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Weak Gamma (Quasi-) Continuous Modules

Ghassan Nieaf Abdulrazaq [a] ***Emad Allawi Shallal*** [b]

[a] *Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah- Iraq..Email: : nayyefghassan @gmail.com*

[b] *Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah- Iraq.Email: : emad.a.shallal@qu.edu.iq*

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ABSTRACT

In this paper, we introduce and study the concept of weak gamma continuous modules as a generalization of continuous gamma modules. A gamma submodule N of a gamma module M is called weak gamma summand if there exists R_Γ −submodule K of M and there exists an ideal I of R where $N + K = M$ and $N \cap I\Gamma K = 0$. An R_Γ −module M is called weak gamma extending module if every R_Γ −submodule N of M is essential in a weak gamma summand. An R_Γ −module is called weak gamma continuous module if is it weak gamma extending module and if for each R_Γ −submodule N of M is isomorphic to a direct summand of M, then N is weak gamma summand of M. Also it is called weak gamma quasi-continuous module if is weak gamma extending module and if A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a weak gamma summand of M. Many properties and results of the these R_Γ −modules are given.

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1. Introduction

An $\boldsymbol{R_\Gamma}$ −submodule $\boldsymbol{T}$ of $\boldsymbol{R_\Gamma}$ −module $\boldsymbol{M}$ is called a direct summand of $\boldsymbol{M}$, if there exists an $\boldsymbol{R_\Gamma}$ −submodule $\boldsymbol{L}$ of $\boldsymbol{M}$ such that $\boldsymbol{M = T + L}$ and $\boldsymbol{T \cap L = 0}$, write by $\boldsymbol{M = T \oplus L}$ [6]. An $\boldsymbol{R_\Gamma}$ −submodule $\boldsymbol{T}$ of $\boldsymbol{M}$ is called weak gamma summand (denoted by $\boldsymbol{T \leq_{W\Gamma S} M}$), if there exists $\boldsymbol{R_\Gamma}$ −submodule $\boldsymbol{L}$ of $\boldsymbol{M}$ and there exists ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ such that $\boldsymbol{M = T + L}$ and $\boldsymbol{T \cap I\Gamma L = 0}$. An $\boldsymbol{R_\Gamma}$ −module $\boldsymbol{M}$ is called $\boldsymbol{\Gamma}$ −extending module if every $\boldsymbol{R_\Gamma}$ −submodule of $\boldsymbol{M}$ is essential in a direct summand of $\boldsymbol{M}$ [3]. As a generalization of $\boldsymbol{\Gamma}$ −extending module, an $\boldsymbol{R_\Gamma}$ −module $\boldsymbol{M}$ is called weak gamma extending module (shortly, $\boldsymbol{W\Gamma E}$ − module) if every $\boldsymbol{R_\Gamma}$ − submodule $\boldsymbol{N}$ of $\boldsymbol{M}$ is an essential in a weak gamma summand. An $\boldsymbol{R_\Gamma}$ −module $\boldsymbol{M}$ is called continuous gamma module if it is a $\boldsymbol{\Gamma}$ −extending module and if $\boldsymbol{N \cong B \leq_\oplus M}$, then $\boldsymbol{N \leq_\oplus M}$ [6]. We say that an $\boldsymbol{R_\Gamma}$ −module $\boldsymbol{M}$ is weak gamma continuous module if $\boldsymbol{M}$ is $\boldsymbol{W\Gamma E}$ −module and if for each $\boldsymbol{R_\Gamma}$ −submodule $\boldsymbol{N}$ of $\boldsymbol{M}$ is isomorphic to a weak gamma summand of $\boldsymbol{M}$, then $\boldsymbol{N}$ is a weak gamma summand of $\boldsymbol{M}$. An $\boldsymbol{R_\Gamma}$ −module $\boldsymbol{M}$ is called quasi- continuous gamma module if $\boldsymbol{M}$ is $\boldsymbol{\Gamma}$ −extending module and if $\boldsymbol{A \leq_\oplus M}$ and $\boldsymbol{B \leq_\oplus M}$ with $\boldsymbol{A \cap B = 0}$, then $\boldsymbol{A \oplus B \leq_\oplus M}$ [6]. As a generalization, an $\boldsymbol{R_\Gamma}$ − module $\boldsymbol{M}$ is called weak quasi-continuous gamma module if an $\boldsymbol{R_\Gamma}$ −module $\boldsymbol{M}$ is $\boldsymbol{W\Gamma E}$ −module and if $\boldsymbol{A}$ and $\boldsymbol{B}$ are direct summands of $\boldsymbol{M}$ with $\boldsymbol{A \cap B = 0}$, then $\boldsymbol{A \oplus B}$ is weak gamma summand of $\boldsymbol{M}$. An $\boldsymbol{R_\Gamma}$ − submodule $\boldsymbol{N}$ of an $\boldsymbol{R_\Gamma}$ − module $\boldsymbol{M}$ is called power essential gamma submodule ($\boldsymbol{PE\Gamma}$ −submodule) in $\boldsymbol{M}$ denoted by $\boldsymbol{N \leq_{pe} M}$, for each $\boldsymbol{m \in M}$ and for each ideal $\boldsymbol{I}$ in $\boldsymbol{R}$ with $\boldsymbol{I\Gamma m \neq 0}$, then $\boldsymbol{I\Gamma(N :_{R_\Gamma} m)\Gamma m \neq 0}$ [7]. If $\boldsymbol{I}$ is ideal of $\boldsymbol{R}$ and $\boldsymbol{K}$ is $\boldsymbol{R_\Gamma}$ −submodule of $\boldsymbol{M}$, then $\boldsymbol{I\Gamma K = \{r\gamma t : r \in R, \gamma \in \Gamma}$ and $\boldsymbol{t \in K\}}$. We use $\boldsymbol{L \leq M}$, $\boldsymbol{L \leq_e M}$, $\boldsymbol{L \leq_\oplus M}$, $\boldsymbol{L \leq_{W\Gamma S} M}$, $\boldsymbol{L \leq_{pe} M}$, $\boldsymbol{WPE}$ − module, $\boldsymbol{CG}$ − module and

