SAS-Injective Modules

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ABSTRACT

We introduce and investigate SAS-injective modules as a generalization of small injectivity. A right module $M$ over a ring $R$ is said to be SAS-$N$-injective (where $N$ is a right $R$-module) if every right $R$-homomorphism from a semiartinian small right submodule of $N$ into $M$ extends to $N$. A module $M$ is said to be SAS-injective, if $M$ is SAS-$R$-injective. Some characterizations and properties of SAS-injective modules are given. Some results on small injectivity are extended to SAS-injectivity.

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1. Introduction

Throughout $R$ is an associative ring with identity and all modules are unitary $R$-modules. If not otherwise specified, by a module (resp. homomorphism) we will mean a right $R$-module (resp. right $R$-homomorphism). For a submodule $N$ of a module $M$, the notations $N \leq M$, $N \ll M$, $N \leq_{ess} M$, $N \leq_{max} M$, and $N \leq^b M$ mean, respectively, that $N$ is a submodule, a small submodule, an essential submodule, a maximal submodule, and a direct summand of $M$, respectively. If $a$ is an element of right $R$-module $M$, then we use $r(a)$ to denote the right annihilator of $a$ in $R$. Also, we use the symbols $J(M)$, $soc(M)$ and $Z(M)$ to denote the Jacobson radical, the socle and singular submodule of $M_R$, respectively. A module $M$ is called semiartinian, if $soc(M/N) \neq 0$, for any proper submodule $N$ of $M$. For a right $R$-module $M_R$, we denote by $Sa(M)$ to the sum of all semiartinian submodules of $M$. We refer the reader to [1,3,4,6,12], for general background materials.

Injective modules have been studied extensively, and several generalizations for these modules are given by many authors (see, for example, [2,10,9,7,8]).
A module $M$ is called small-injective if every homomorphism from a small right ideal of $R$ into $M$ can be extended to a homomorphism from $R_0$ into $M$ [10].

In this article, a proper generalization of small-injectivity is introduced and investigated, namely SAS-injective modules. Let $N$ be a right $R$-module. A right $R$-module $M$ is said to be SAS-$N$-injective if every $R$-homomorphism from a semiartinian small right submodule of $N$ into $M$ extends to $N$. If $M$ is SAS-$R$-injective, then we say that $M$ is SAS-injective. Firstly, we give an example to show that SAS-injective modules need not be small-injective. Several properties of the class of SAS-injective modules are given. For example, we show that the class of SAS-$N$-injective modules is closed under isomorphic copies, direct products, finite direct sums and summands. Some characterizations of SAS-injective modules are given. We prove the equivalence of the following statements: (1) Every right $R$-module is SAS-injective; (2) Every simple right $R$-module is SAS-injective (3) Every semiartinian small submodule of any right $R$-module SAS-injective; (4) Every semiartinian small right ideal of $R$ is SAS-injective; (5) Every semiartinian small right ideal of $R$ is a summand of $R$; (6) $\text{Sa}(R_0) \cap J(R) = 0$. Conditions under which quotient of SAS-injective right $R$-modules is SAS-injective are given. For instance, we prove that the equivalence of the following: (1) The class of SAS-injective right $R$-modules is closed under quotient; (2) For any right $R$-module $M$, the sum of any two SAS-injective submodules of $M$ is SAS-injective; (3) All semiartinian small submodules of $R_0$ are projective. Finally, we give conditions such that the class of SAS-injective right $R$-modules is closed under direct sums. For instance, we prove that the equivalence of the following conditions: (1) $\text{Sa}(R_0) \cap J(R)$ is Noetherian; (2) All direct sums of injective modules are SAS-injective; (3) The class of SAS-injective modules is closed under direct sums.

2. SAS-Injective Modules

As a generalization of small injective modules, we introduce the concept of SAS-injective modules.

**Definition 2.1.** A right $R$-module $M$ is said to be SAS-$N$-injective (where $N$ is a right $R$-module), if any right $R$-homomorphism $f: K \to M$ extends to $N$, where $K$ is any semiartinian small submodule of $N$. If $M$ is SAS-$R$-injective, then $M$ is said to be SAS-injective.

**Examples 2.2.**

(1) All small-injective modules are SAS-injective, but the converse is not true in general, for example: let $R$ be the localization ring of $\mathbb{Z}$ at the prime $p$, that is $R = \mathbb{Z}_{(p)} = \{ \frac{m}{n} : p \text{ does not divide } n \}$. Then $R$ is not small injective with $\text{soc}(R_0) = 0$ (see [13, Example 4]). Since $\text{soc}(R_0) = 0$, we have that $\text{Sa}(R_0) = 0$ and hence the zero ideal is the only semiartinian small right ideal in $R_0$. Thus $R_0$ is SAS-injective and hence SAS-injectivity is a proper generalization of small injectivity.

