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A Certain Families of Bi-Univalent Functions with Respect to Symmetric Conjugate Points Defined by Beta Negative Binomial Distribution Series

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ABSTRACT

The objective of this paper is to introduce and investigate two families of analytical and bi-univalent functions, $T_{\Sigma}^{sc}(\alpha,\beta,\eta,\lambda,\theta;\mu)$ and $W_{\Sigma}^{sc}(\alpha,\beta,\eta,\lambda,\theta;v)$, with respect to symmetric conjugate points that are defined in the open unit disk U and connected to a series of beta-negative binomial distributions. For functions in each of these families, we look into upper bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

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1. Introduction

We indicate by $\mathcal{A}$ the family of functions which are holomorphic in the open unit disk $U=\{z\in\mathbb{C} : |z|<1\}$ and have the following normalized from :

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \ . \tag{1.1}$$

We also indicate by S the subclass of $\mathcal{A}$ consisting of functions which are also univalent in U. According to the Koebe one-quarter theorem [3], every function $f\in S$ has an inverse f^{-1} defined by

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$$f^{-1}\big(f(z)\big) = z, (z \in U)$$

and

$$f\big(f^{-1}(w)\big) = w, \qquad \left(|w| < r_0(f), r_0(f) \geq \tfrac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \qquad (1.2)$$

If both f and f^{-1} are univalent in U, then a function $f \in \mathcal{A}$ is said to be bi-univalent in U. The category of bi-univalent functions in U provided by (1.1) is denoted by the symbol. See the ground-breaking study on this topic by Srivastava et al. [13] for a brief history and fascinating instances of functions in the class. This work really rekindled the study of bi-univalent in more recent years. Here, we select to recollect the following instances of functions in the class from the work of Srivastava et al. [13]:

$$\frac{z}{1-z}\ ,\ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \text{ and } -\log(1-z)$$

We notice that the class Σ is not empty . However the Koebe function is not a member of Σ.

Many authors introduced and studied various subclasses of the bi-univalent function class in sequels to the work of Srivastava et al. [13] (see, for example, [1,5,6,10,11,16,17,18]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor -Maclaurin expansion (1.1) were obtained in many of these recent papers. The challenge of determining the Taylor-Maclaurin coefficients' universal bounds $|a_n|$ ($n \in \mathbb{N} \setminus\{1,2\}; \mathbb{N} := \{1,2,3,\dots\}$),

for functions $f \in \Sigma$ is still not completely addressed for many of the subclasses of the bi-univalent function class Σ(see, for example, [12,14,15]).

From a theoretical standpoint, the Geometric Function Theory has explored some basic distributions, including the Poisson, Pascal, Logarithmic, Binomial, and Borel (for examples, see [2,4,7,9,20]).

If a discrete random variable x takes the values 0, 1, 2, 3,... with the probability, it is said to have a beta negative binomial distribution.

$\frac{\beta(\eta+\theta,\lambda)}{\beta(\eta,\lambda)}, \theta\frac{\beta(\eta+\theta,\lambda+1)}{\beta(\eta,\lambda)}, \frac{1}{2}\theta(\theta+1)\frac{\beta(\eta+\theta,\lambda+2)}{\beta(\eta,\lambda)}, \dots$ respectively, where η, θ, λ are named the parameters.

$$Prob(x=\tau) = \binom{\theta+\tau-1}{\tau}\frac{\beta(\eta+\theta,\lambda+\tau)}{\beta(\eta,\lambda)} = \frac{\Gamma(\theta+\tau)}{\tau!\ \Gamma(\theta)}\ \frac{\Gamma(\eta+\theta)\Gamma(\lambda+\tau)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+\tau)\Gamma(\eta)\Gamma(\lambda)} = \frac{(\eta)_\theta\ (\theta)_\tau\ (\lambda)_\tau}{(\eta+\lambda)_\theta\ (\theta+\eta+\lambda)_\tau\ \tau!},$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} 1 & (k=0), \\ \alpha(\alpha+1)\dots(\alpha+k-1) & (k \in \mathbb{N}). \end{cases}$$

The following power series, whose coefficients are probabilities of the beta negative binomial distribution, was very recently introduced by Wanas and Al-Ziadi [19].

$$\wp_{\eta,\lambda}^{\theta}(z) = z + \sum_{k=2}^{\infty}\frac{(\eta)_\theta\ (\theta)_{k-1}\ (\lambda)_{k-1}}{(\eta+\lambda)_\theta\ (\theta+\eta+\lambda)_{k-1}\ (k-1)!}z^k, \quad (z \in U, \eta, \lambda, \theta > 0).$$

We observe using the well-known Ratio Test that the above series' radius of convergence is infinite.

