Approximate Solution for Some Continuous Optimal Control Problems Using Veita-Pell Polynomials

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ABSTRACT

In this paper, Veita-Pell polynomials (VPPs) are first presented with some interesting properties. These properties are utilized to construct a general explicit formula of their operational matrix of derivative. Then an appropriate direct technique is suggested for solving quadratic optimal control problems based on VPPs and the idea of state parameterization algorithm. The resulting performance index optimal value shows the proposed method is able to provide a good treatment with fast convergence. The effectiveness of the presented method is illustrated by solving three numerical examples. The obtained results show that as the number of basis functions VPPs increase the error in the solution by the present method will be decreased and it may exactly close with the analytical one. This is the main modification of the algorithm and this contribution in the using special basis functions in obtaining an approximate solution with minimum number of VPPs and satisfactory accuracy.

MSC.

1. Introduction

Optimal control problems (OCPs) can be found in many disciplines based on mathematical modeling physics, economy and chemistry [1, 2]. Because of the complexity in most applications, the OCPs are solved either approximately or numerically. An important type of basic functions named basis orthogonal polynomials and wavelets functions [3-5]. Many researchers applied orthogonal polynomials and wavelets functions along with the
direct techniques to solve OCPs approximately. The optimal solution can be found by directly minimizing the cost function subject to constraints by approximating the dynamic optimal control problem by a finite dimensional nonlinear programming problem using either the parameterization technique or discretization technique [6-9].

Motivated by the above presentation, we are interested in applying Veita-Pell Polynomials (VPPs) to find the approximate solution for optimal control problems. As regards the properties of Veita-Pell family Functions, many authors had been applied them with their important results [10-16]. A novel algorithm is suggested in the present work to solve OCPs approximately. The proposed method is utilized together with VPPs to parameterize the states functions. The proposed method is aimed to obtain the accuracy and efficiency simultaneously. Hence the first goal of this work is to use VPPs basis functions to perform a parameterization technique for the system state variables. Some specific numerical test examples are included.

Rest of the paper is constructed as follows: In Section 2, Veita-Pell polynomials and their important properties are presented. Section 3 deals with the suggested direct algorithm for approximate solution of optimal control problem. Then the convergence of the suggested VPPs method is illustrated in section 4 by the numerical results. The chapter ends with section 5 by concluding the remarks.

### 2. Veita-Pell Polynomials and Their Properties

The Veita Pell Polynomials (VPPs) can be defined recursively as below:

\[ V_P(n)(\tau) = 2\tau V_P(n-1)(\tau) - V_P(n-2)(\tau) \]  

with initial conditions \( V_P(0)(\tau) = 0, \ V_P(1)(\tau) = 1. \)

In other words

\[ V_P(n)(\tau) = \begin{cases} 
0 & \text{if } n = 0, \\
1 & \text{if } n = 1, \\
2\tau V_P(n)(\tau) - V_P(n)(\tau) & \text{if } n > 1. 
\end{cases} \]

In addition, the general term of the VPPs can be constructed in terms the sums and the product respectively as below

\[ V_P(n)(\tau) = \sum_{i=0}^{[\frac{n}{2}]} (-1)^i \binom{n-2i}{i} 2^{n-2i}, \]  

\[ V_P(n)(\tau) = 2^n \prod_{i=1}^{n} \left( \tau - \cos \left( \frac{i\pi}{n+1} \right) \right). \]

Note that the general matrix form of VPPs can be written as below

\[ V_P(\tau) = GT(\tau)^T \]

where \( V_P(\tau) = [V_P_1(\tau)V_P_2(\tau) \cdots V_P_n(\tau)], \ T(\tau) = [1 \ \tau \ \tau^2 \ \ldots \ \tau^n] \)

and \( G \) is the lower triangle matrix constructed as

For odd \( n \)

\[ G = \begin{pmatrix} 
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 2^1 & 0 & 0 & \cdots & 0 \\
-1 & g_{2,1} & 2^2 & 0 & \cdots & 0 \\
0 & g_{3,1} & g_{3,2} & 2^3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cos \frac{n\pi}{2} & g_{n,1} & g_{n,2} & g_{n,3} & \cdots & 2^n 
\end{pmatrix} \]

For even \( n \), the last row in matrix \( G \) can be defined as \( (0 \ g_{n,1} \ g_{n,2} \ g_{n,3} \ \cdots \ g_{n,n-1} \ 2^n). \)

The entries \( g_{ij} \) in matrix \( G \) can be constructed as
\[ g_{i,j} = \begin{cases} 2(g_{i-1,j-1} - g_{i-2,j}) & j < i, \\ 0 & \text{otherwise}. \end{cases} \]

### 3. The VPPs Derivatives

The formulation of the derivative matrix of VPPs is discussed throughout this section.

