Let $R$ be a ring. A right $R$-module $M$ is called JS-N-injective (where $N$ is any right $R$-module) if every right $R$-homomorphism from a submodule of $J(N)(R_R)$ into $M$ extends to $N$ [9]. A ring $R$ is called right JS-injective if $R_R$ is JS-R-injective. The right JS-injective rings are studied in this paper. Many characterizations and properties of this type of rings are obtained.

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1. Introduction

This paper assumes that $R$ is an associative ring with identity $1 \neq 0$ and any module is unitary. By a module (resp. homomorphism) we mean a right $R$-module (resp. right $R$-homomorphism), if not otherwise specified. The class of right $R$-modules is denoted by Mod-$R$. We use soc($M$) and $J(M)$ to denote, respectively, the socle and the Jacobson radical of a right $R$-module $M$. We write $Z(R_R)$ for the right singular ideal of a ring $R$. We denote to $J(M)(R_R)$ by JS($M$) for any right $R$-module $M$. For any $a \in R$, we use $l_R(a)$ (resp. $r_R(a)$) to denote the left (resp. right) annihilator of $a$ in $R$.

Injective modules play important role in module theory, and extensively many authors were studied their generalizations (see, for example, [5], [6], and [7]). If every $R$-homomorphism from a right ideal of $R$ into $R_R$ can be extended to $R_R$, then a ring $R$ is called right self-injective ring [4, p.64]. Let $N$ be a right $R$-module. A right $R$-module $M$ is called JS-N-injective, if every right $R$-homomorphism from a submodule of $J(N)(R_R)$ into $M$ extends to $N$. If a right $R$-module $M$ is JS-R-injective, then $M$ is called JS-injective. A ring $R$ is called right JS-injective if the right $R$-module $R_R$ is JS-injective [9]. JS-injective rings are studied in this paper. We give many characterizations and properties of right JS-injective rings. For examples, we prove that a ring $R$ is a right JS-injective if and only if for any
$N \in \text{Mod-R}$ and a nonzero $R$-monomorphism $f$ from $N$ to $R$ with $f(N) \subseteq J(S(R))$, then $\text{Hom}_R(R, N) = Rf$. Also, we show that if $R$ is a right JS-injective ring, then $l_p(A_1 \cap A_2) = l_p(A_1) + l_p(A_2)$, for all submodules $A_1$ and $A_2$ of $J(S(R))$.

In Proposition 2.4, we prove that if $R$ is a right JS-injective ring, then $J(S(R)) \subseteq Z(R)$. Moreover, we show that if $R$ is a simple left ideal of a right JS-injective ring $R$, then $J(S(aR)) \cap \text{soc}(aR)$ is zero or simple, for any $a \in R$. Condition under which JS-injectivity implies injectivity is given. We get that if $R$ is a ring such that for any right ideal $K$ of $R$, we have $K = eR \cap B$ for some right ideal $B$ of $R$ with $B \subseteq J(S(R))$ and an idempotent $e^2 = e \in K$, then $R$ is a right JS-injective ring if and only if $R$ is a right self-injective ring. Then, we prove that if $R$ is a right JS-injective ring, and $a, b \in R$ with $b \in J(S(R))$ and $bR \cong aR$, then $Ra \cong Rb$. Finally, we prove that every right $R$-module is JS-injective if and only if $R$ is a right JS-injective ring and every cyclic submodule of $J(S(R))$ is projective.

### 2. JS-Injective Rings

Let $N$ be a right $R$-module. A right $R$-module $M$ is called JS-$N$-injective, if every right $R$-homomorphism from a submodule of $J(N)(J(S(R)))$ into $M$ extends to $N$. A right $R$-module $M$ is called JS-injective if $M$ is JS-$R$-injective. A ring $R$ is called right JS-injective if the right $R$-module $R_0$ is JS-injective [9]. In this section, right JS-injective rings are studied extensively. Many characterizations and properties of this type of rings are given.

Recall that a right $R$-module $M$ is called multiplication if any submodule $N$ of $M$ takes the form $MI$, for some ideal $I$ of $R$ [8, p. 3839].