The linear operator $\mathfrak{P}_{\eta,\lambda}^{\theta} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined as follows (see [19])

$$\mathfrak{P}_{\eta,\lambda}^{\theta}f(z) = \wp_{\eta,\lambda}^{\theta}(z) * f(z) = z + \sum_{k=2}^{\infty}\frac{(\eta)_\theta\ (\theta)_{k-1}\ (\lambda)_{k-1}}{(\eta+\lambda)_\theta\ (\theta+\eta+\lambda)_{k-1}\ (k-1)!}a_k z^k, \qquad z \in U,$$

where $*$ indicate the Hadamard product (or convolution) of two series.

We now think back to the lemma that will be utilized to support our key findings.

Lemma 1.1 [3]. If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h analytic in U for which

$$Re(h(z)) > 0, (z \in U),$$

with

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, (z \in U).$$

2. Coefficient Bounds for the Class $\mathbf{T}_{\Sigma}^{sc}(\alpha, \beta, \eta, \lambda, \theta; \mu)$

Definition 2.1. A function $f \in \Sigma$ from (1.1) is said to be in the class $T_{\Sigma}^{sc}(\alpha, \beta, \eta, \lambda, \theta; \mu)$ if it satisfies the following conditions:

$$\left| \arg \left(\frac{2\left[\alpha\beta z^3 \left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z)\right)''' + (\alpha + \beta(2\alpha - 1)) z^2 \left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z)\right)'' + z \left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z)\right)'\right]}{\alpha\beta z^2 \left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z) - \overline{\mathfrak{P}_{\eta,\lambda}^{\theta} f(-\bar{z})}\right)'' + (\alpha - \beta) z \left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z) - \overline{\mathfrak{P}_{\eta,\lambda}^{\theta} f(-\bar{z})}\right)' + (1 - \alpha + \beta)\left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z) - \overline{\mathfrak{P}_{\eta,\lambda}^{\theta} f(-\bar{z})}\right)} \right) \right| < \frac{\mu\pi}{2}, \tag{2.1}$$

and

$$\left| \arg \left(\frac{2\left[\alpha\beta w^3 \left(\mathfrak{P}_{\eta,\lambda}^{\theta} g(w)\right)''' + (\alpha + \beta(2\alpha - 1)) w^2 \left(\mathfrak{P}_{\eta,\lambda}^{\theta} g(w)\right)'' + w \left(\mathfrak{P}_{\eta,\lambda}^{\theta} g(w)\right)'\right]}{\alpha\beta w^2 \left(\mathfrak{P}_{\eta,\lambda}^{\theta} g(w) - \overline{\mathfrak{P}_{\eta,\lambda}^{\theta} g(-\bar{w})}\right)'' + (\alpha - \beta) w \left(\mathfrak{P}_{\eta,\lambda}^{\theta} g(w) - \overline{\mathfrak{P}_{\eta,\lambda}^{\theta} g(-\bar{w})}\right)' + (1 - \alpha + \beta)\left(\mathfrak{P}_{\eta,\lambda}^{\theta} g(w) - \overline{\mathfrak{P}_{\eta,\lambda}^{\theta} g(-\bar{w})}\right)} \right) \right| < \frac{\mu\pi}{2}, (w \in U), \tag{2.2}$$

where $z, w \in U, 0 < \mu \leq 1, \eta, \lambda, \theta > 0, 0 \leq \beta \leq \alpha \leq 1$ and $g = f^{-1}$ is given by (1.2).

Theorem 2.1 below states that our first main finding.

