University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

Approximate Solution for Some Continuous Optimal Control Problems Using Veita-Pell Polynomials

Suha N. Shihab [a] **,** ***Eman H. Ouda*** [b] **,** ***Rasha J. Mitlif*** [a,b] *****

[a] *Department of Applied Sciences, University of Technology, Baghdad, Iraq.*

Email: Suha.N.shihab@uotechnology.edu.iq.

[b] *Department of Applied Sciences, University of Technology, Baghdad, Iraq..*

Email: Eman.H.Ouda@uotechnology.edu.iq.

[a,b]*Department of Applied Sciences, University of Technology, Baghdad, Iraq.*

Email: rasha.j.mitlif@uotechnology.edu.iq.

ARTICLE INFO

Article history:
Received: 11 /03/2023
Rrevised form: 10 /04/2023
Accepted: 12 /04/2023
Available online: 30 /06/2023

Keywords:

Vieta-Pell polynomials, continuous optimal control problem, direct method, Lagrange multipliers, and quadratic programming technique.

ABSTRACT

In this paper, Veita-Pell polynomials (VPPs) are first presented with some interesting properties. These properties are utilized to construct a general explicit formula of their operational matrix of derivative. Then an appropriate direct technique is suggested for solving quadratic optimal control problems based on VPPs and the idea of state parameterization algorithm. The resulting performance index optimal value shows the proposed method is able to provide a good treatment with fast convergence. The effectiveness of the presented method is illustrated by solving three numerical examples. The obtained results show that as the number of basis functions VPPs increase the error in the solution by the present method will be decreased and it may exactly close with the analytical one. This is the main modification of the algorithm and this contribution in the using special basis functions in obtaining an approximate solution with minimum number of VPPs and satisfactory accuracy.

MSC..

https://doi.org/ 10.29304/jqcm.2023.15.2.1246

1. Introduction

Optimal control problems (OCPs) can be found in many disciplines based on mathematical modeling physics, economy and chemistry [1, 2]. Because of the complexity in most applications, the OCPs are solved either approximately or numerically. An important type of basic functions named basis orthogonal polynomials and wavelets functions [3-5]. Many researchers applied orthogonal polynomials and wavelets functions along with the

∗Corresponding author : Suha N. Shihab

Email addresses: Suha.N.shihab@uotechnology.edu.iq.

Communicated by 'sub etitor'

direct techniques to solve OCPs approximately. The optimal solution can be found by directly minimization the cost function subject to constraints by approximating the dynamic optimal control problem by a finite dimensional nonlinear programming problem using either the parameterization technique or discretization technique [6-9].

Motivated by the above presentation, we are interested in applying Veita-Pell Polynomials (VPPs) to find the approximate solution for optimal control problems. As regards the properties of Veita-Pell family Functions, many authors had been applied them with their important results [10-16].A novel algorithm is suggested in the present work to solve OCPs approximately. The proposed method is utilized together with VPPs to parameterize the states functions. The proposed method is aimed to obtain the accuracy and efficiency simultaneously. Hence the first goal of this work is to use VPPs basis functions to perform a parameterization technique for the system state variables. Some specific numerical test examples are included.

Rest of the paper is constructed as follows: In Section 2, Veita-Pell polynomials and their important properties are presented. Section 3 deals with the suggested direct algorithm for approximate solution of optimal control problem. Then the convergence of the suggested VPPs method is illustrated in section 4 by the numerical results. The chapter ends with section5 by concluding the remarks.

2. Veita-Pell Polynomials and Their Properties

The Veita Pell Polynomials (VPPs) can be defined **recursively** as below:

$$VP_n(\tau) = 2\tau VP_{n-1}(\tau) - VP_{n-2}(\tau) \quad (1)$$

with initial conditions $VP_0(\tau) = 0,\ VP_1(\tau) = 1$.