∗Corresponding author

Email addresses:

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QCG −module, for R_Γ −submodule, essential, direct summand, weak gamma summand, weak power essential gamma, gamma continuous module and gamma quasi-continuous module, respectively. For more basis refer to [1,2,4,5] and [7].

2. Weak Gamma Extending Modules

Definition (2.1): If N is an R_Γ −submodule of R_Γ −module M, then we say that N is weak gamma summand of M (denoted by $N \leq_{W\Gamma S} M$), if there exists an R_Γ −submodule K of M and an ideal I of R such that $N + K = M$ and $N \cap I\Gamma K = 0$. In this case $M = N \boxplus K$.

Clearly that, if $K \leq_{\oplus} M$, then $K \leq_{W\Gamma S} M$. The converse is not true see Examples (3.6)(6).

In the following proposition, we list some fundamental properties of weak gamma summand modules.

Proposition (2.2):

(1) If $N \leq_{W\Gamma S} K$ and $K \leq_{W\Gamma S} M$, then $N \leq_{W\Gamma S} M$.

(2) If $A \leq B$ and $A \leq D$ with $M = B \boxplus D$, then $B / A \leq_{W\Gamma S} M / A$.

(3) If $A \leq_{W\Gamma S} M_1$ and $B \leq_{W\Gamma S} M_2$, then $A \oplus B \leq_{W\Gamma S} M_1 \oplus M_2$.

(4) If $M = A \boxplus B$ and $f: M \to N$ is an epimorphism, then $f(A) \leq_{W\Gamma S} N$.

Proof:

(1) Since $N \leq_{W\Gamma S} K$, then there exists $K_1 \leq K$ such that $N + K_1 = K$ and an ideal I of R such that $N \cap I\Gamma K_1 = 0$. Since $K \leq_{W\Gamma S} M$, then there exists $K_2 \leq M$ such that $K + K_2 = M$ and an ideal J of R such that $K \cap J\Gamma K_2 = 0$. So $(N + K_1) + K_2 = M$, and for the ideal $I\Gamma J$ of R, $N \cap (I\Gamma J)\Gamma(K_1 + K_2) = (N \cap I\Gamma(J\Gamma K_1)) + (N \cap J\Gamma(I\Gamma K_2)) \subseteq N \cap I\Gamma K_1 + K \cap J\Gamma K_2 = 0$, so $N \leq_{W\Gamma S} M$.

(2) Since $M = B \boxplus D$, it follows that $M = B + D$ and there exists an ideal I of R such that $B \cap I\Gamma D = 0$. Thus $(B/A) + (D/A) = (B + D)/A = M/A$ and $(B/A) \cap I\Gamma(D/A) = B \cap I\Gamma D/A = (B/A) \cap (I\Gamma D/A) = 0$.