Suppose that \( x(\tau) \) has an approximate solution in the truncated VPPs series form as

\[ x(\tau) = \sum_{i=0}^{n} a_i V_P(\tau) \]

Eq. 5 can be converted into a matrix form as

\[ x(\tau) = V_P(\tau)A \]

or \( x(\tau) = G T(\tau)^{T} A \) (6)

The first derivative of \( x(\tau) \) is defined as

\[ x(\dot{\tau}) = G \dot{T}(\tau)^{T} A = GBT(\tau)^{T} A \] (7)

where \( \dot{T}(\tau) = \begin{bmatrix} 0 & 1 & 2\tau & \cdots & n\tau^{n-1} \end{bmatrix} \)

and \( B^{T} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \)

### 4. The VPPs Algorithm for Solving Optimal Control Problems

Suppose that the process described by the system of nonlinear differential equations on \([-1,1]\)

\[ u(t) = f(x(t), \dot{x}(t)) \] (8)

with initial conditions

\[ x(-1) = \alpha, \ x(1) = \beta \] (9)

where: \( x(\cdot): [-1,1] \rightarrow \mathbb{R} \) is the state variable,

\( u(\cdot): [-1,1] \rightarrow \mathbb{R} \), is the control variable and \( f \) is a real-valued continuously differentiable function yielding the performance index \( J \) which is given by

\[ J(x(\tau), u(\tau)) = \int_{-1}^{1} F(x^2(\tau), u^2(\tau))d\tau \] (10)

Approximating the state variable \( x(\tau) \) using VPPs, gives

\[ x(\tau) = a^{T} V_P(\tau), \]

where \( a = [a_1, a_2, \ldots, a_N]^{T} \), is \( (N + 1) \times 1 \) vector of unknown parameters, then \( \ddot{x}(\tau) \) can be expressed as

\[ \ddot{x}(\tau) = a^{T} V_P(\tau) \] (12)
where $VP(\tau)$ is the derivative vector of $VP(\tau)$. Then obtain the approximation for the control variable by substituting Eq. 11 and Eq. 12 into Eq. 8 to obtain

$$u(\tau) = f \left( a^T VP(\tau), a^T VP(\tau) \right)$$

(13)

Then obtain $J$ as a function of the unknown $a_1, a_2, ..., a_n$ by calculating

$$J(a_1, a_2, ..., a_n) = \int_{-1}^{1} F \left( a^T VP(\tau)VP(\tau)^T a, a^T VP(\tau)VP(\tau)^T a \right) d\tau$$

The functional $J$ represents a nonlinear mathematical programming problem of unknown parameters $a_1, a_2, ..., a_n$.

The resulting nonlinear mathematical programming problem can be simplified as below

$$J(a_1, a_2, ..., a_n) = \frac{1}{2} a^T Ha$$

where $H = 2 \int_{-1}^{1} F \left( VP(\tau)VP(\tau)^T, VP(\tau)VP(\tau)^T \right) d\tau$.

The boundary conditions in Eq. (9) can be rewritten as

$$x(-1) = a^T VP(-1) = \alpha, \quad x(1) = a^T VP(1) = \beta.$$  

Finally, the obtained quadratic programming problem can be rewritten as follows

$$J = \frac{1}{2} a^T Ha$$

subject to $Fa - b = 0$

where $F = \begin{bmatrix} VP^T(-1) \\ VP^T(1) \end{bmatrix}$, $b = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

Using Lagrange multiplier technique, one can obtain the optimal values of the unknown parameters $a^*$,

$$a^* = H^{-1} F^T (FH^{-1}F^T)^{-1} b.$$  

5. Numerical Results

The suggested modification in the state direct parameterization algorithm based on VPPs is applied to solve optimal control problems. This algorithm starts with an approximation to the state variable in terms of VPPs which satisfy the given boundary conditions. The modification in the assumption has succeeded to give an approximate solution with less number of VPPs terms. Numerical test examples are solved and the obtained results show that the suggested method is efficient and only small numbers of terms are used to reach the convergence.

All problems considered in this paper have analytical solution to allow the validation of the algorithm comparing with exact solution results.

Example 1: This problem is concerned with minimization of

$$J = \frac{1}{2} \int_{-1}^{1} (u(\tau)^2 + x(\tau)^2) d\tau, \quad \tau \in [-1, 1],$$

subject to $u(\tau) = 2 \dot{x}(\tau)$,

with the conditions $x(-1) = 0, \quad x(1) = 0.5$, 
and the exact solution \( x(\tau) = \frac{e \sinh(\tau)}{e^{2-1}}, \; u(\tau) = \frac{e \cosh(\tau)}{e^{2-1}} \).

Table1 presents the approximate optimal values of the performance index at different values of \( n \). In this table, we have computed the absolute error between the exact performance index value and the approximate values. In Figures 1 and 2, the exact and approximate solutions for \( x(\tau) \) and \( u(\tau) \) respectively for various values of \( n \).

**Table1:** The approximate values of \( J \) and absolute error for Example1.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( J_{\text{approximate}} )</th>
<th>Absolute Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.333333333333</td>
<td>0.0050745119580</td>
</tr>
<tr>
<td>3</td>
<td>0.3285984848485</td>
<td>3.3966347365e-04</td>
</tr>
<tr>
<td>4</td>
<td>0.3282593375616</td>
<td>5.1618682400e-07</td>
</tr>
<tr>
<td>5</td>
<td>0.328258307090</td>
<td>9.3341760432e-09</td>
</tr>
<tr>
<td>6</td>
<td>0.328258213798</td>
<td>5.039024753e-012</td>
</tr>
</tbody>
</table>

Fig.1 The behaviour of \( x(\tau) \) for Example1 at \( m = 3, 4, 5, 6 \) and exact solution.