In other words

$$VP_n(\tau) = \begin{cases} 0 & if\ n = 0, \\ 1 & if\ n = 1, \\ 2\tau VP_n(\tau) - VP_n(\tau) & if\ n > 1. \end{cases}$$

In addition, the general term of the **VPPs** can be constructed in terms the sums and the product respectively as below

$$VP_n(\tau) = \sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor} (-1)^i \binom{n-i}{i} 2^{n-2i}\tau^{n-2i}. \quad (2)$$

$$VP_n(\tau) = 2^n \prod_{i=1}^{n} \left(\tau - cos\left(\frac{i\pi}{n+1}\right)\right). \quad (3)$$

Note that the general matrix form of VPPs can be written as below

$$VP(\tau) = GT(\tau)^T \quad (4)$$

where $VP(\tau) = [VP_1(\tau)VP_2(\tau)\cdots VP_n(\tau)]$, $T(\tau) = [1 \quad \tau \quad \tau^2 \quad \cdots \quad \tau^n]$

and G is the lower triangle matrix constructed as

For odd n

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2^1 & 0 & 0 & \cdots & 0 \\ -1 & g_{2,1} & 2^2 & 0 & \cdots & 0 \\ 0 & g_{3,1} & g_{3,2} & 2^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ cos\frac{n}{2}\pi & g_{n,1} & g_{n,2} & g_{n,3} & \cdots & 2^n \end{pmatrix}$$

For even n, the last row in matrix G can be defined as $(0 \quad g_{n,1} \quad g_{n,2} \quad g_{n,3} \quad \cdots \quad g_{n,n-1} \quad 2^n)$.

The entries g_{ij} in matrix G can be constructed as

$$g_{i,j}=\begin{cases}2\left(g_{i-1,j-1}-g_{i-2,j}\right) & j<i,\\ 0 & \text{otherwise.}\end{cases}$$

3. The VPPs Derivatives

The formulation of the derivative matrix of VPPs is discussed throughout this section.

Suppose that $x(\tau)$ has an approximate solution in the truncated VPPs series form as

$$x(\tau)=\sum_{i=0}^{n} a_i VP_i(\tau) \qquad (5)$$

Eq. 5 can be converted into a matrix form as

$x(\tau)=VP(\tau)A$ where $A=[a_0\ a_1\ \ a_2\ \ \ldots a_n]^T$,

$$\text{or}\quad x(\tau)=GT(\tau)^T A \qquad (6)$$

The first derivative of $x(\tau)$ is defined as

$$\dot{x}(\tau)=G\dot{T}(\tau)^T A=GBT(\tau)^T A \qquad (7)$$

where $\dot{T}(\tau)=[0\quad 1\quad 2\tau\quad \cdots\quad n\tau^{n-1}]$

and $B^T=\begin{bmatrix}0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & n\\ 0 & 0 & 0 & \ldots & 0\end{bmatrix}$

4. The VPPs Algorithm for Solving Optimal Control Problems

Suppose that the process described by the system of nonlinear differential equations on $[-1,1]$

$$u(t)=f\left(x(t),\dot{x}(t)\right) \qquad (8)$$

with initial conditions

$$x(-1)=\alpha,\ x(1)=\beta \qquad (9)$$

where: $x(\cdot):[-1,1]\to\Re$ is the state variable,

$u(\cdot):[-1,1]\to\Re$, is the control variable and f is a real-valued continuously differentiable function yielding the performance index J which is given by

$$J[x(\tau),u(\tau)]=\int_{-1}^{1} F\left(x^2(\tau),u^2(\tau)\right)d\tau \qquad (10)$$

Approximating the state variable $x(\tau)$ using VPPs, gives

$$x(\tau)=a^T VP(\tau), \qquad (11)$$

where $a=[a_1,a_2,\ldots,a_N]^T$, is $(N+1)\times 1$ vector of unknown parameters, then $\dot{x}(\tau)$ can be expressed as

$$\dot{x}(\tau)=a^T V\dot{P}(\tau) \qquad (12)$$

where $\dot{VP}(\tau)$ is the derivative vector of $VP(\tau)$. Then obtain the approximation for the control variable by substituting Eq. 11 and Eq. 12 into Eq. 8 to obtain

$$u(\tau) = f\left(a^T VP(\tau), a^T \dot{VP}(\tau)\right) \qquad (13)$$

Then obtain J as a function of the unknown $a_1, a_2, \dots, a_n$ by calculating

$$J(a_1, a_2, \dots, a_n) = \int_{-1}^{1} F\left(a^T VP(\tau) VP(\tau)^T a, a^T \dot{VP}(\tau) \dot{VP}(\tau)^T a\right) d\tau$$

The functional J represents a nonlinear mathematical programming problem of unknown parameters $a_1, a_2, \dots, a_N$.