(2) Clearly, if $\text{soc}(N_0) = 0$, then $0$ is the only semiartinian small submodule of $N$ and hence every module is SAS-$N$-injective. Particularly, all $\mathbb{Z}$-modules are SAS-injective.

Some properties of SAS-$N$-injective modules are given in the following theorem.

**Theorem 2.3.** Let $M, N$ and $K$ be right $R$-modules. Then the following statements hold:

(1) Let $\{M_i : i \in I\}$ be a class of modules. Then the direct product $\prod_{i \in I} M_i$ is SAS-$N$-injective if and only if all $M_i$ are SAS-$N$-injective.

(2) If $K \subseteq N$ and $M$ is SAS-$N$-injective, then $M$ is SAS-$K$-injective.

(3) If $M$ is SAS-$K$-injective and $M \cong N$, then $N$ is SAS-$K$-injective.

(4) If $M$ is SAS-$K$-injective and $K \cong N$, then $M$ is SAS-$N$-injective.


**Proof.** Obvious. □
Corollary 2.4. The next statements hold:

1. A finite direct sum of SAS-N-injective modules is SAS-N-injective, for any module \( N \). Moreover, a finite direct sum of SAS-injective modules is SAS-injective.
2. A summand of an SAS-injective module is again SAS-injective.

Proof. (1) By applying Theorem 2.3 (1), when index \( I \) is taken to be a finite set.

(2) This directly by using Theorem 2.3 (5). □

If for any submodule \( N \) of a right \( R \)-module \( M \), there exists an ideal \( I \) of \( R \) such that \( N = MI \), then \( M \) is called a multiplication module \([11, \text{p.} 3839]\).

Proposition 2.5. Let \( \{N_i : i \in I\} \) be a family of right \( R \)-modules and \( M \) be a right \( R \)-module. If \( \bigoplus_{i \in I} N_i \) is a multiplication module, then \( M \) is SAS-\( \bigoplus_{i \in I} N_i \)-injective if and only if \( M \) is SAS-\( N_i \)-injective, for all \( i \in I \).

Proof. (\( \Rightarrow \)) By Theorem 2.3 ((2), (4)).

(\( \Leftarrow \)) Let \( K \) be a semiartinian small submodule of \( \bigoplus_{i \in I} N_i \). Since \( \bigoplus_{i \in I} N_i \) is a multiplication module (by hypothesis), we have from \([11, \text{Theorem} \ 2.2, \text{p.} \ 3844]\) that \( K = \bigoplus_{i \in I} K_i \) with \( K_i \) is a submodule of \( N_i \), for all \( i \in I \). By \([4, \text{Lemma} \ 5.1.3(\text{c}), \text{p.} \ 108]\), \( K_i \ll N_i \). Since \( K \) is a semiartinian module, we have from \([4, \text{Exercises} \ (7)(8), \text{p.} \ 238]\) that \( K_i \) is a semiartinian module and hence \( K_i \) is a semiartinian submodule of \( N_i \). For \( i \in I \), consider the following diagram:

\[
\begin{array}{ccc}
K = \bigoplus_{i \in I} K_i & \xrightarrow{i_1} & \bigoplus_{i \in I} N_i \\
\downarrow i_{K_i} & & \downarrow h \\
K_i & \xrightarrow{i_2} & N_i \\
\downarrow i_{N_i} & & \\
M & \xrightarrow{f} & \\
\end{array}
\]

where \( i_{K_i}, i_{N_i} \) are injection maps and \( i_1, i_2 \) are inclusion maps. The hypothesis implies that there exists a homomorphism \( h: N_i \rightarrow M \) such that \( h_i \circ i_2 = f \circ i_{K_i} \). By \([4, \text{Theorem} \ 4.1.6(2)]\), there exists exactly one homomorphism \( h: \bigoplus_{i \in I} N_i \rightarrow M \) satisfying \( h_i = h \circ i_{N_i} \). Thus \( f \circ i_{K_i} = h_1 \circ i_2 = h \circ i_{N_i} \circ i_2 = h \circ i_1 \circ i_{K_i} \) for all \( i \in I \). Let \( (a_i)_{i \in I} \in \bigoplus_{i \in I} K_i \), thus \( a_i \in K_i \) for all \( i \in I \) and \( f((a_i)_{i \in I}) = f(\sum_{i \in I} K_i(a_i)) = (h \circ i_1)((a_i)_{i \in I}) \) and hence \( f = h \circ i_1 \). □

If all right ideals of a ring \( R \) are ideals in \( R \), then \( R \) is called right invariant \([11, \text{p.} \ 3839]\).

Corollary 2.6. Let \( R \) be a right invariant ring and let \( 1 = s_1 + s_2 \ldots + s_n \) in \( R \), where the \( s_i \) are orthogonal idempotents, then a right \( R \)-module \( M \) is SAS-injective if and only if \( M \) is SAS-\( s_i R \)-injective for every \( i = 1, 2, \ldots, n \).