Theorem 2.1. Let $f \in T_{\Sigma}^{sc}(\alpha, \beta, \eta, \lambda, \theta; \mu)$ $(0 < \mu \leq 1, \eta, \lambda, \theta > 0)$ be given by (1.1). Then

$$|a_2| \leq \frac{\mu\Gamma(\eta + \theta + \lambda + 1)\Gamma(\eta)\Gamma(\lambda)\sqrt{\Gamma(\eta + \theta + \lambda + 2)}}{\sqrt{\left|\begin{array}{c}\mu\theta(\theta + 1)(6\alpha\beta + 2(\alpha - \beta) + 1)\Gamma^2(\eta + \theta + \lambda + 1)\Gamma(\eta + \theta)\Gamma(\lambda + 2)\Gamma(\eta + \lambda)\Gamma(\eta)\Gamma(\lambda) \\ +2(1 - \mu)\theta^2(2\alpha\beta + \alpha - \beta + 1)^2\Gamma(\eta + \theta + \lambda + 2)\Gamma^2(\eta + \theta)\Gamma^2(\lambda + 1)\Gamma^2(\eta + \lambda)\end{array}\right|}}$$

and

$$|a_3| \leq \frac{\mu^2\Gamma^2(\eta + \theta + \lambda + 1)\Gamma^2(\eta)\Gamma^2(\lambda)}{(2\alpha\beta + \alpha - \beta + 1)^2\theta^2\Gamma^2(\eta + \theta)\Gamma^2(\lambda + 1)\Gamma^2(\eta + \lambda)} + \frac{2\Gamma(\eta + \theta + \lambda + 2)\Gamma(\eta)\Gamma(\lambda)}{\theta(\theta + 1)(6\alpha\beta + 2(\alpha - \beta) + 1)\Gamma(\eta + \theta)\Gamma(\lambda + 2)\Gamma(\eta + \lambda)}.$$

Proof. Conditions (2.1) and (2.2) lead to the conclusion that

$$\frac{2\left[\alpha\beta z^3 \left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z)\right)''' + (\alpha + \beta(2\alpha - 1)) z^2 \left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z)\right)'' + z \left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z)\right)'\right]}{\alpha\beta z^2 \left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z) - \overline{\mathfrak{P}_{\eta,\lambda}^{\theta} f(-\bar{z})}\right)'' + (\alpha - \beta) z \left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z) - \overline{\mathfrak{P}_{\eta,\lambda}^{\theta} f(-\bar{z})}\right)' + (1 - \alpha + \beta)\left(\mathfrak{P}_{\eta,\lambda}^{\theta} f(z) - \overline{\mathfrak{P}_{\eta,\lambda}^{\theta} f(-\bar{z})}\right)} = [p(z)]^{\mu} \tag{2.3}$$

and

$$\frac{2\left[\alpha\beta w^3\left(\mathfrak{P}^{\theta}_{\eta,\lambda}g(w)\right)''' + (\alpha+\beta(2\alpha-1))w^2\left(\mathfrak{P}^{\theta}_{\eta,\lambda}g(w)\right)'' + w\left(\mathfrak{P}^{\theta}_{\eta,\lambda}g(w)\right)'\right]}{\alpha\beta w^2\left(\mathfrak{P}^{\theta}_{\eta,\lambda}g(w) - \overline{\mathfrak{P}^{\theta}_{\eta,\lambda}g(-\overline{w})}\right)'' + (\alpha-\beta)w\left(\mathfrak{P}^{\theta}_{\eta,\lambda}g(w) - \overline{\mathfrak{P}^{\theta}_{\eta,\lambda}g(-\overline{w})}\right)' + (1-\alpha+\beta)\left(\mathfrak{P}^{\theta}_{\eta,\lambda}g(w) - \overline{\mathfrak{P}^{\theta}_{\eta,\lambda}g(-\overline{w})}\right)} = [q(w)]^{\mu} \tag{2.4}$$

where $g = f^{-1}$ and p, q in $\mathcal{P}$ have the representations of the following series:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.5}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{2.6}$$