The resulting nonlinear mathematical programming problem can be simplified as below

$$J(a_1, a_2, \dots, a_n) = \frac{1}{2} a^T H a$$

where $H = 2\int_{-1}^{1} F\left(VP(\tau) VP^T(\tau), \dot{VP}(\tau)\dot{VP}^T(\tau)\right) d\tau$,

The boundary conditions in Eq. (9) can be rewritten as

$x(-1) = a^T VP(-1) = \alpha,\ \ x(1) = a^T VP(1) = \beta.$

Finally, the obtained quadratic programming problem can be rewritten as follows

$$J = \frac{1}{2} a^T H a$$

subject to $Fa - b = 0$

where $F = \begin{bmatrix} VP^T(-1) \\ VP^T(1) \end{bmatrix},\ \ b = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

Using Lagrange multiplier technique, one can obtain the optimal values of the unknown parameters a^*,

$$a^* = H^{-1}F^T(FH^{-1}F^T)^{-1}b.$$

5. Numerical Results

The suggested modification in the state direct parameterization algorithm based on VPPs is applied to solve optimal control problems. This algorithm starts with an approximation to the state variable in terms of VPPs which satisfy the given boundary conditions. The modification in the assumption has succeeded to give an approximate solution with less number of VPPs terms. Numerical test examples are solved and the obtained results show that the suggested method is efficient and only small numbers of terms are used to reach the convergence.

All problems considered in this paper have analytical solution to allow the validation of the algorithm comparing with exact solution results.

Example 1: This problem is concerned with minimization of

$$J = \frac{1}{2}\int_{-1}^{1} (u(\tau)^2 + x(\tau)^2) d\tau, \qquad \tau \epsilon [-1,1],$$

subject to $u(\tau) = 2\,\dot{x}(\tau)$,

with the conditions $x(-1) = 0,\ x(1) = 0.5$, and the exact solution $x(\tau) = \frac{e\,sinh(\tau)}{(e^2-1)}$, $u(\tau) = \frac{e\,cosh(\tau)}{(e^2-1)}$.

Table1 presents the approximate optimal values of the performance index at different values of n. In this table, we have computed the absolute error between the exact performance index value and the approximate values. In Figures 1 and 2, the exact and approximate solutions for $x(\tau)$ and $u(\tau)$ respectively for various values of n.

Table1: The approximate values of J and absolute error for Example1.

n	$J_{approximate}$	Absolute Errors
2	0.3333333333333	0.0050745119580
3	0.3285984848485	3.3966347365e-04
4	0.3282593375616	5.1618682400e-07
5	0.3282588307090	9.3341760432e-09
6	0.3282588213798	5.039024753e-012

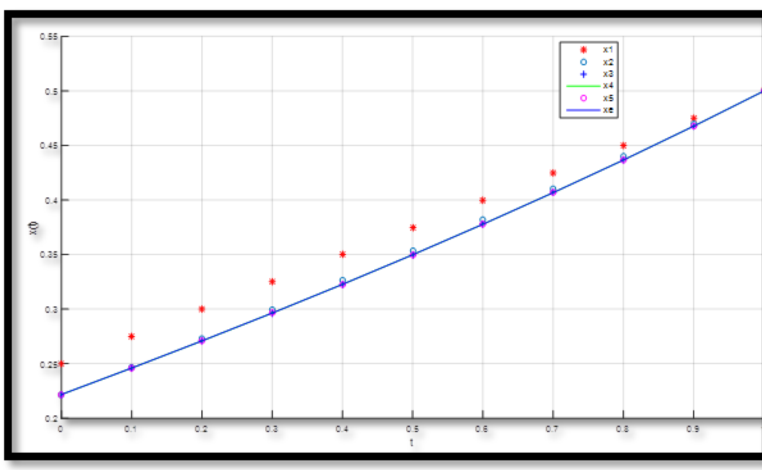

Fig.1 The behaviour of $x(\tau)$ for Example1 at $m = 3, 4, 5, 6$ and exact solution.

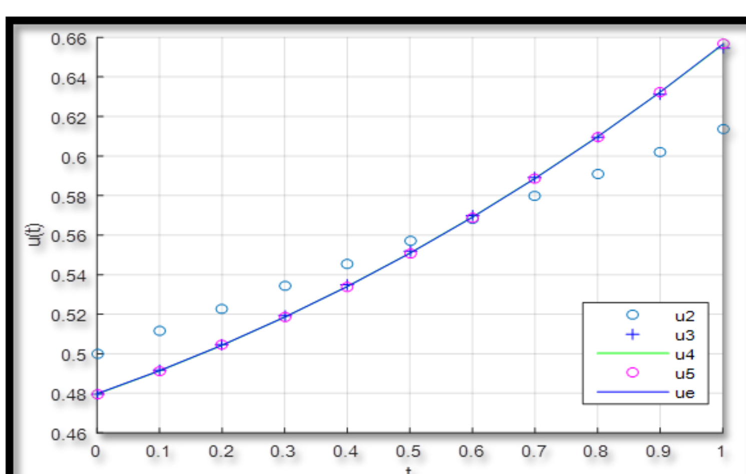

Fig.2 The behaviour of $u(\tau)$ for Example1 at $m = 3, 4, 5, 6$ and exact solution.