Proof. By \([1, \text{Corollary} \ 7.3, \text{p.} \ 96]\), we have \( R = \bigoplus_{i=1}^n s_i R \). Since \( R \) is a right invariant ring, we get from \([11, \text{Proposition} \ 3.1, \text{p.} \ 3855]\) that \( R \) is a multiplication module and hence Proposition 2.5 implies that \( M \) is SAS-injective if and only if \( M \) is SAS-\( s_i R \)-injective. □

The following proposition gives characterizations of SAS-injective modules.

Proposition 2.7. The next conditions are equivalent for a right \( R \)-module \( M \):
(1) $M$ is SAS-injective.

(2) The sequence $0 \to \text{Hom}_R(R/N, M) \overset{\pi^*}{\to} \text{Hom}_R(R, M) \overset{i^*}{\to} \text{Hom}_R(N, M) \to 0$ is exact, for all submodule $N$ of $\text{Sa}(R_R) \cap J(R)$, where $i$ and $\pi$ are the inclusion and canonical maps, respectively.

(3) $\text{Ext}^1(R/N, M) = 0$, for all submodule $N \subseteq \text{Sa}(R_R) \cap J(R)$.

(4) For each semiartinian small right ideal $N$ of $R$ and for any $R$-homomorphism $f: N \to M$, there exists an element $m \in M$ such that $f(r) = mr$ for all $r \in N$.

Proof. $(1) \Rightarrow (2)$ Let $N$ be a submodule of $\text{Sa}(R_R) \cap J(R)$. It is clear that the sequence

$$0 \to \text{Hom}_R(R/N, M) \overset{\pi^*}{\to} \text{Hom}_R(R, M) \overset{i^*}{\to} \text{Hom}_R(N, M) \to 0$$

is exact. Let $g \in \text{Hom}_R(N, M)$. Since $M$ is SAS-injective, there exists a right $R$-homomorphism $f: R \to M$ such that $fi = g$ and hence $i^*(f) = g$. Thus $i^*$ is an $R$-epimorphism and hence the sequence $0 \to \text{Hom}_R(R/N, M) \overset{\pi^*}{\to} \text{Hom}_R(R, M) \overset{i^*}{\to} \text{Hom}_R(N, M) \to 0$ is exact.

$(2) \Rightarrow (3)$ By [5, Theorem 4.4(3), p.491], there is an exact sequence $0 \to \text{Hom}_R(R/N, M) \overset{\pi^*}{\to} \text{Hom}_R(R, M) \overset{i^*}{\to} \text{Hom}_R(N, M) \to 0$. Since $R_R$ is projective, it follows from [5, Theorem 4.4(1), p.491] that $\text{Ext}^1(R, M) = 0$ and hence the sequence $0 \to \text{Hom}_R(R/N, M) \overset{\pi^*}{\to} \text{Hom}_R(R, M) \overset{i^*}{\to} \text{Hom}_R(N, M) \to 0$ is exact. By hypothesis, the sequence $0 \to \text{Hom}_R(R/N, M) \overset{\pi^*}{\to} \text{Hom}_R(R, M) \overset{i^*}{\to} \text{Hom}_R(N, M) \to 0$ is exact and hence $\text{Ext}^1(R/N, M) = 0$.

$(3) \Rightarrow (4)$ Let $f: N \to M$ be a $R$-homomorphism where $N$ is a semiartinian small right ideal of $R$. Thus $N \subseteq \text{Sa}(R_R) \cap J(R)$. As the proof of $(2) \Rightarrow (3)$ we have that the sequence $0 \to \text{Hom}_R(R/N, M) \overset{\pi^*}{\to} \text{Hom}_R(R, M) \overset{i^*}{\to} \text{Hom}_R(N, M) \to \text{Ext}^1(R/N, M) = 0$ is exact. By hypothesis, $\text{Ext}^1(R/N, M) = 0$ and hence the sequence $0 \to \text{Hom}_R(R/N, M) \overset{\pi^*}{\to} \text{Hom}_R(R, M) \overset{i^*}{\to} \text{Hom}_R(N, M) \to 0$ is exact. Thus there is a right $R$-homomorphism $g \in \text{Hom}_R(R, M)$ with $i^*(g) = f$, this means $gi = f$. Let $r \in N$, thus $f(r) = g(r) = g(1)r = mr$, where $m = g(1)$.

$(4) \Rightarrow (1)$ It is clear. □

**Proposition 2.8.** For a module M, the next statements are equivalent:

(1) All modules are SAS-$M$-injective.

(2) All semiartinian small submodules of any module is SAS-$M$-injective.

(3) All semiartinian small submodules of $M$ are SAS-$M$-injective.

(4) Every semiartinian small submodule of $M$ is a summand of $M$.

(5) $\text{Sa}(M) \cap J(M) = 0$.