We begin this section with the following theorem, which gives some characterizations of right JS-injective rings.

**Theorem 2.1.** Consider the following statements for a ring $R$:

1. $R$ is a right JS-injective ring.
2. If $N$ and $M$ are finitely generated projective right $R$-modules, then $M$ is JS-$N$-injective.
3. If $N \in \text{Mod-R}$ and $f: N \to R$ is an $R$-monomorphism with $f(N) \subseteq J(S(R))$, then $\text{Hom}_R(N, R) = Rf$.

Then $2 \Rightarrow 1$ and $1 \Leftrightarrow 3$. Moreover, if $J(S(R)^m)$ is a multiplication module for any positive integer $m$, then $1 \Rightarrow 2$.

**Proof.** $2 \Rightarrow 1$ Clear.

$(1) \Rightarrow (2)$ Suppose that $R$ is a right JS-injective ring with $J(S(R)^m)$ is a multiplication module, for any $m \in Z^+$. Let $N$ and $M$ be finitely generated projective right $R$-modules and $K$ a submodule of $J(S(N))$. Let $f: K \to M$ be any $R$-homomorphism. Since $M$ is finitely generated, there exists a right $R$-epimorphism $\alpha_1: R^n \to M$ for some positive integer $n$. Since $M$ is projective, there is a right $R$-monomorphism $\alpha_2: M \to R^n$ with $\alpha_1\alpha_2 = l_M$, where $l_M: M \to M$ is the identity homomorphism. Since $R$ is a right $J(S)$-injective ring, we have from [9, Proposition 2.5 and Corollary 2.4] that $R^n$ is a right JS-$R^n$-injective $R$-module, for any $m \in Z^+$. Since $N$ is finitely generated, $N$ is a direct summand of $R^k$ for some $k$. By [9, Proposition 2.3(2)], $R^n$ is a right JS-$N$-injective $R$-module. Then $h_i = \alpha_2 f_i$, for some $h_i \in \text{Hom}_R(N, R^n)$. Put $g_i = \alpha_1 h_i: N \to M$. Then $gi = (\alpha_1 h_i) i = \alpha_1 (h_i) = \alpha_1 (\alpha_2 f_i) = (\alpha_1 \alpha_2) f_i = l_M f_i = f$. Therefore, $gi = f$ for some $R$-homomorphism $g: N \to M$.

$(1) \Rightarrow (3)$ Suppose that $R$ is a right JS-injective ring. Let $N$ be any right $R$-module and $f: N \to R$ be a nonzero $R$-monomorphism with $f(N) \subseteq J(S(R))$. Define $f: N \to f(N)$ by $\hat{f}(a) = f(a)$, for all $a \in N$. It is clear that $\hat{f}$ is an isomorphism. Let $g \in \text{Hom}_R(N, R)$, then we have $gf^{-1}: f(N) \to R$ is an $R$-homomorphism. Since a ring $R$ is right JS-injective and $f(N) \subseteq J(S(R))$, there is $c \in R$ with $(gf^{-1})(k) = ck$, for all $k \in f(N)$ (by [9, Proposition 2.7]). Let $n \in N$, then $f(n) \in f(N)$ and hence $(gf^{-1})(f(n)) = cf(n)$. Since $(gf^{-1})(f(n)) = g(n)$, it follows that $g(n) = cf(n)$, for all $n \in N$. Thus $\text{Hom}_R(N, R) = Rf$.

$(3) \Rightarrow (1)$ Let $K$ be a submodule of $J(S(R))$, $f: K \to R$ a right $R$-homomorphism, and $i: K \to R$ the inclusion map. Then by hypothesis, we have $\text{Hom}_R(K, R) = Ri$ and hence $f = ci$ for some $c \in R$. Thus there exists $c \in R$ such that $f(a) = ca$ for all $a \in K$. Then $R$ is a right $J(S)^m$-injective ring, by [9, Proposition 2.7].

**Theorem 2.2.** Let $R$ be a right JS-injective ring, then the following statements hold:
(1) \( l_gr_R(m) = Rm \), for all \( m \in JS(R_R) \).