Example 2: Consider the following optimal control problem

$$min\, J = \frac{1}{4}\int_{-1}^{1}(u^2\,(\tau) + x^2(\tau))d\tau, \tau \in [-1,1],$$

when $u(\tau) = 2\dot{x}(\tau) + x(\tau)$,

and $x(-1) = 1,\ x(1) = 0.2819695348$,

are satisfied. We have obtained the analytical solution

$$x(\tau) = Ae^{\sqrt{2}\tau} + (1-A)e^{-\sqrt{2}\,\tau}$$

$$u(\tau) = A(\sqrt{2}+1)e^{\sqrt{2}\tau} - (1-A)e^{-\sqrt{2}\tau}$$

$$J = \frac{e^{-2\sqrt{2}}}{2}\left((\sqrt{2}+1)(e^{4\sqrt{2}}-1)\right)A^2 + \frac{e^{-2\sqrt{2}}}{2}(\sqrt{2}-1)(e^{2\sqrt{2}}-1)(1-A^2)$$

where $A = \frac{2\sqrt{2}-3}{2\sqrt{2}-3-\left(e^{\sqrt{2}}\right)^2}$.

Table 2 shows the approximate optimal values of the performance index at different values of n. In this table, we have computed the absolute error between the exact performance index value and the approximate values. In Figures 3 and 4, the exact and approximate solutions for $x(\tau)$ and $u(\tau)$ respectively for various values of n.

Table2: The approximate values of J and absolute error for Example2.

n	$J_{approximate}$	Absolute Errors
2	0.192909298093	0.058453436781
3	0.194298641535	0.001389342577
4	0.192909298093	2.230774388e-05
5	0.192909445024	1.469309399e-07
6	0.192909280931	8.645119897e-10

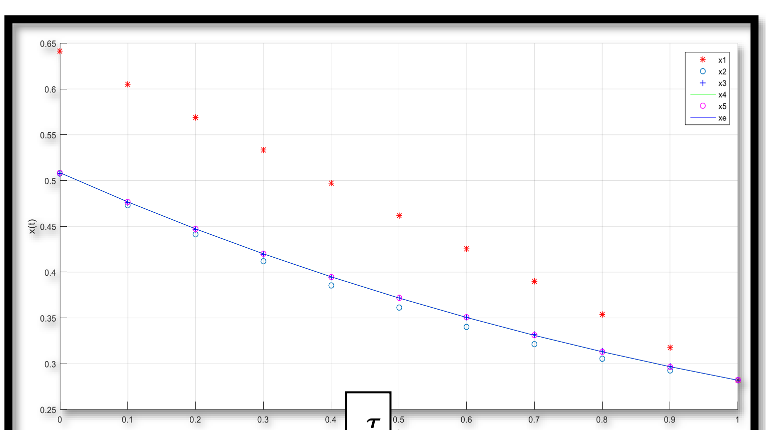

Fig.3 The behaviour of $x(\tau)$ for Example 2 at $m = 3, 4, 5, 6$ and exact solution.