**Proof.** $(1) \Rightarrow (2) \Rightarrow (3)$ and $(5) \Rightarrow (1)$ are clear.

$(3) \Rightarrow (4)$ Let $W$ be a semiartinian small submodule of $M$. By hypothesis, $W$ is SAS-$M$-injective and hence $g \circ i = i_W$ for some a homomorphism $g: M \to W$, where $i$ is the inclusion and $i_W$ is the identity homomorphism. Hence $i$ is split and this implies that $W$ is a summand of $M$.

$(4) \Rightarrow (5)$ Let $x \in \text{Sa}(M) \cap J(M)$, thus $x \in \text{Sa}(M)$ and $x \in J(M)$. By [4, Exercises (7)(2), p.238], $\text{Sa}(M)$ is a semiartinian module and hence from [4, Exercises (7)(8), p.238] we have $xR$ is a semiartinian submodule of $M$. By [4, Corollary 9.1.3, p.214], $xR$ is a small submodule of $M$. By hypothesis $xR$ is a summand of $M$ and hence $xR \oplus K = M$ for some submodule $K$ of $M$. Since $xR$ is a small submodule of $M$, we have that $K = M$ and hence $xR = 0$. So, $x = 0$ and hence $\text{Sa}(M) \cap J(M) = 0$. □

**Corollary 2.9.** For a ring $R$, the next conditions are equivalent:

...
(1) All right $R$-modules are SAS-injective.
(2) All semiartinian small submodules of any right $R$-module are SAS-injective.
(3) All semiartinian small right ideals of $R$ are SAS-injective.
(4) All semiartinian right ideals of $R$ are summands of $R$.
(5) $\text{Sa}(R_R) \cap J(R_R) = 0$.

**Proof.** By applying Proposition 2.8 with $M = R$. □

**Proposition 2.10.** Let $M$ be a right $R$-module. Then $\text{Sa}(M) \cap \{M\}$ is a semisimple summand of $M$ if and only if all modules are SAS-$M$-injective.

**Proof.** (⇒) Let $\text{Sa}(M) \cap \{M\}$ be a semisimple summand of $M$ and let $N$ be a module. Let $K$ be a semiartinian small submodule of $M$. Since $\text{Sa}(M) \cap \{M\}$ is a semisimple summand of $M$, it follows that $M = (\text{Sa}(M) \cap \{M\}) \oplus W$ for some submodule $W$ of $M$. Since $K$ is a submodule of $\text{Sa}(M) \cap \{M\}$ and $\text{Sa}(M) \cap \{M\}$ is semiartinian, $\text{Sa}(M) \cap \{M\} = K \oplus U$ for some submodule $U$ of $\text{Sa}(M) \cap \{M\}$. We obtain $M = K \oplus U \oplus W$ and hence all semiartinian small submodules of $M$ are summands of $M$. By Proposition 2.8, all modules are SAS-$M$-injective.

(⇐) Suppose that every right $R$-module is SAS-$M$-injective. By Proposition 2.8, $\text{Sa}(M) \cap \{M\} = 0$ and hence $\text{Sa}(M) \cap \{M\}$ is a semisimple summand of $M$. □

**Theorem 2.11.** If all simple singular modules are SAS-injective, then $r(a) \leq^R aR$ and $aR$ is projective, for every $a \in \text{Sa}(R_R) \cap J(R_R)$.

**Proof.** Let $a \in \text{Sa}(R_R) \cap J(R_R)$ and let $L = RaR + r(a)$. Thus there exists $N \leq_R R$ such that $L \oplus N \leq^{ess} R_R$. Assume that $L \oplus N \neq R_R$, then there exists $I \leq^{max} R_R$ with $L \oplus N \leq I$, and hence $I \leq^{ess} R_R$. By [6, Example 7.6 (3), p. 247], $R/I$ is a singular right $R$-module. By [4, Corollary 3.1.14, p. 49], $R/I$ is a simple module and hence the hypothesis implies that $R/I$ is SAS-injective. Clearly, $a$ is a well-defined $R$-homomorphism, where $a: aR \to R/I$ is defined by $a(at) = t + I$, for any $t \in R$. It is obvious that $aR$ is a semiartinian small right ideal of $R$. By SAS-injectivity of $R/I$, there is a right $R$-homomorphism $g: R \to R/I$ with $g(x) = f(x)$ for any $x \in aR$. Thus $1 + I = f(a) = g(a) = g(1)a = (c + I)a = ca + I$, for some $c \in R$ and hence $1 - ca \in I$. But $ca \in RaR \subseteq I$, so $1 \in I$, a contradiction. Hence $L \oplus N = R$ and so $RaR + (r(a) \oplus N) = R$ and this implies that $r(a) \oplus N = R$ (since $RaR \not\subseteq R_R$). We will prove that $aR$ is projective. Since $r(a)$ is a summand of $R_R$, it follows that $r(a) = (1 - e)R$ for some idempotent element $e$ in $R$ (by [12, 23(3), p.8]) with $R = eR \oplus (1 - e)R$. Define $\lambda: eR \to aeR$ by $\lambda(er) = aer$, for all $r \in R$. It is clear that $\lambda$ is an epimorphism. Let $x \in \ker(\lambda)$, thus $\lambda(x) = 0$ and so $x = er$ for some $r \in R$ and $aer = 0$. Hence $er \in r(a)$ and $er \in eR$, and this implies that $x \in eR \cap r(a)$ and so $\ker(\lambda) \subseteq eR \cap r(a)$. Let $y \in R \cap r(a)$, then $y = er$ and $ay = 0$. So $aer = 0$ and hence $\lambda(y) = 0$. Thus $y \in \ker(\lambda)$ and so $eR \cap r(a) \subseteq \ker(\lambda)$. Thus $\ker(\lambda) = eR \cap r(a)$. Since $R = eR \oplus (1 - e)R$, we have $eR \cap (1 - e)R = 0$. Since $r(a) = (1 - e)R$, we have $eR \cap r(a) = 0$. Since $\ker(\lambda) = eR \cap r(a)$, we have $\ker(\lambda) = 0$. Thus $\lambda: eR \to aeR$ is an isomorphism. Clearly $aR = aeR$, since $aeR \subseteq aR$ and if $x \in eR$, then $x = a \cdot r$ for some $r \in R$. So $x = ar = aer + a(1 - e)r$. Since $r(a) = (1 - e)R$, we have $a(1 - e)r = 0$ and so $x = aer \in aeR$. Thus $aR \subseteq aeR$ and hence $aR = aeR$. Since $R = eR \oplus (1 - e)R$, we have $eR$ is projective. Since $eR \cong aeR$, we have $aeR$ is projective. Since $aR = aeR$, we have that $aR$ is projective. □