We discover that by comparing the corresponding coefficients of (2.3) and (2.4),

$$\frac{2(2\alpha\beta+\alpha-\beta+1)\theta\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)} a_2 = \mu p_1, \tag{2.7}$$

$$\frac{(6\alpha\beta+2(\alpha-\beta)+1)\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)} a_3 = \mu p_2 + \frac{\mu(\mu-1)}{2} p_1^2, \tag{2.8}$$

$$-\frac{2(2\alpha\beta+\alpha-\beta+1)\theta\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)} a_2 = \mu q_1 \tag{2.9}$$

and

$$\frac{(6\alpha\beta+2(\alpha-\beta)+1)\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)} (2a_2^2 - a_3) = \mu q_2 + \frac{\mu(\mu-1)}{2} q_1^2. \tag{2.10}$$

Making use of (2.7) and (2.9), we obtain

$$p_1 = -q_1 \tag{2.11}$$

and

$$\frac{8\theta^2(2\alpha\beta+\alpha-\beta+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)} a_2^2 = \mu^2(p_1^2 + q_1^2). \tag{2.12}$$

If we add (2.8) to (2.10), we have

$$\frac{2\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)} a_2^2 = \mu(p_2+q_2) + \frac{\mu(\mu-1)}{2}(p_1^2 + q_1^2). \tag{2.13}$$

After performing various calculations and substituting the value of $p_1^2 + q_1^2$ from (2.12) into the right-hand side of (2.13), we conclude that

$$a_2^2 = \frac{\mu^2\Gamma(\eta+\theta+\lambda+2)\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)(p_2+q_2)}{2\mu\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)\Gamma(\eta)\Gamma(\lambda) + 4(1-\mu)\theta^2(2\alpha\beta+\alpha-\beta+1)^2\Gamma(\eta+\theta+\lambda+2)\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}. \tag{2.14}$$

Now, taking the absolute value of (2.14) and applying Lemma 1.1 for the coefficients p_2 and q_2, we obtain

$$|a_2| \leq \frac{\mu\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\sqrt{\Gamma(\eta+\theta+\lambda+2)}}{\sqrt{\left|\mu\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)\Gamma(\eta)\Gamma(\lambda) + 2(1-\mu)\theta^2(2\alpha\beta+\alpha-\beta+1)^2\Gamma(\eta+\theta+\lambda+2)\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\right|}}$$

In order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8), we get

$$\frac{2\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}(a_3-a_2^2)=\mu(p_2-q_2)+\frac{\mu(\mu-1)}{2}(p_1^2-q_1^2). \quad (2.15)$$

It follows from (2.11), (2.12) and (2.15) that

$$a_3=\frac{\mu^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)(p_1^2+q_1^2)}{8(2\alpha\beta+\alpha-\beta+1)^2\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}+\frac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)(p_2-q_2)}{2\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}. \quad (2.16)$$

Taking the absolute value of (2.16) and applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we obtain

$$|a_3|\le\frac{\mu^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}{(2\alpha\beta+\alpha-\beta+1)^2\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}+\frac{2\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)},$$

which completes the proof of Theorem 2.1.

3. Coefficient Bounds for the Class $W_\Sigma^{sc}(\alpha,\beta,\eta,\lambda,\theta;v)$

Definition 3.1. A function $f\in\Sigma$ given by (1.1) is said to be in the class $W_\Sigma^{sc}(\alpha,\beta,\eta,\lambda,\theta;v)$ if it satisfies the following conditions:

$$Re\left\{\frac{2\left[\alpha\beta z^3\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)\right)'''+(\alpha+\beta(2\alpha-1))z^2\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)\right)''+z\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)\right)'\right]}{\alpha\beta z^2\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}f(-\bar z)}\right)''+(\alpha-\beta)z\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}f(-\bar z)}\right)'+(1-\alpha+\beta)\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}f(-\bar z)}\right)}\right\}>v \quad (3.1)$$

and

$$Re\left\{\frac{2\left[\alpha\beta w^3\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)\right)'''+(\alpha+\beta(2\alpha-1))w^2\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)\right)''+w\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)\right)'\right]}{\alpha\beta w^2\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}g(-\bar w)}\right)''+(\alpha-\beta)w\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}g(-\bar w)}\right)'+(1-\alpha+\beta)\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}g(-\bar w)}\right)}\right\}>v, \quad (3.2)$$

where

$z,w\in U$, $0\le v<1,\eta,\lambda,\theta>0$ and the function $g=f^{-1}$ is given by (1.2).

