}}{\nrightarrow} 1 = \mathcal{F}(\mathcal{U}_0)$

$\therefore \mathcal{F}$ is not statistically continuous function at $\mathcal{U}_0$

2- If $\mathcal{U}_0 \in \mathbb{Q}'$then $\mathcal{F}(\mathcal{U}_0) = 5$, So there exist a $\mathcal{U}_n \in \mathbb{Q}$such that $\{\mathcal{U}_n\} \to \mathcal{U}_0$

$\therefore \{\mathcal{U}_n\} \xrightarrow{\text{s.c.}} \mathcal{U}_0$

Since$\{\mathcal{F}(\mathcal{U}_n)\} = \{1\} \overset{\text{s.c.}}{\nrightarrow} 5 = \mathcal{F}(\mathcal{U}_0)$

$\therefore \mathcal{F}$ is not statistically continuous Function at $\mathcal{U}_0$

3.1 2 Theorem: every continuous function is statistically continuous

Proof: Let $\mathcal{F}: \Re \to \Re$ be a continuous function, at $\mathcal{U}_o \in \Re$

To proof $\mathcal{F}$ is statistically continuous at $\mathcal{U}_o$

Suppose $\{\mathcal{U}_n\} \xrightarrow{s.c.} \mathcal{U}_o$

Then there exist $K = \{k_1 < k_2 < \cdots < k_n < \cdots\}$ such that $\delta(K) = 1$ and $\left\{\frac{1}{a_{k_n}}\right\}_{n=1}^{\infty} \to \mathcal{U}_o$ by (theorem(2.3))

$\therefore \{\mathcal{F}(\mathcal{U}_{k_n})\}$ Is sub sequence of $\{\mathcal{F}(\mathcal{U}_n)\}$ converge to $\mathcal{F}(\mathcal{U}_o)$ and $S(K) = 1$

$\therefore \{\mathcal{F}(\mathcal{U}_n)\} \xrightarrow{s.c.} \mathcal{F}(\mathcal{U}_o)$ as $n \to \infty$

$\therefore \mathcal{F}$ Is statistically continuous

3.13 Theorem: if $\mathcal{F}: \mathcal{X} \to \mathcal{Y}$ is statistically continuous and $\mathcal{G}: \mathcal{Y} \to \mathcal{Z}$ is statistically continuous at $\mathcal{F}(\mathcal{U}_o)$ then $\mathcal{G} \circ \mathcal{F}$ is statistically continuous at $\mathcal{U}_o$

Proof: Let $\{\mathcal{U}_n\}$ be a sequence in $\mathcal{X}$ statistically convergent to $\mathcal{U}_O$

Since $\mathcal{F}$ is statistically continuous at $\mathcal{U}_O$ then$\{\mathcal{F}(\mathcal{U}_n)\} \xrightarrow{s.c.} \{\mathcal{F}(\mathcal{U}_o)\}$ in $(\mathcal{Y})$

Since $\mathcal{G}$ is statistically continuous at $\mathcal{F}(\mathcal{U}_o)$

Then $\{\mathcal{G}(\mathcal{F}(\mathcal{U}_n))\} \xrightarrow{s.c.} \{\mathcal{G}(\mathcal{F}(\mathcal{U}_o))\}$

$\therefore \{(\mathcal{G} \circ \mathcal{F})(\mathcal{U}_n)\} \xrightarrow{s.c.} (\mathcal{G} \circ \mathcal{F})(\mathcal{U}_o)$

$\therefore (\mathcal{G} \circ \mathcal{F})$ Is statistically continuous at $\mathcal{U}_o, \forall\, \mathcal{U}_o \in \mathcal{U}$

The prove of the next Corollary is consequence from the fact every continuous function is statistically continuous

3.14 Corollary: If $\mathcal{F}: \mathcal{X} \to \mathcal{Y}$ is statistically continuous and $\mathcal{G}: \mathcal{Y} \to \mathcal{Z}$ is continuous then $\mathcal{G} \circ \mathcal{F}$ is statistically continuous

3.15 Corollary: If $\mathcal{F}: \mathcal{X} \to \Re$ is statistically continuous and $\mathcal{F}(\mathcal{U}) \neq 0, \forall \mathcal{U} \in \mathcal{X}$ then $\frac{1}{\mathcal{F}}$ is statistically continuous

Proof: Define $\mathcal{G}: \Re/\{0\} \to \Re$ by $\mathcal{G}(\mathcal{U}) = \frac{1}{\mathcal{U}}$, then $\mathcal{G}$ is continuous $\forall\, \mathcal{U} \in \Re\ /\{0\}$

$\therefore \mathcal{G} \circ \mathcal{F}$ is statistically continuous by (Corollary 3.12)