**Corollary 2.12.** If all simple singular right $R$-modules are SAS-injective, then $Z(R_R) \cap \text{Sa}(R_R) \cap J(R_R) = 0$.

**Proof.** Assume that $Z(R_R) \cap \text{Sa}(R_R) \cap J(R_R) \neq 0$, then there exists $0 \neq a \in Z(R_R) \cap \text{Sa}(R_R) \cap J(R_R)$. Since $a \in Z(R_R)$, we have $r(a) \leq^{ess} R_R$. By Proposition 2.11, $r(a) \leq^{R} R_R$ and so $r(a) \cap K = 0$ and $r(a) + K = R$ for some $K \leq R_R$. Since $r(a) \leq^{ess} R_R$, which implies that $K = 0$ and so $r(a) = R$ and hence $a = 0$ but this a contradiction. Thus $Z(R_R) \cap \text{Sa}(R_R) \cap J(R_R) = 0$. □
A ring \( R \) is named zero insertive, if for any \( a, b \in R \) with \( ab = 0 \), then \( aRb = 0 \) \([10]\).

**Lemma 2.13.** \([10, \text{Lemma 2.11}]\) \( RaR + r(a) \leq_{\text{ess}} R_{aR} \) for any element \( a \) in a zero insertive ring \( R \).

**Proposition 2.14.** If all simple singular right \( R \)-modules are SAS-injective and \( R \) is a zero insertive ring, then \( Sa(R_a) \cap (R_a) = 0 \).

**Proof.** Assume that \( Sa(R_a) \cap (R_a) \neq 0 \). Thus there is \( 0 \neq a \in Sa(R_a) \cap (R_a) \), and hence \( RaR \ll R_{aR} \). If \( RaR + r(a) \not\subseteq R \), then \( RaR + r(a) \subseteq K \) for some a maximal right ideal \( K \) of \( R \). Using Lemma 2.13, we have \( RaR + r(a) \) is an essential in \( R_{aR} \) and hence \( K \) is an essential in \( R_{aR} \) and so \( R/K \) is a simple singular right \( R \)-module \([6, \text{Example 7.6(3) p. 247}]\). By hypothesis, \( R/K \) is an SAS-injective module. Consider the mapping \( f: aR \to R/K \) defined by \( f(ar) = r + K \) for all \( r \in R \). Thus \( f \) is a well-defined right \( R \)-homomorphism. Since \( aR \) is a semiartinian small right ideal of \( R \), it follows from SAS-injectivity of \( R/K \), there is a right \( R \)-homomorphism \( g: R \to R/K \) with \( g(x) = f(x) \) for any \( x \in aR \). Thus \( 1 + K = f(a) = g(a) = g(1)a = (c + K)a = ca + K \), for some \( c \in R \) and hence \( 1 - ca \in K \). Since \( ca \in RaR \subseteq K \), we have \( 1 \in K \) and so \( K = R \) and this is a contradiction. Therefore, \( RaR + r(a) = R \). Since \( RaR \ll R_{aR} \), which implies that \( r(a) = R \) and so \( a = 0 \) and this is a contradiction. Thus \( Sa(R_a) \cap (R_a) = 0 \). \( \Box \)

**Corollary 2.15.** If all simple singular right modules over a zero insertive ring \( R \) are SAS-injective, then all right \( R \)-modules are SAS-injective.