Fig.2 The behaviour of \( u(\tau) \) for Example1 at \( m = 3, 4, 5, 6 \) and exact solution.
Example 2: Consider the following optimal control problem

\[
\min J = \frac{1}{4} \int_{-1}^{1} (u^2(\tau) + x^2(\tau)) d\tau, \tau \in [-1, 1],
\]

when \( u(\tau) = 2\dot{x}(\tau) + x(\tau) \),
and \( x(-1) = 1, x(1) = 0.2819695348 \),
are satisfied. We have obtained the analytical solution

\[
x(\tau) = A e^{\sqrt{2}\tau} + (1 - A) e^{-\sqrt{2}\tau}
\]
\[
u(\tau) = A(\sqrt{2} + 1) e^{\sqrt{2}\tau} - (1 - A) e^{-\sqrt{2}\tau}
\]

\[
J = \frac{e^{-2\sqrt{2}}}{2} \left( (\sqrt{2} + 1)(e^{4\sqrt{2}} - 1) \right) A^2 + \frac{e^{-2\sqrt{2}}}{2} \left( \sqrt{2} - 1 \right)(e^{2\sqrt{2}} - 1)(1 - A^2)
\]

where \( A = \frac{2\sqrt{2} - 3}{2\sqrt{2} - 3 - (e^{\sqrt{2}})} \).

Table 2 shows the approximate optimal values of the performance index at different values of \( n \). In this table, we have computed the absolute error between the exact performance index value and the approximate values. In Figures 3 and 4, the exact and approximate solutions for \( x(\tau) \) and \( u(\tau) \) respectively for various values of \( n \).

Table 2: The approximate values of \( J \) and absolute error for Example 2.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( J_{\text{approximate}} )</th>
<th>( \text{Absolute Errors} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.192909298093</td>
<td>0.058453436781</td>
</tr>
<tr>
<td>3</td>
<td>0.194298641535</td>
<td>0.001389342577</td>
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<tr>
<td>4</td>
<td>0.192909298093</td>
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<td>5</td>
<td>0.192909445024</td>
<td>1.469309399e-07</td>
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<tr>
<td>6</td>
<td>0.192909280931</td>
<td>8.645119897e-10</td>
</tr>
</tbody>
</table>

Fig. 3 The behaviour of \( x(\tau) \) for Example 2 at \( m = 3, 4, 5, 6 \) and exact solution.
Example 3: Consider the following optimal control problem
\[
\min J = \frac{1}{2} \int_{-1}^{1} \left( \frac{1}{2} u^2(\tau) + \frac{5}{8} x^2(\tau) + \frac{1}{2} x(\tau) u(\tau) \right) d\tau, \quad \tau \in [-1,1],
\]
where \( u(\tau) = -2\dot{x}(\tau) + \frac{1}{2} x(\tau) \),
and \( x(-1) = 1, \ x(1) = 0.2819695348 \),
are satisfied. We have obtained the analytical solution
\[
x(\tau) = \frac{\cosh(1 - \tau)}{\cosh 1}, \quad u(\tau) = \frac{-(\tanh(1 - \tau) + 0.5)cosh(1 - \tau)}{cosh 1}
\]
and \( J = 0.387970779 \).

Table 3 shows the approximate optimal values of the performance index at different values of \( n \). In this table, we have computed the absolute error between the exact performance index value and the approximate values. In Figures 5 and 6, the exact and approximate solutions for \( x(\tau) \) and \( u(\tau) \) respectively for various values of \( n \).

Table 3: The approximate values of \( J \) and absolute error for Example 3.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( J_{\text{approximate}} )</th>
<th>Absolute Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.38088385111</td>
<td>8.677314204e-05</td>
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<tr>
<td>4</td>
<td>0.380799833623</td>
<td>2.75564915e-06</td>
</tr>
<tr>
<td>5</td>
<td>0.380797080316</td>
<td>2.33830569e-09</td>
</tr>
<tr>
<td>6</td>
<td>0.380797078005</td>
<td>2.71849675e-011</td>
</tr>
</tbody>
</table>
6. Conclusion

An efficient numerical method based on Veita-Pell polynomials for solving continuous optimal control problem are considered in this paper. In the proposed method, a power series solution in terms of VPPs has been chosen such that it satisfies the given boundary conditions. Plugging this series solution into the given optimal control problem and using appropriate a special technique, an optimization problem with unknown Veita-Pell coefficients is obtained. These are the two main modifications and novelty of the procedure and this small contribution in the assumption of power series solution in terms of VPPs results in obtaining the approximate solution with less number of terms with good accuracy. Three numerical examples are provided to confirm the reliability and effectiveness of the suggested method.
References