(2) If \( r_R(m) \subseteq r_R(n) \), where \( m \in JS(R_R) \) and \( n \in R \), then \( Rn \subseteq Rm \).

(3) \( l_g(mR \cap r_R(a)) = l_g(m) + Ra \), for all \( m, a \in R \) with \( am \in JS(R_R) \).

**Proof.** (1) Let \( m \in JS(R_R) \) and let \( n \in l_gr_R(m) \). By [1, Proposition 2.15, p. 37], \( r_R(m) = r_R(l_gr_R(m) \subseteq r_R(n) \). Define \( f: mR \to R \) by \( f(mr) = nr \) for any \( r \in R \), thus \( f \) is a well-defined right \( R \)-homomorphism. By hypothesis, there exists an endomorphism \( g \) of \( R \) such that \( g(x) = f(x) \), for all \( x \in mR \). Then \( n = n \cdot 1 = f(m \cdot 1) = f(m) = g(m) = g(1)m \in Rm \). Hence \( l_g(m) \subseteq Rm \). Conversely, let \( rm \in Rm \). Thus \( r \in R \). Thus \( rmk = 0 \) for all \( k \in r_R(m) \) and hence \( rm \in l_g(m) \). Therefore, \( l_gr_R(m) = Rm \).

(2) Let \( n \in R \) and \( m \in JS(R_R) \) such that \( r_R(m) \subseteq r_R(n) \). Since \( r_R(m) \subseteq r_R(n) \) (by hypothesis), \( l_gr_R(n) \subseteq l_gr_R(m) \) (by [1, Proposition 2.15, p. 37]). So, \( n \in l_gr_R(m) \). By (1), \( n \in Rm \) and this implies that \( Rn \subseteq Rm \).

(3) Let \( a, m \in R \) such that \( am \in JS(R_R) \). If \( x \in l_g(m) + Ra \), then \( x = x_1 + x_2 \) such that \( x_1, m = 0 \) and \( x_2 = sa \) for some \( s \in R \). For all \( b \in mR \cap r_R(a) \), we have \( b = nr \) and \( ab = 0 \) for some \( r \in R \). Since \( x_1b = x_1(mr) = (x_1m)r = 0 \) and \( x_2b = (sa)b = s(ab) = 0 \), it is follows that \( x \in l_g(mR \cap r_R(a)) \). This implies that \( l_g(m) + Ra \subseteq l_g(mR \cap r_R(a)) \). Let \( y \in l_g(mR \cap r_R(a)) \). If \( r \in r_R(am) \), then \( (am)r = 0 \) and hence \( a(mr) = 0 \). Thus \( mr \in mR \cap r_R(a) \) and hence \( ym = y(mr) = 0 \) and \( ym \in l_g(r_R(am)) \). Thus \( l_gr_R(r_R(am)) \subseteq l_gr_R(ym) \). By [1, Proposition 2.15, p. 37], \( r_R(am) \subseteq l_gr_R(ym) \). By hypothesis, \( am \in JS(R_R) \). By (2), \( Rym \subseteq Ram \). Thus \( ym = sam \), for some \( s \in R \) and hence \( (y - sa)m = 0 \) and this implies that \( y - sa \in l_g(m) \). Thus \( y \in l_g(m) + Ra \) and hence \( l_g(mR \cap r_R(a)) = l_g(m) + Ra \). □

**Proposition 2.3.** If \( R \) is a right \( JS \)-injective ring, then \( l_g(A_1 \cap A_2) = l_g(A_1) + l_g(A_2) \), for all submodules \( A_1 \) and \( A_2 \) of \( JS(R_R) \).