**Proof.** By Proposition 2.14 and Corollary 2.9. \( \Box \)

**Theorem 2.16.** Let \( R \) be ring. Then the following conditions are equivalent:

1. \( Sa(R_a) \cap (R_a) = 0 \).
2. All right \( R \)-modules are SAS-injective.
3. All simple right \( R \)-modules are SAS-injective.

**Proof.** Clearly, we have (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3).

(3) \( \Rightarrow \) (1). Assume that \( Sa(R_a) \cap (R_a) \neq 0 \). Thus there is \( 0 \neq a \in Sa(R_a) \cap (R_a) \), and hence \( aR \ll R_{aR} \). If \( Sa(R_a) \cap (R_a) + r(a) \not\subseteq R \), then \( (Sa(R_a) \cap (R_a)) + r(a) \subseteq I \), for some maximal right ideal \( I \) of \( R \). Thus \( R/I \) is a simple right \( R \)-module. By hypothesis, \( R/I \) is an SAS-injective module. We define \( f: aR \to R/I \) by \( f(ax) = x + I \) for any element \( x \) in \( R \). Then clearly \( f \) is a well-defined right \( R \)-homomorphism. Since \( aR \not\subseteq Sa(R_a) \cap (R_a) \) it follows that \( aR \) is a semiartinian small right ideal of \( R \). By SAS-injectivity of \( R/I \), there is a right \( R \)-homomorphism \( g: R \to R/I \) with \( g(x) = f(x) \) for any \( x \in aR \). Thus \( 1 + I = f(a) = g(a) = g(1)a = (c + I)a = ca + I \), for some \( c \in R \) and hence \( 1 - ca \in I \). Since \( Sa(R_a) \) and \( (R_a) \) are predicals, we have that \( Sa(R_a) \) and \( (R_a) \) are two-sided ideals. Thus \( ca \in Sa(R_a) \cap (R_a) \subseteq I \) and hence \( 1 \in I \), and so \( I = R \) and this is a contradiction. Therefore, \( (Sa(R_a) \cap (R_a)) + r(a) = R \). Since \( Sa(R_a) \cap (R_a) \) is a small ideal in \( R_{aR} \), we have that \( r(a) = R \) and so \( a = 0 \) and this is a contradiction. Thus \( Sa(R_a) \cap (R_a) = 0 \). \( \Box \)

**Remark 2.17.** It is not necessary that all semiartinian small submodules of a projective module are projective, for example \( < 2 > \) is a semiartinian small sub-module of the projective \( Z_4 \)-module \( Z_4 \) but it is not projective, because it is not a summand of \( Z_4^{(0)} \), for any index \( I \).

**Theorem 2.18.** The following conditions are equivalent for a projective module \( M \):

1. All epimorphic images of SAS-M-injective modules are SAS-M-injective.
2. All epimorphic images of small-M-injective modules are SAS-M-injective.
3. All epimorphic images of injective modules are SAS-M-injective.
4. All sums of two SAS-M-injective submodules of any module are SAS-M-injective.
5. All sums of two small-M-injective submodules of any module are SAS-M-injective.
(6) All sums of two injective submodules of any module are SAS-M-injective.
(7) All semiartinian small submodules of $M$ are projective.

**Proof.** (1)$\Rightarrow$(2)$\Rightarrow$(3) and (4)$\Rightarrow$(5)$\Rightarrow$(6) are clear.

(3)$\Rightarrow$(7) Let $D$ and $N$ be modules and $U$ be a semiartinian small submodule of $M$. Consider the following diagram:

$$
\begin{array}{ccc}
N & \xrightarrow{f} & D \\
& \searrow_{h} & \downarrow \\
& & 0 \\
0 & \xrightarrow{i} & U \\
& \nearrow_{\tilde{i}} & \\
& & M
\end{array}
$$

where $f$ is epimorphism, $h$ is a homomorphism, and $i$ is the inclusion homomorphism. We can take $N$ to be an injective $R$-module (by [3, Proposition 5.2.10, p. 148]). By hypothesis, $D$ is SAS-M-injective and hence $ai = h$ for some a homomorphism $\alpha: M \to D$. By projectivity of $M$, we get that $\alpha$ can be lifted to an $R$-homomorphism $\bar{\alpha}: M \to N$ with $f\bar{\alpha} = \alpha$. Let $\tilde{h}: U \to N$ be the restriction of $\bar{\alpha}$ over $U$. It is clear that $f\tilde{h} = h$ and hence $U$ is projective.

(7)$\Rightarrow$(1) Let $h: A \to B$ be an $R$-epimorphism, where $A$ and $B$ are right $R$-modules and $A$ is an SAS-M-injective. Let $K$ be a semiartinian small submodule of $M$, $f: K \to B$ be an $R$-homomorphism and $i: K \to M$ the inclusion homomorphism. By (7), $K$ is projective and hence $hg = f$ for some a homomorphism $g: K \to A$. By SAS-M-injectivity of $A$, we get $\bar{g}i = g$ for some a homomorphism $\bar{g}: M \to A$. Put $\alpha = h\bar{g}: M \to B$. Thus $ai = h\bar{g}i = hg = f$. Hence $B$ is an SAS-M-injective right $R$-module.