3.16 Theorem: If $\mathcal{F}, \mathcal{G}: \mathcal{X} \to \Re$ are real valued statistically continuous mapping then

1. $\mathcal{F} + \mathcal{G}$ is statistically continuous where $(\mathcal{F} + \mathcal{G})(\mathcal{U}) = \mathcal{F}(\mathcal{U}) + \mathcal{G}(\mathcal{U}) \forall \in \mathcal{X}$
2. $\mathcal{F}.\mathcal{G}$ is statistically continuous where$(\mathcal{F}.\mathcal{G})(\mathcal{U}) = \mathcal{F}(\mathcal{U}).\mathcal{G}(\mathcal{U})\ \forall \mathcal{U} \in \mathcal{X}$
3. If $\mathcal{G}(\mathcal{U}) \neq 0\ \forall\, \mathcal{U} \in \mathcal{X}$ then $\frac{\mathcal{F}}{\mathcal{G}}$ is statistically continuous where $\left(\frac{\mathcal{F}}{\mathcal{G}}\right)\mathcal{U} = \frac{\mathcal{F}(\mathcal{U})}{\mathcal{G}(\mathcal{U})}$
4. $|\mathcal{F}|$ is statistically continuous where$|\mathcal{F}|(\mathcal{U}) = |\mathcal{F}(\mathcal{U})|\ \forall \in \mathcal{X}$

Proof 1: let $\mathcal{U}_O \in \mathcal{X}$ and $\{\mathcal{U}_n\}$ be a sequence in $\mathcal{U}$ such that $\mathcal{U}_n \xrightarrow{s.c.} \mathcal{U}_O$

Since $\mathcal{F}$ Is statistically continuous then $\{\mathcal{F}(\mathcal{U}_n) \xrightarrow{s.c.} \mathcal{F}(\mathcal{U}_o)$

Also $\mathcal{G}$ Is statistically continuous then $\mathcal{G}(\mathcal{U}_n) \xrightarrow{s.c.} \mathcal{G}(\mathcal{U}_o)$

$\mathcal{F}(\mathcal{U}_n) + \mathcal{G}(\mathcal{U}_n) \xrightarrow{s.c.} \mathcal{F}(\mathcal{U}_o) + \mathcal{G}(\mathcal{U}_o)$

$(\mathcal{F} + \mathcal{G})(\mathcal{U}_n) \xrightarrow{s.c.} (\mathcal{F} + \mathcal{G})(\mathcal{U}_o)$

$\therefore \mathcal{F} + \mathcal{G}$ Is statistically continuous at $\mathcal{U}_O \in \mathcal{X}$

Proof 2: suppose $\mathcal{U}_n \xrightarrow{s.c.} \mathcal{U}_O$ since $\mathcal{F}$ is statistically continuous at $\mathcal{U}_O$ we have $\{\mathcal{F}(\mathcal{U}_n)\} \xrightarrow{s.c.} \mathcal{F}(\mathcal{U}_o)$

$\mathcal{G}$ Is statistically continuous at $\mathcal{U}_O$ we have $\mathcal{G}(\mathcal{U}_n) \xrightarrow{s.c.} \mathcal{G}(\mathcal{U}_o)$ and therefore

We have $\mathcal{F}.\mathcal{G}(\mathcal{U}_n) = \mathcal{F}(\mathcal{U}_n).\mathcal{G}(\mathcal{U}_n) \xrightarrow{s.c.} \mathcal{F}(\mathcal{U}_o).\mathcal{G}(\mathcal{U}_o) = \mathcal{F}.\mathcal{G}(\mathcal{U}_o)$

$\therefore \mathcal{F}.\mathcal{G}$ Is statistically continuous at $\mathcal{U}_o$

Proof 3: Since $\mathcal{G}$ is statistically continuous then $\frac{1}{\mathcal{G}}$ is statistically continuous

Then $\mathcal{F}\frac{1}{\mathcal{G}}$ is statistically continuous by (theorem (3.16),part2)

So that $\frac{\mathcal{F}}{\mathcal{G}}$ is statistically continuous

Proof 4: Define $\mathcal{G}: \mathfrak{R} \to \mathfrak{R}$ by $\mathcal{G}(\mathcal{U}) = |\mathcal{U}|$.It clear that $\mathcal{G}$ is continuous function

Since $\mathcal{F}$ Is statistically continuous and is $\mathcal{G}$ continuous

then $\mathcal{G} \circ \mathcal{F}$ Is statistically continuous by(corollary (3.14))

But $(\mathcal{G} \circ \mathcal{F})(\mathcal{U}) = \mathcal{G}(\mathcal{F}(\mathcal{U})) = |\mathcal{F}(\mathcal{U})| \; \forall \, \mathcal{U} \in \mathcal{X}$

$\therefore \mathcal{G} \circ \mathcal{F} = |\mathcal{F}|$ is statistically continuous function at u

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