Journal of Al-Qadisiyah for Computer Science and Mathematics Vol. 15(2) 2023 , pp Math. 111–117

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

On JS-Injective Rings

Zahraa Abbas Zone[a], Akeel Ramadan Mehdi[b]

[a]*Mathematical Department / College of Education / University of Al-Qadisiyah / Al-Diwaniya City/ Iraq. Email: edu-math.post32@qu.edu.iq*

[b]*Mathematical Department / College of Education / University of Al-Qadisiyah / Al-Diwaniya City/ Iraq. Email: akeel.mehdi@qu.edu.iq*

ARTICLE INFO

Article history:
Received: 22 /04/2023
Rrevised form: 05 /06/2023
Accepted: 07 /06/2023
Available online: 30 /06/2023

Keywords:

JS-injective ring.
Finitely generated module.
Injective ring.

ABSTRACT

Let R be a ring. A right R-module M is called JS-N-injective (where N is any right R-module) if every right R-homomorphism from a submodule of J(N)J(R_R) into M extends to N [9]. A ring R is called right JS-injective if R_R is JS- R -injective. The right JS-injective rings are studied in this paper. Many characterizations and properties of this type of rings are obtained.

2020MSC: Primary: 16D50, 16D10; Secondary: 16D25, 16S90.

https://doi.org/10.29304/jqcm.2023.15.2.1253

1. Introduction

This paper assumes that R is an associative ring with identity $1 \neq 0$ and any module is unitary. By a module (resp. homomorphism) we mean a right R-module (resp. right R-homomorphism), if not otherwise specified. The class of right R-modules is denoted by Mod-R. We use $\mathrm{soc}(M)$ and $\mathrm{J}(M)$ to denote, respectively, the socle and the Jacobson radical of a right R-module M. We write $Z(R_R)$ for the right singular ideal of a ring R. We denote to $\mathrm{J}(M)\mathrm{J}(R_R)$ by $\mathrm{JS}(M)$ for any right R-module M. For any $a \in R$, we use $l_R(a)$ (resp. $r_R(a)$) to denote the left (resp. right) annihilator of a in R.

Injective modules play important role in module theory, and extensively many authors were studied their generalizations (see, for example, [5], [6], and [7]). If every R-homomorphism from a right ideal of R into R_R can be extended to R_R, then a ring R is called right self-injective ring [4, p.64]. Let N be a right R-module. A right R-module M is called JS-N-injective, if every right R-homomorphism from a submodule of $\mathrm{J}(N)\mathrm{J}(R_R)$ into M extends to N. If a right R-module M is JS-R-injective, then M is called JS-injective. A ring R is called right JS-injective if the right R-module R_R is JS-injective [9]. JS-injective rings are studied in this paper. We give many characterizations and properties of right JS-injective rings. For examples, we prove that a ring R is a right JS-injective if and only if for any

*Corresponding author: Akeel Ramadan Mehdi

Email addresses: akeel.mehdi@qu.edu.iq

Communicated by ‘sub etitor’

$N \in$ Mod-R and a nonzero R-monomorphsim f from N to R with $f(N) \subseteq \mathrm{JS}(R_R)$, then $\mathrm{Hom}_R(R, N) = Rf$. Also, we show that if R is a right JS-injective ring, then $l_R(A_1 \cap A_2) = l_R(A_1) + l_R(A_2)$, for all submodules A_1 and A_2 of $\mathrm{JS}(R_R)$. In Proposition 2.4, we prove that if R is a right JS-injective ring, then $\mathrm{JS}(R_R) \subseteq Z(R_R)$. Moreover, we show that if Ra is a simple left ideal of a right JS-injective ring R, then $\mathrm{JS}(aR) \cap \mathrm{soc}(aR)$ is zero or simple, for any $a \in R$. Condition under which JS-injectivity implies injectivity is given. We get that if R is a ring such that for any right ideal K of R, we have $K = eR \oplus B$ for some right ideal B of R with $B \subseteq \mathrm{JS}(R_R)$ and an idempotent $e^2 = e \in K$, then R is a right JS-injective ring if and only if R is a right self-injective ring. Then, we prove that if R is a right JS-injective ring, and $a, b \in R$ with $b \in \mathrm{JS}(R_R)$ and $bR \cong aR$, then $Ra \cong Rb$. Finally, we prove that every right R-module is JS-injective if and only if R is a JS-injective ring and every cyclic submodule of $\mathrm{JS}(R_R)$ is projective.