(1)$\Rightarrow$(4) Let $K_1$ and $K_2$ be two SAS-M-injective submodules of a right $R$-module $K$. Then $K_1 + K_2$ is a homomorphic image of $K_1 \oplus K_2$. Since $K_1 \oplus K_2$ is SAS-M-injective (by Corollary 2.4.(1)), it follows from hypothesis that $K_1 + K_2$ is SAS-M-injective.

(6)$\Rightarrow$(3) Let $E$ be an injective module and $N \leq E$. Let $Q = E \oplus E$, $H = \{(x, x) \mid x \in N\}$, $\tilde{Q} = Q/H$, $K_1 = \{y + H \in \tilde{Q} \mid y \in E \oplus 0\}$ and $K_2 = \{y + H \in \tilde{Q} \mid y \in 0 \oplus E\}$. Then $\tilde{Q} = K_1 + K_2$. Since $(E \oplus 0) \cap H = 0$ and $(0 \oplus E) \cap H = 0$, it follows that $E \cong K_i$, $i = 1, 2$. Clearly, $K_1 \cap K_2 \cong N$ under $y \mapsto y + H$ for all $y \in E \oplus 0$. By hypothesis, $\tilde{Q}$ is SAS-M-injective. Injectivity of $K_i$ implies that $\tilde{Q} = K_i \oplus A$ for some submodule $A$ of $\tilde{Q}$ and hence $A \cong (K_1 + K_2)/K_i \cong K_2/(K_1 \cap K_2) \cong E/N$. By Theorem 2.3 (3),(5), $E/N$ is SAS-M-injective. □

**Corollary 2.19.** The following statements are equivalent for a ring $R$:

(1) Every epimorphic image of an SAS-injective right $R$-module is SAS-injective.
(2) Every epimorphic image of a small injective right $R$-module is SAS-injective.
(3) Every epimorphic image of an injective right $R$-module is SAS-injective.
(4) Every sum of two SAS-injective submodules of any right $R$-module is SAS-injective.
(5) Every sum of two small injective submodules of any right $R$-module is SAS-injective.
(6) Every sum of two injective submodules of any right $R$-module is SAS-injective.
(7) Every semiartinian small submodule of $R_\alpha$ is projective.

**Proof.** By taking $M = R$ and applying Theorem 2.18. □

Let $N$ be a right $R$-module. A right $R$-module $M$ is called a rad-$N$-injective, if for any submodule $K$ of $\{N\}$, any right $R$-homomorphism $f: K \to M$ extends to $N$ [14, p.412].

**Theorem 2.20.** If $M$ is a finitely generated right $R$-module, then the following statements are equivalent:

(1) $Sa(M) \cap J(M)$ is a Noetherian $R$-module.
(2) Any direct sum of SAS-M-injective right $R$-modules is SAS-M-injective.
(3) Any direct sum of rad-M-injective right $R$-modules is SAS-M-injective.
(4) Any direct sum of small $M$-injective right $R$-modules is SAS-M-injective.
(5) Any direct sum of injective right $R$-modules is SAS-M-injective.
(6) $K^{(L)}$ is SAS-M-injective, for any injective right $R$-module $K$ and for any index set $L$.
(7) $K^{(m)}$ is SAS-M-injective, for any injective right $R$-module $K$.

Proof. (2)⇒(3)⇒(4)⇒(5)⇒(6)⇒(7) are clear.

(1)⇒(2) Let $E = \bigoplus_{i \in I} M_i$ be a direct sum of SAS-M-injective right $R$-modules. Let $K$ be a semiartinian small submodule of $M$ and $f: K \to E$ be a homomorphism. Thus $K \subseteq \text{Sa}(M) \cap \{M\}$. Since $\text{Sa}(M) \cap \{M\}$ is a Noetherian (by hypothesis), $K$ is finitely generated and hence $f(K) \subseteq \bigoplus_{j \in J} M_j$, for some finite subset $J$ of $I$. Since a finite direct sum of SAS-M-injective modules is SAS-M-injective (by Corollary 2.4(1)), we have $\bigoplus_{j \in J} M_j$ is SAS-M-injective. Define $\alpha: K \to \bigoplus_{j \in J} M_j$ by $\alpha(x) = f(x)$, for every $x \in K$. It is clear that $\alpha$ is a right $R$-homomorphism. By SAS-M-injectivity of $\bigoplus_{j \in J} M_j$, there exists a right $R$-homomorphism $g: M \to \bigoplus_{j \in J} M_j$ such that $g(\alpha) = (\alpha)(a)$, for all $a \in K$. Define $h: M \to E = \bigoplus_{i \in I} M_i$ by $h(x) = (ig)(x)$ for every $x \in M$, where $i: \bigoplus_{j \in J} M_j \to \bigoplus_{i \in I} M_i$ is the inclusion. Thus, for all $a \in K$, we have that $h(a) = ig(a) = g(\alpha) = (\alpha)(a) = f(a)$ and hence $E$ is SAS-M-injective.