**Proof.** Let \( A_1 \) and \( A_2 \) be any two submodules of \( JS(R_R) \). Let \( r \in l_g(A_1 \cap A_2) \), thus \( r \cdot (A_1 \cap A_2) = 0 \). Consider the mapping \( f: A_1 + A_2 \to R \) is given by \( f(a_1 + a_2) = ra_1 \), for all \( a_1 \in A_1, a_2 \in A_2 \). Thus \( f \) is a well-defined right \( R \)-homomorphism, since if \( a_1 + a_2 = b_1 + b_2 \), where \( a_1, b_1 \in A_1 \), \( a_2, b_2 \in A_2 \), then \( a_1 - b_1 = a_2 - b_2 \). Since \( r \cdot (A_1 \cap A_2) = 0 \), we have that \( r(a_1 - b_1) = 0 \) and hence \( ra_1 = rb_1 \), so \( f(a_1 + a_2) = f(b_1 + b_2) \) and this implies that \( f \) is a well-defined. Also, for every \( a_1 + a_2, b_1 + b_2 \in A_1 + A_2 \) where \( a_1, b_1 \in A_1 \), \( a_2, b_2 \in A_2 \), and \( t \in R \), we have \( f((a_1 + a_2) + (b_1 + b_2)) = f((a_1 + b_1) + (a_2 + b_2)) = r(a_1 + b_1) + rb_1 = f(a_1 + a_2) + f(b_1 + b_2) \) and \( f((a_1 + a_2) + t) = f(a_1 + t + a_2 + t) = r(a_1 + t) = (ra_1 + t) = (f(a_1 + a_2) + t) \). Thus, \( f \) is a well-defined right \( R \)-homomorphism. By \( JS \)-injectivity of \( R_R \), there is a right \( R \)-homomorphism \( g: R \to R \) such that \( g(a_1 + a_2) = f(a_1 + a_2) \), for all \( a_1 \in A_1 \), \( a_2 \in A_2 \). Thus \( g(a_1 + a_2) = ra_1 \), so \( r_1 - g(a_1) = g(a_2) = g(0 + a_2) = r_1 + 0 = 0 \) and hence \( (r - g(1))a_1 = 0 \), for all \( a_1 \in A_1 \). So \( r - g(1) \in l_g(A_1) \). Since \( g(1) \in l_g(A_2) \) (because \( g(1)A_2 = g(A_2) = 0 \)), we have that \( r \in l_g(A_1) + l_g(A_2) \) and hence \( l_g(A_1 \cap A_2) \subseteq l_g(A_1) + l_g(A_2) \). The other inclusion is obtained from [1, Proposition 2.16, p. 38]. □

**Proposition 2.4.** If \( R \) is a right \( JS \)-injective ring, then \( JS(R_R) \subseteq Z(R_R) \).

**Proof.** Let \( a \in JS(R_R) = \{JS(R_R)\}(R_R) \) and \( bR \cap r_R(a) = 0 \) for any \( b \in R \). By Theorem 2.2(3), we have that \( l_g(b) + Ra = l_g(bR \cap r_R(a)) = l_g(0) = R \), it follows that \( l_g(b) + Ra = R \). Since \( a \in JS(R_R) \), \( JS(R_R) \subseteq J(R_R) \), it follows from [3, Corollary 9.1.3, p. 214] that \( l_g(b) = R \) and hence that \( b = 0 \). So, \( r_R(a) \) is an essential in \( R_R \) and hence \( a \in Z(R_R) \). Therefore, \( JS(R_R) \subseteq Z(R_R) \). □

A ring \( R \) is called reduced if \( R \) has no nonzero nilpotent elements [4, p.249].

**Corollary 2.5.** If \( R \) is a \( JS \)-injective reduced ring, then every right \( R \)-module is \( JS \)-injective.

**Proof.** Let \( R \) be a \( JS \)-injective reduced ring. By [4, Lemma 7.8, p. 249], \( Z(R_R) = 0 \). Since \( R \) is a right \( JS \)-injective ring, it follows from Proposition 2.4 that \( JS(R_R) \subseteq Z(R_R) \) and hence \( JS(R_R) = 0 \). By [9, Corollary 2.9], every right \( R \)-module is \( JS \)-injective. □
A subset $K$ of a ring $R$ is said to be right $t$-nilpotent if for each sequence $a_1, a_2, a_3, \ldots$ of elements of $K$, $a_n \cdots a_2 a_1 = 0$, for some $n \in \mathbb{N}$ [2, p.239].