(3) Since $A \leq_{W\Gamma S} M_1$, there exists $A_1 \leq M_1$ such that $A + A_1 = M_1$ and an ideal I of R such that $A \cap I\Gamma A_1 = 0$. Since $B \leq_{W\Gamma S} M_2$, there exists $B_1 \leq M_2$ such that $B + B_1 = M_2$ and an ideal J of R such that $B \cap J\Gamma B_1 = 0$. Now $(A \oplus B) + (A_1 \oplus B_1) = M_1 \oplus M_2$ and $(A + B) \cap (I\Gamma J)\Gamma(A_1 + B_1) = (A + B) \cap (I\Gamma(J\Gamma A_1) + I\Gamma(J\Gamma B_1) = (A + B) \cap I\Gamma(J\Gamma A_1) + (A + B) \cap I\Gamma(J\Gamma B_1) = A \cap I\Gamma(J\Gamma A_1) + B \cap I\Gamma(J\Gamma A_1) + A \cap I\Gamma(J\Gamma B_1) + B \cap I\Gamma(J\Gamma B_1) \subseteq A \cap I\Gamma A_1 + B \cap J\Gamma B_1 = 0$, hence $A \oplus B \leq_{W\Gamma S} M_1 \oplus M_2$.

(4) Let $M = A \boxplus B$, so $A + B = M$ and $A \cap I\Gamma B = 0$ for some ideal I of R. Since $f(M) = N$, we have $f(A) + f(B) = N$ and $f(A) \cap I\Gamma f(B) = f(A \cap I\Gamma B) = 0$.

Definition(2.3): We say that an R_Γ −module M is weak gamma extending module (for shortly $W\Gamma E$ −module), if it satisfies (WGC_1): Every R_Γ −submodule N of R_Γ −module M is essential in a weak gamma summand.

It is clear that every gamma extending module is weak gamma extending module but the converse is not true, see Examples (3.6)(1).

It is well known that every direct summand is closed, but weak gamma summand may not be closed in general see Examples (3.6)(7).

The following proposition show that if M is a weak gamma extending, then the closed R_Γ −submodule of M is weak gamma summand of M.

Proposition(2.4): If M is a weak gamma extending module, then every closed R_Γ −submodule of M is a weak gamma summand.

Proof: Suppose M is $W\Gamma E$ −module, let N be closed R_Γ −submodule of M, so $N \leq_e B \leq_{W\Gamma S} M$ for some $B \leq M$ but N is closed R_Γ −submodule of M, then $N = B$.

Closed R_Γ −submodule need not be weak gamma summand, for Example see Example (3.6)(7).

Proposition(2.5): If every closed R_Γ −submodule of R_Γ −module M is weak gamma summand of M, then M is weak gamma extending.

Proof: Let N be an R_Γ −submodule of M, by Zorn's Lemma then N has a maximal essential extension such that $N \leq_e B \leq M$, so B is closed R_Γ −submodule of M, then B is weak gamma summand in M which implies that $N \leq_e B \leq_{W\Gamma S} M$.

Every direct summand of Γ −extending module is Γ −extending module[6]. Examples (3.6)(8) shows that a weak gamma summand of weak gamma extending module may not be weak gamma extending module .

3-Weak Gamma (Quasi-)Continuous Module

We introduce the concept of weak gamma (quasi-)continuous module

Definition(3.1): An R_Γ −module M is called weak gamma continuous module (shortly $W\Gamma C$ −module) if it is has (WGC_1) and (WGC_2) where WGC_2 means that if for each $N \cong K \leq_\oplus M$, then $N \leq_{W\Gamma S} M$.

Every continuous gamma module is weak continuous gamma module but converse not true, see Examples (3.6)(6).

Definition(3.2): An R_Γ −module M is called weak gamma quasi-continuous module (shortly $W\Gamma QC$ −module) if has (WGC_1) and (WGC_3) where WGC_3 means that if $T \leq_\oplus M$ and $L \leq_\oplus M$ with $T \cap L = (0)$, then $T \oplus L \leq_{W\Gamma S} M$.

It is clear that QCG −module is $W\Gamma QC$ −module, the converse not true see Examples (3.6)(6).

Lemma(3.3): Isomorphic image of $W\Gamma C$ −module is also $W\Gamma C$ −module.

The prove of the following propositions direct [6].

Proposition(3.4): Every cyclic fully R_Γ −idempotent is weak gamma continuous module.

Proposition(3.5): Every quasi-injective R_Γ −module is weak gamma continuous module.

Examples (3.6):

1- The Z_2 and Z_8 as Z_Z −modules are $W\Gamma E$ −modules, since they are Γ −extending modules [1]. Let $M = Z_2 \oplus Z_8 = \{(0,0), (0,1), (0,2), (0,3), (0,4)\ (0,5)\ , (0,6), (0,7), (1,0), (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7)\}$.