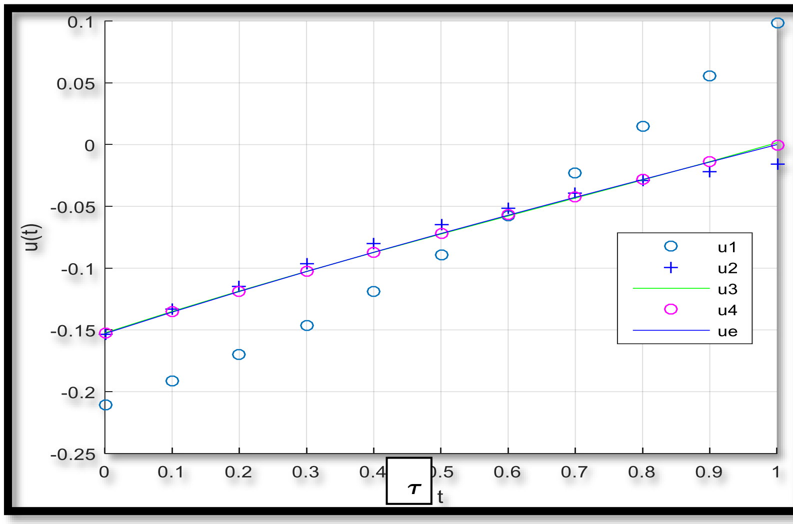

Fig. 4 The behaviour of $u(\tau)$ for Example 2 at $m = 3, 4, 5, 6$ and exact solution.

Example 3: Consider the following optimal control problem

$$min\, J = \frac{1}{2}\int_{-1}^{1}(\frac{1}{2}u^2\,(\tau) + \frac{5}{8}x^2(\tau) + \frac{1}{2}x(\tau)u(\tau))d\tau, \tau \in [-1,1],$$

where $u(\tau) = -2\dot{x}(\tau) + \frac{1}{2}x(\tau)$,

and $x(-1) = 1,\ x(1) = 0.2819695348$,

are satisfied. We have obtained the analytical solution

$$x(\tau) = \frac{\cosh(1-\tau)}{\cosh 1}, u(\tau) = \frac{-(tanh(1-\tau) + 0.5\ )\cosh(1-\tau)}{\cosh 1}$$

and $J = 0.387970779$.

Table3 shows the approximate optimal values of the performance index at different values of n. In this table, we have computed the absolute error between the exact performance index value and the approximate values. In Figures 5 and 6, the exact and approximate solutions for $x(\tau)$ and $u(\tau)$ respectively for various values of n.

Table 3: The approximate values of J and absolute error for Example3.

n	$J_{approximate}$	Absolute Errors
3	0.38088385111	8.677314204e-05
4	0.380799833623	2.75564915e-06
5	0.380797080316	2.33830569e-09
6	0.380797078005	2.71849675e-011

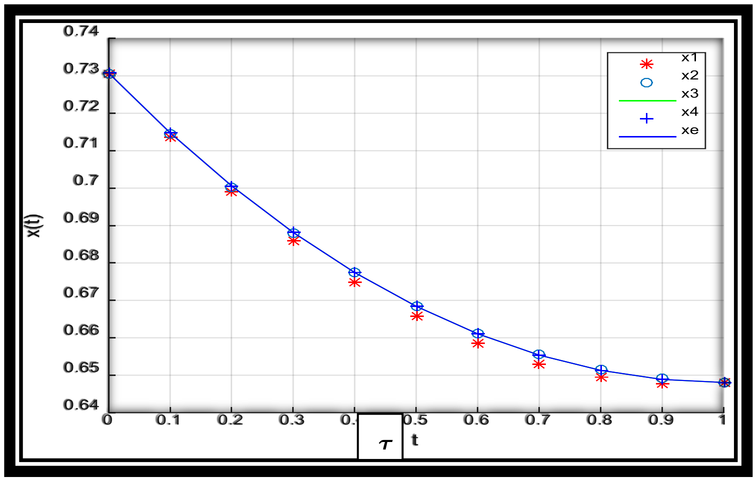

Fig. 5 The behaviour of $x(\tau)$ for Example3 at $m = 3, 4, 5, 6$ and exact solution.

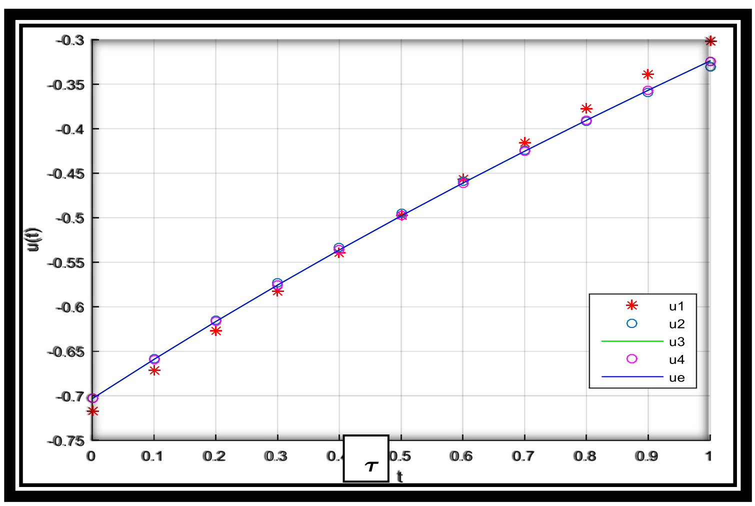

Fig. 6 The behaviour of $u(\tau)$ for Example3 at $m = 3, 4, 5, 6$ and exact solution.