(7)⇒(1) Let $K_1 \subseteq K_2 \ldots$ be a chain of submodules of $\text{Sa}(M) \cap \{M\}$. For each $i \geq 1$, let $E_i = E(M/K_i)$ and $E = \bigoplus_{i=1}^\infty E_i$. For every $i \geq 1$, we put $M_i = \prod_{j=i}^\infty E_j = E_i \bigoplus \left( \bigcap_{j=i}^\infty E_j \right)$, then $M_i$ is injective. By hypothesis, $\bigoplus_{i=1}^\infty M_i = (\bigoplus_{i=1}^\infty E_i) \bigoplus \left( \bigcap_{i=1}^\infty E_i \right)$ is SAS-M-injective. By using Theorem 2.3(5) we obtain that $E$ is SAS-M-injective. Define $f: H = \bigcup_{i=1}^\infty K_i \to E$ by $f(x) = (x + K_i)$. Obviously, $f$ is a well-defined right $R$-homomorphism. Since $M$ is finitely generated, $\text{Sa}(M) \cap \{M\}$ is a semiartinian small submodule of $M$, and so $\bigcup_{i=1}^\infty K_i$ is a semiartinian small submodule of $M$. By SAS-M-injectivity of $E$, there exists a right $R$-homomorphism $g: M \to E = \bigoplus_{i=1}^\infty E_i$ such that $g(i) = f$, where $i: H \to M$ is the inclusion homomorphism. Since $M$ is finitely generated, $g(M) \subseteq \bigoplus_{i=1}^n E(M/K_i)$ for some $n$ and hence $f(H) \subseteq \bigoplus_{i=1}^n E(M/K_i)$. Let $\pi_i: \bigoplus_{i=1}^n E(M/K_i) \to E(M/K_i)$ be the projection homomorphism. Thus $\pi_i f(x) = \pi_i((x + K_i)_{i \geq 2}) = x + K_i$ for all $x \in H$ and $i \geq 1$ and hence $\pi_i f(H) = H/K_i$ for all $i \geq 1$. Since $f(H) \subseteq \bigoplus_{i=1}^n E(M/K_i)$, we have that $H/K_i = \pi_i f(H) = 0$ for all $i \geq n + 1$. So $H = K_i$ for all $i \geq n + 1$ and hence the chain $K_1 \subseteq K_2 \subseteq \cdots$ terminates at $K_{n+1}$. Thus $\text{Sa}(M) \cap \{M\}$ is a Noetherian $R$-module.

Corollary 2.21. If $N$ is a finitely generated right $R$-module, then the following statements are equivalent:

(1) $\text{Sa}(N) \cap \{N\}$ is a Noetherian $R$-module.
(2) $M^{(L)}$ is SAS-N-injective, for each SAS-N-injective right $R$-module $M$ and for any index set $L$.
(3) $M^{(L)}$ is SAS-N-injective, for each rad-N-injective right $R$-module $M$ and for any index set $L$.
(4) $M^{(L)}$ is SAS-N-injective, for each small $N$-injective right $R$-module $M$ and for any index set $L$.
(5) $M^{(N)}$ is SAS-N-injective, for each SAS-N-injective right $R$-module $M$.
(6) $M^{(N)}$ is SAS-N-injective, for each small $N$-injective right $R$-module $M$.
(7) $M^{(N)}$ is SAS-N-injective, for each small $N$-injective right $R$-module $M$.

Proof. By Theorem 2.20.

Corollary 2.22. For a ring $R$, the following conditions are equivalent:

(1) $\text{Sa}(R) \cap \{R\}$ is a Noetherian right $R$-module.
(2) All direct sums of SAS-injective right $R$-modules are SAS-injective.
(3) All direct sums of small-injective right $R$-modules are SAS-injective.
(4) All direct sums of injective right $R$-modules are SAS-injective.
(5) If $M$ is an injective right $R$-module, then $M^{(L)}$ is SAS-injective, for any index set $L$.

(6) If $M$ is a small-injective right $R$-module, then $M^{(L)}$ is SAS-injective, for any index set $L$.

(7) $M^{(L)}$ is SAS-injective, for any SAS-injective right $R$-module $M$ and for any index set $L$.

(8) $M^{(N)}$ is SAS-injective, for any injective right $R$-module $M$.

(9) $M^{(N)}$ is SAS-injective, for any small-injective right $R$-module $M$.

(10) $M^{(N)}$ is SAS-injective, for any SAS-injective right $R$-module $M$.

**Proof.** By applying Theorem 2.20 and Corollary 2.21. □

**References**