The R_Γ −submodules of M are: $K_1 = \langle(0,1)\rangle$, $K_2 = \langle(0,2)\rangle$, $K_3 = \langle(0,4)\rangle$, $K_4 = \langle(1,0)\rangle$, $K_5 = \langle(1,1)\rangle$, $K_6 = \langle(1,2)\rangle$ and $K_7 = \langle(1,4)\rangle$. Note that K_1 , K_4 , K_5 and K_7 are weak gamma summand of M since the are direct summands, K_2 is essentials of K_1, and K_3 , K_6 are essential in K_5, , then M is $W\Gamma E$ −module, note that M is not Γ −Extending module since K_6 is closed but not direct summand of M.

2- Every simple R_Γ −module is $W\Gamma E$ −module because every simple R_Γ −module is Γ −extending module, the converse is not true, for example $Z_2 \oplus Z_4$ is $W\Gamma E$ −module but it is not simple .

3- $W\Gamma QC$ − module is $W\Gamma E$ − module, but the converse not true. Let $M = Z_2 \oplus Z_4 = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3)$ $A = \langle(1,0)\rangle \leq_\oplus M$ and $B = \langle(1,2)\rangle \leq_\oplus M$, but $A \oplus B = \{(0,0), (1,2), (1,0), (0,2)\}$ not weak gamma summand, therefore M is $W\Gamma E$ −module but not $W\Gamma QC$ −module.

4- $W\Gamma C$ −module is $W\Gamma QC$ −module but the converse is not true . The Q as Z_Q −module is $W\Gamma QC$ −module but not $W\Gamma C$ −module. $2Z \cong Z \leq_{\oplus} Z$, but $2Z$ is not weak gamma summand of Z, since $2n(\neq 0) \in 2Z.Z.nZ$ for any ideal nZ of Z.

5- Every $PE\Gamma$ −module is $W\Gamma E$ −module because every $PE\Gamma$ −module is Γ −extending module [7], the converse is not true, let $R = \Gamma = Z$ and $M = Z_4 \oplus Z_2$, then M is Γ −Extending module, so M is $W\Gamma E$ −module but M is not $PE\Gamma$ −module[7].

6- Let $R = \left\{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y, z \in F\right\}$ and $\Gamma = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in F\right\}$ where F is a field, then R is Γ −ring with usual multiplication of matrices. If consider $A = \left\{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in F\right\}$, $B = \left\{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} : y \in F\right\}$, $C = \left\{\begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix} : y, z \in F\right\}$, then $R = A \oplus C$ and $B \cong A \leq_{\oplus} R$ while B can not be direct summand of R, so R has not CG −module. Since $R = B + R$ and $C\Gamma R = \left\{\begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix} : w \in F\right\}$, such that $B \cap C\Gamma R = 0$, therefore B is weak gamma summand of R, thus R is $W\Gamma C$ −module but not CG −module also $B \leq_{W\Gamma S} R$ but $B \nleq_{\oplus} R$

7- Let $M = Z_3 \oplus Z_4$ as Z_Z −module, take $A = \langle(0,1)\rangle$, and $F = \langle(1,2)\rangle$. Then A and F are R_Γ −submodules of M with $F + A = M$, $F \cap 4Z \cdot Z \cdot A = 0$, therefore $F \leq_{W\Gamma S} M$, but F is not closed in M.

8- Let $M = Z_4 \oplus Z_4 = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3), (2,0)\ (2,1), (2,2), (2,3), (3,0), (3,1), (3,2), (3,3)\}$ and $P = \langle 2 \rangle \oplus Z_4 = \{(0,0), (0,1), (0,2), (0,3), (2,0), (2,1), (2,2), (2,3)\}$, since M is Γ −extending module, then M is $W\Gamma E$ −module. Let $C = \langle(1,0)\rangle$, then $P + C = M$, and $P \cap 4Z.Z.C = 0$ therefore $P \leq_{W\Gamma S} M$. Take $B = \{(0,0), (0,2)\}$ is an R_Γ −submodule of P and $B \nleq_e K \leq P$ for any $K \leq P$ thus P is not $W\Gamma E$ −module.

9- Direct sum of two $W\Gamma C$ −modules may not be $W\Gamma C$ −module, see examples (3) Z_2 and Z_4 are $W\Gamma C$ −modules because Z_2 and Z_4 are CG −modules and $(0) \oplus (2) \cong Z_2 \oplus (0) \leq_{\oplus} M$ but $(0) \oplus (2) \nleq_{W\Gamma S} M$, hence M is not $W\Gamma C$ −module.

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