6. Conclusion

An efficient numerical method based on Veita-Pell polynomials for solving continuous optimal control problem are considered in this paper. In the proposed method, a power series solution in terms of VPPs has been chosen such that it satisfies the given boundary conditions. Plugging this series solution into the given optimal control problem and using appropriate a special technique, an optimization problem with unknown Veita-Pell coefficients is obtained. These are the two main modifications and novelty of the procedure and this small contribution in the assumption of power series solution in terms of VPPs results in obtaining the approximate solution with less number of terms with good accuracy. Three numerical examples are provided to confirm the reliability and effectiveness of the suggested method.

References

[1] Hang F., Chao M., “Asynchronous resource-aware control for uncertain semi-Markov jump systems with application to robot arm”, optimal Control Application and Methods, (2022).

[2] Court S., Kunisch K., Laurent Pfeiffer, “Optimal control problem for viscous systems of conservation laws, with geometric parameter, and application to the Shallow-Water equations”, European Mathematical Society, Vol. 21, No. 3, (2019), pp.273-311.

[3] Eman H., Samaa F., Shihab N., “Boubaker Wavelets Functions: properties and Applications”, Baghdad Science Journal,Vol. 18, No. 4, (2021) , pp. 1226-1233.

[4] Alabdali O., Shihab S., Rasheed M., “Orthogonal Bouboker-Turki Polynomials Algorithm for Problems ” , Arising in Engineering, AIP Conference Proceedings 2386, 050019, (2022).

[5] Jalal R., Shihab S., AlhadI M., Rasheed M., “Spectral Numerical Algorithm for Solving Optimal Control Using Boubaker-Turki Operational Matrices”, Journal of Physics: Conference Series 1660 (1), 012090, (2021).

[6] Zbigniew M., Jacek B. and Mohammad K., “Genetic algorithms and optimal control problems”, Proceedings of the IEEE Conference on Decision and Control, Vol. 3, (1991), pp. 1664-1666.

[7] Ayat O., Mirkamal M., “Solving optimal control problems by using Hermite polynomials, Computational Methods for Differential Equations” , Vol. 8, No. 2, (2020), pp. 314-329.

[8] Ejlali N. and Hosseini S., “Adaptive control parameterization method by density functions for optimal control problems” , in IMA Journal of Mathematical Control and Information, Vol. 37, No. 1, (2020), pp. 493-508,.

[9] Mehne H. and Borzabadi H., "A numerical method for solving optimal control problems using state parameterization, Nnmer. Algorithms” , Vol. 42, (2006), pp. 165-169.

[10] Feyza N., Tas D. and Erkus E., “Generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials”, Math. Commun, Vol. 20, (2015), pp. 241–251.

[11] Kuloğlu B, Özkan E and Anthony G.,” Incomplete generalized Vieta-Pell and Vieta-Pell-Lucas polynomials Notes on Number”, Theory and Discrete Mathematics , Vol. 27, No. 4, (2021) , pp. 245-256.

[12] Uygun S, Karatas, H. , “ Notes on Generalization of Vieta-Pell and Vieta-Pell Lucas polynomials”, International Journal of Mathematics Research, Vol. 12, No. 1, (2020) ,pp. 5-22.

[13] Hassan S., Shihab S, Rasheed M, “On Some Properties of Pell Polynomials”, Al-Qadisiyah Journal of Pure Science Vol.26, Issue 1, (2021), pp. 39-54.

[14] Tasci D., Yalcin F., “Vieta-Pell and Vieta-Pell-Lucas polynomials” , Adv. Differ. Eq., 224, (2013).

[15] Sriprad W., Srisawat S. And Naklor P., “Vieta–Fibonacci-like polynomials and some identities”, Annuals Mathematicae et Informaticae, Vol. 54, (2021), pp. 97–108.

[16] Boughaba S., Saba N., Bouss ayoud A., “Construction of generating functions of the products of Vieta polynomials with Gaussian numbers and polynomials”, Int. J. Nonlinear Anal. Appl., Vol. 12, No. 1, (2021) , pp. 649-668.