Our second main result is asserted by Theorem 3.1 below.

Theorem 3.1. Let $f\in W_\Sigma^{sc}(\alpha,\beta,\eta,\lambda,\theta;v)$ $(0\le v<1,\eta,\lambda,\theta>0)$ be given by (1.1). Then

$$|a_2|\le\sqrt{\frac{2(1-v)\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{|\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)|}} \quad (3.3)$$

and

$$|a_3| \le \frac{(1-v)^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}{\theta^2(2\alpha\beta+\alpha-\beta+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)} + \frac{2(1-v)\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}. \tag{3.4}$$

Proof. It results from (3.1) and (3.2) that there exist $p, q \in \mathcal{P}$ such that

$$\frac{2\left[\alpha\beta z^3\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)\right)''' + \left(\alpha+\beta(2\alpha-1)\right)z^2\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)\right)'' + z\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)\right)'\right]}{\alpha\beta z^2\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}f(-\bar{z})}\right)'' + (\alpha-\beta)z\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}f(-\bar{z})}\right)' + (1-\alpha+\beta)\left(\mathfrak{P}_{\eta,\lambda}^{\theta}f(z)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}f(-\bar{z})}\right)} = v + (1-v)p(z) \tag{3.5}$$

and

$$\frac{2\left[\alpha\beta w^3\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)\right)''' + \left(\alpha+\beta(2\alpha-1)\right)w^2\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)\right)'' + w\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)\right)'\right]}{\alpha\beta w^2\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}g(-\bar{w})}\right)'' + (\alpha-\beta)w\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}g(-\bar{w})}\right)' + (1-\alpha+\beta)\left(\mathfrak{P}_{\eta,\lambda}^{\theta}g(w)-\overline{\mathfrak{P}_{\eta,\lambda}^{\theta}g(-\bar{w})}\right)} = v + (1-v)q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ get the forms (2.5) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$\frac{2(2\alpha\beta+\alpha-\beta+1)\theta\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}a_2 = (1-v)p_1, \tag{3.7}$$

$$\frac{(6\alpha\beta+2(\alpha-\beta)+1)\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}a_3 = (1-v)p_2, \tag{3.8}$$

$$-\frac{2(2\alpha\beta+\alpha-\beta+1)\theta\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}a_2 = (1-v)q_1 \tag{3.9}$$

and

$$\frac{(6\alpha\beta+2(\alpha-\beta)+1)\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}(2a_2^2-a_3) = (1-v)q_2. \tag{3.10}$$

From (3.7) and (3.9), we have

$$p_1 = -q_1 \tag{3.11}$$

and

$$\frac{8\theta^2(2\alpha\beta+\alpha-\beta+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}a_2^2 = (1-v)^2(p_1^2+q_1^2). \tag{3.12}$$

Adding (3.8) and (3.10), we obtain

$$\frac{2\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}a_2^2 = (1-v)(p_2+q_2). \tag{3.13}$$

Therefore, we obtain

$$a_2^2 = \frac{(1-v)\mu(\mu-1)\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)(p_2+q_2)}{2\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}.$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we have

$$|a_2| \le \sqrt{\frac{2\,(1-v)\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{|\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)|}}.$$

This gives the desired estimate for $|a_2|$ as asserted in (3.3).

In order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we have

$$\frac{2\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}(a_3-a_2^2)=(1-v)(p_2-q_2),$$

or equivalently

$$a_3=a_2^2+\frac{(1-v)\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)(p_2-q_2)}{2\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}.$$

Upon substituting the value of a_2^2 from (3.12), it follows that

$$a_3=\frac{(1-v)^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)(p_1^2+q_1^2)}{8\theta^2(2\alpha\beta+\alpha-\beta+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}+\frac{(1-v)\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)(p_2-q_2)}{2\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}.$$

Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2 , we have

$$|a_3|\le\frac{(1-v)^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}{\theta^2(2\alpha\beta+\alpha-\beta+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}+\frac{2(1-v)\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{\theta(\theta+1)(6\alpha\beta+2(\alpha-\beta)+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)},$$

which completes the proof of Theorem 3.1.

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