**Proposition 2.6.** Let $R$ be a right JS-injective ring. If the ascending chain $r_R(a_1) \subseteq r_R(a_2 a_1) \subseteq \cdots \subseteq r_R(a_n \cdots a_2 a_1) \subseteq \cdots$ terminates for any sequence $a_1, a_2, \ldots$ in $|J(R) \cap Z(R)|$, then $|J(R) \cap Z(R)|$ is a right $t$-nilpotent and $JS(R) \subseteq |J(R) \cap Z(R)|$.

**Proof.** Let $a_1, a_2, \ldots$ be any sequence in $|J(R) \cap Z(R)|$, then we have $r_R(a_1) \subseteq r_R(a_2 a_1) \subseteq \cdots$. By hypothesis, there exists $m \in \mathbb{N}$ such that $r_R(a_m \cdots a_2 a_1) = r_R(a_{m+1} a_m \cdots a_2 a_1)$. Assume that $a_{m+1} a_m \cdots a_2 a_1 \neq 0$. Since $r_R(a_{m+1}) \subseteq \text{ess} R$, then $(a_m \cdots a_2 a_1) R \cap r_R(a_{m+1}) \neq 0$ and hence $0 \neq a_m \cdots a_2 a_1 r \in r_R(a_{m+1})$ for some $r \in R$. Then $a_{m+1} a_m \cdots a_2 a_1 r = 0$ and this means that $a_m \cdots a_2 a_1 r = 0$ and this is a contradiction. Hence $|J(R) \cap Z(R)|$ is a right $t$-nilpotent. Since $JS(R) \subseteq Z(R)$ by Proposition 2.4 and $JS(R) \subseteq |J(R)|$, we have that $JS(R) \subseteq |J(R) \cap Z(R)|$. □

**Proposition 2.7.** If $Ra$ is a simple left ideal of a right JS-injective ring $R$, then $JS(aR) \cap soc(aR)$ is zero or simple, for any $a \in R$.

**Proof.** Suppose that $JS(aR) \cap soc(aR)$ is a nonzero. Assume that $JS(aR) \cap soc(aR)$ is not simple. Then there exist simple submodules $x_1 R$ and $x_2 R$ of $JS(aR)$ with $x_1 \in aR$, $i = 1, 2$. Thus $x_1 R \cap x_2 R = 0$. By Proposition 2.3, $l_R(x_1 R \cap x_2 R) = l_R(x_1 R) + l_R(x_2 R)$. Since $l_R(0) = 0$, it implies $l_R(x_1 R) + l_R(x_2 R) = R$. Since $x_1, x_2 \in aR$, we have $x_1 = ar_1$ for some $r_1 \in R$, $i = 1, 2$ and hence $l_R(a) \subseteq l_R(ax_1) = l_R(x_1)$, $i = 1, 2$. Since $Ra$ is a simple (by assumption), $l_R(a)$ is a maximal left ideal in $R$, that is $l_R(x_1 R) = l_R(x_2 R) = l_R(a)$ (because $l_R(x) \subseteq R$) and hence $l_R(a) = R$. Therefore, $a = 0$ and this is a contradiction with minimality of $Ra$. Hence $JS(aR) \cap soc(aR)$ is simple. □

**Proposition 2.8.** Let $R$ be a right JS-injective ring with $JS(R)$ is a semisimple module. Then $r_R l_R(JS(R)) = JS(R)$ if and only if $r_R l_R(K) = K$ for all submodule $K$ of $JS(R)$.

**Proof.** ($\Rightarrow$) Suppose that $r_R l_R(JS(R)) = JS(R)$ and let $K$ be a submodule of $JS(R)$. First, we have $K \subseteq r_R l_R(K)$ by [1, Proposition 2.15, p.37]. We will prove that $K$ is essential in $r_R l_R(K)$. If $K \cap x R = 0$ for some $x \in r_R l_R(K)$, then by Proposition 2.3, $l_R(K \cap x R) = l_R(K) + l_R(x R) = l_R(0) = 0$, since $x \in r_R l_R(K) \subseteq r_R l_R(JS(R)) = JS(R)$, we must have $x = 0$. Let $a \in l_R(K)$, then $ax = 0$. Thus $a(x R) = 0$ for any $r \in R$ and so $a \in l_R(x R)$. Hence $l_R(K) \subseteq l_R(x R)$. Thus $l_R(x R) = l_R(0) = R$ and hence $x = 0$ and this implies that $K$ is essential in $r_R l_R(K)$. Since $r_R l_R(K) \subseteq r_R l_R(JS(R)) = JS(R)$ and $JS(R)$ is semisimple (by hypothesis), we have $r_R l_R(K)$ is semisimple and hence $K = r_R l_R(K)$.

($\Leftarrow$) Suppose that $r_R l_R(K) = K$ for all submodule $K$ of $JS(R)$. Thus $r_R l_R(JS(R)) = JS(R)$. □

**Proposition 2.9.** Let $K$ be a right ideal of $R$ such that $K = e R \oplus B$ for some right ideal $B$ of $R$ with $B \subseteq JS(R_a)$ and an idempotent $e^2 = e \in K$. If $R$ is a right JS-injective ring, then each $R$-homomorphism from $K$ into $R$ is extended to $R$.

**Proof.** Let $K$ be a right ideal of $R$ such that $K = e R \oplus B$ for a right ideal $B$ of $R$ with $B \subseteq JS(R_a)$ and an idempotent $e^2 = e \in K$. Let $f : K \to R$ be a homomorphism. We will prove that $K = e R \oplus (1-e) B$. It is clear that $e R + (1-e) B$ is direct sum, since if $x \in e R \cap (1-e) B$, then $x = e r$ and $x = (1-e) b$, for some $b \in B$ and hence $b = e r + e b \in e R \cap (1-e) B = 0$. Thus $b = 0$ and hence $x = 0$, so $e R \cap (1-e) B = 0$. Let $x \in K$, then $x = a + b$, for some $a \in e R, b \in B$, we can write $x = a + e b + (1-e) b$ and so $x \in e R \oplus (1-e) B$. The converse, if $x \in e R \oplus (1-e) B$, then $x = a + (1-e) b$, for some $a \in e R$ and $(1-e) b \in (1-e) B$, we obtain $x = a + (1-e) b = a - e b + b \in e R \oplus B$. Hence $K = e R \oplus (1-e) B$. It is obvious that $(1-e) B \subseteq JS(R_a)$ and $(1-e) B$ is a right ideal of $R$. Let $f' : (1-e) B \to R$ be a right $R$-homomorphism defined by $f'(x) = f(x)$, for all $x \in (1-e) B$. JS-injectivity of a ring $R$ implies that there exists a right $R$-homomorphism $g : R \to R$ with $g((1-e) b) = f'((1-e) b)$ for all $(1-e) b \in (1-e) B$. Define $\alpha : R \to R$ by $\alpha(y) = f(ey) + g((1-e) y)$, for any $y \in R$. Then $\alpha$ is a well-defined $R$-homomorphism. If $x \in K$, then $x = a + b$ where $a \in e R$ and $b \in (1-e) B$. So $\alpha(x) = f(ex) + g((1-e) x) = f(a) + f(b) = f(a) + g(b) = f(a + b) = f(x)$. Then we get the result. □
Corollary 2.10. Let $R$ be a ring such that for any right ideal $K$ of $R$, we have $K = eR \oplus B$ for some right ideal $B$ of $R$ with $B \subseteq JS(R_R)$ and an idempotent $e^2 = e \in K$. Then $R$ is a right JS-injective ring if and only if $R$ is a right self-injective ring.

Proof. Let $R$ be a ring in which any right ideal $K$ of $R$, we have $K = eR \oplus B$ for some right ideal $B$ of $R$ with $B \subseteq JS(R_R)$ and an idempotent $e^2 = e \in K$.

$(\Rightarrow)$ Suppose that $R$ is a right JS-injective ring. Let $I$ be a right ideal of $R$, $i: I \rightarrow R$ an inclusion mapping and $f: I \rightarrow R$ any right $R$-homomorphism. By hypothesis, $I = eR \oplus B$ for some right ideal $B$ of $R$ with $B \subseteq JS(R_R)$ and an idempotent $e^2 = e \in K$. By Proposition 2.9, there is a right $R$-homomorphism $g: R \rightarrow R$ with $g(a) = f(a)$, for all $a \in I$. Thus $R$ is a right injective ring.

$(\Leftarrow)$ It is clear. □

Theorem 2.11. Let $R$ be a right JS-injective ring, and let $a, b \in R$ with $b \in JS(R_R)$.

(1) If $bR$ embeds in $aR$, then $Ra$ embeds in $Rb$.

(2) If $aR$ is an image of $bR$, then $Ra$ embeds in $Rb$.

(3) If $bR \cong aR$, then $Ra \cong Rb$.

Proof. Let $a, b \in R$ with $b \in JS(R_R)$ and let $f \in \text{Hom}_R(bR, aR)$. Since $b \in JS(R_R)$ (by hypothesis), it follows from JS-injectivity of $R$ that there is a right $R$-homomorphism $g: R \rightarrow R$ with $g_i = i_2g$, where $i_1: bR \rightarrow R$ and $i_2: aR \rightarrow R$ are the inclusion maps. Thus $f(b) = g(b) = g(1)b = vb$, where $v = g(1)$. Since $f(b) \in aR$, it follows that $vb \in aR$ and hence there is $u \in R$ with $vb = au$. Define $\theta: Ra \rightarrow Rb$ by $\theta(ra) = (ra)u = r(vb)$, for all $r \in R$. Thus $\theta$ is a well-defined left $R$-homomorphism.

(1) If $f$ is a right monomorphism, then we have $r(a) \subseteq r(b)$. By Theorem 2.2(2), $Rb \subseteq Rvb$. Thus $b = r(vb) = \theta(ra)$ (for some $r \in R$). Hence $\theta$ is a left $R$-epimorphism.

(2) If $f$ is an epimorphism, then there is $s \in R$ with $f(bs) = a$ and hence $a = f(b)s = vbs$. We will prove $\ker(\theta) = 0$. Let $x \in \ker(\theta)$, thus $\theta(x) = 0$. Since $x \in Ra$, we have $x = ra$, for some $r \in R$. Thus $\theta(ra) = 0$ and hence $r(vb) = 0$. So, $x = ra = r(bvs) = (r(vb)s = 0$ and hence $\ker(\theta) = 0$. Therefore, $\theta$ is a left $R$-monomorphism.

(3) If $f$ is an isomorphism, then by the proofs of (1) and (2), we have that $\theta$ is a left $R$-isomorphism. □

The class of JS-injective right $R$-modules is denoted by $JSI_R$.

Proposition 2.12. The following two statements are equivalent for a ring $R$:

(1) $\text{Mod}R = JSI_R$.

(2) (i) $R$ is a JS-injective ring;
(ii) every cyclic submodule of $JS(R_R)$ is projective.

Proof. (1) $\Rightarrow$ (2). Suppose that every right $R$-module is JS-injective. Thus $R_R$ is a JS-injective module and every epimorphic image of JS-injective module is JS-injective. By [9, Corollary 2.19], every submodule of $JS(R_R)$ is projective.

(2) $\Rightarrow$ (1). Let $aR$ be a cyclic submodule of $JS(R_R)$. By (2)(ii), $aR$ is projective. Define $h: R \rightarrow aR$ by $h(r) = ar$, for any $r \in R$. It is clear that $h$ is a right epimorphism. By projectivity of $aR$, there is a homomorphism $f: aR \rightarrow R$ such that $(hf)(x) = x$, for all $x \in aR$. Thus $(hf)(a) = a$ and hence $af(a) = a$. Since $R_R$ is JS-injective (by hypothesis), there is a homomorphism $g: R \rightarrow R$ such that $g(x) = f(x)$, for all $x \in aR$. Thus $a = af(a) = ag(a) = ag(1)a = aba$, where $b = g(1)$. Put $e = ab$. Thus $e^2 = abab = ab = e$ and $ea = aba = a$. Let $x \in aR$, then $x = ar$, for some $r \in R$. Thus $x = ar = ear \in eR$ and hence $aR \subseteq eR$. Let $y \in eR$, thus $y = et$ for some $t \in R$ and hence $y = abt \in aR$. 


Thus $eR = aR$. Since $R = eR \oplus (1-e)R$, it follows that $aR$ is a direct summand of $R_R$. Since $R_R$ is JS-injective module, we have from [9, Corollary 2.4] that $aR$ is a JS-injective module. By [9, Theorem 2.15], every right $R$-module is JS-injective. \[\square\]

**Lemma 2.13.** Let $R$ be a ring, then $D(R) = \{a \in R \mid r_R(a) \cap mR \neq 0 \text{ for each } 0 \neq m \in JS(R_R)\}$ is a left ideal of $R$.

**Proof.** It is obviously that $D(R)$ is a non-empty set, since $0 \in D(R)$. If $\mathbf{a} \in D(R)$ and $0 \neq m \in JS(R_R)$, thus $mb \in r_R(a) \cap mR$, for some $b \in R$ and so $a(mb) = 0$. Since $(-a)(mb) = -(amb) = 0$, then $mb \in r_R(-a)$ and hence $r_R(-a) \cap mR \neq 0$. Thus $-a \in D(R)$. Now, let $a_1, a_2 \in D(R)$ and $0 \neq m \in JS(R_R)$. We have that $0 \neq mb \in r_R(a_1) \cap mR$ for some $b \in R$. Since $a_2 \in D(R)$, it follows that $-r_2 \in D(R)$ and hence $0 \neq mbc \in r_R(-a_2) \cap mR$ for some $c \in R$. Therefore, $0 \neq mbc \in r_R(a_1) \cap r_R(-a_2) \cap mR$. Since $r_R(a_1) \cap r_R(-a_2) = r_R(a_1 + (-a_2)) = r_R(a_1 - a_2)$ (by [1, Proposition 2.16, p. 38]), we have $r_R(a_1 - a_2) \cap mR \neq 0$ for all $0 \neq m \in JS(R_R)$ and hence $a_1 - a_2 \in D(R)$. Also, let $x \in R$ and $a \in D(R)$. Since $r_R(a) \subseteq r_R(xa)$, it follows that $r_R(xa) \cap mR \neq 0$ for all $0 \neq m \in JS(R_R)$, that is $xa \in D(R)$. Thus $D(R)$ is a left ideal of $R$. \[\square\]

**Proposition 2.14.** Let $R$ be a right JS-injective ring. Then $r_R(a) \subseteq r_R(a - axa)$, for all $a \notin D(R)$ and for some $x \in R$.

**Proof.** For all $a \notin D(R)$, we can find $0 \neq m \in JS(R_R)$ such that $r_R(a) \cap mR = 0$. Clearly, $r_R(ama) = r_R(m)$, so $Rm = Ram$ by Theorem 2.2(2). Thus $m = xam$ for some $x \in R$ and this implies that $m - xam = 0$ and hence $(1 - xa)(m) = 0$. Thus $a, (1 - xa)(m) = a, 0$ and so $(a - axa)m = 0$. Therefore, $m \in r_R(a - axa)$, but $m \notin r_R(a)$ because $r_R(a) \cap mR = 0$ and hence the inclusion is strictly. \[\square\]

**Proposition 2.15.** Let $R$ be a right JS-injective ring, then the set $\{a \in R \mid r_R(1 - sa) = 0 \text{ for all } s \in R\}$ is contained in $D(R)$.

**Proof.** We will prove that by contradiction. Assume that there is $a$ such that $r_R(1 - sa) = 0$ for all $s \in R$ with $a \notin D(R)$. Then there exists $0 \neq m \in JS(R_R)$ with $r_R(a) \cap mR = 0$. If $r \in r_R(am)$, then $(am)r = 0$ and hence $a(mr) = 0$ and so $mr \in r_R(a)$. Since $r_R(a) \cap mR = 0$, it follows that $mr = 0$ and so $r \in r_R(m)$. Hence $r_R(am) \subseteq r_R(m)$. By Theorem 2.2(2), $Rm \subseteq Ram$. Thus $m = sam$, for some $s \in R$. Therefore, $(1 - sa)m = 0$ and hence $m \in r_R(1 - sa) = 0$ so $m = 0$ and this is a contradiction. Thus the statement is hold. \[\square\]

**References**