2. JS-Injective Rings

Let N be a right R-module. A right R-module M is called JS-N-injective, if every right R-homomorphism from a submodule of $\mathrm{J}(N)\mathrm{J}(R_R)$ into M extends to N. A right R-module M is called JS-injective if M is JS-R-injective. A ring R is called right JS-injective if the right R-module R_R is JS-injective [9]. In this section, right JS-injective rings are studied extensively. Many characterizations and properties of this type of rings are given.

Recall that a right R-module M is called multiplication if any submodule N of M takes the form MI, for some ideal I of R [8, p. 3839].

We begin this section with the following theorem, which gives some characterizations of right JS-injective rings.

Theorem 2.1. Consider the following statements for a ring R:

(1) R is a right JS-injective ring.

(2) If N and M are finitely generated projective right R-modules, then M is JS-N-injective.

(3) If $N \in$ Mod-R and $f: N \to R$ is an R-monomorphsim with $f(N) \subseteq \mathrm{JS}(R_R)$, then $\mathrm{Hom}_R(N, R) = Rf$.

Then (2) $\Longrightarrow$ (1) and (1) $\Longleftrightarrow$ (3). Moreover, if $\mathrm{JS}((R_R)^m)$ is a multiplication module for any positive integer m, then (1) $\Longrightarrow$ (2).

Proof. (2) $\Longrightarrow$ (1) Clear.

(1) $\Longrightarrow$ (2) Suppose that R is a right JS-injective ring with $\mathrm{JS}((R_R)^m)$ is a multiplication module, for any $m \in \mathbb{Z}^+$. Let N and M be finitely generated projective right R-modules and K a submodule of $\mathrm{JS}(N)$. Let $f: K \longrightarrow M$ be any R-homomorphism. Since M is finitely generated, there exists a right R-epimorphism $\alpha_1: R^n \longrightarrow M$ for some positive integer number n. Since M is projective, there is a right R-homomorphism $\alpha_2: M \longrightarrow R^n$ with $\alpha_1\alpha_2 = I_M$, where $I_M: M \longrightarrow M$ is the identity homomorphism. Since R is a right JS-injective ring, we have from [9, Proposition 2.5 and Corollary 2.4] that R^n is a right JS-R^m-injective R-module, for any $m \in \mathbb{Z}^+$. Since N is finitely generated projective, N is a direct summand of R^k for some k. By [9, Proposition 2.3(2)], R^n is a right JS-N-injective R-module. Then $hi = \alpha_2 f$, for some $h \in \mathrm{Hom}_R(N, R^n)$. Put $g = \alpha_1 h: N \longrightarrow M$. Then $gi = (\alpha_1 h)i = \alpha_1(hi) = \alpha_1(\alpha_2 f) = (\alpha_1\alpha_2)f = I_M f = f$. Therefore, $gi = f$ for some R-homomorphism $g: N \longrightarrow M$.

(1) $\Longrightarrow$ (3) Suppose that R is a right JS-injective ring. Let N be any right R-module and $f: N \longrightarrow R$ be a nonzero R-monomorphism with $f(N) \subseteq \mathrm{JS}(R_R)$. Define $\acute{f}: N \longrightarrow f(N)$ by $\acute{f}(a) = f(a)$, for all $a \in N$. It is clear that $\acute{f}$ is an isomorphism. Let $g \in \mathrm{Hom}_R(N, R)$, then we have $g\acute{f}^{-1}: f(N) \longrightarrow R$ is an R-homomorphism. Since a ring R is right JS-injective and $f(N) \subseteq \mathrm{JS}(R_R)$, there is $c \in R$ with $(g\acute{f}^{-1})(k) = ck$, for all $k \in f(N)$ (by [9, Proposition 2.7]). Let $n \in N$, then $f(n) \in f(N)$ and hence $(g\acute{f}^{-1})(f(n)) = cf(n)$. Since $(g\acute{f}^{-1})(f(n)) = g(n)$, it follows that $g(n) = cf(n)$, for all $n \in N$. Thus $\mathrm{Hom}_R(N, R) = Rf$.

(3) $\Longrightarrow$ (1) Let K be a submodule of $\mathrm{JS}(R_R)$, $f: K \longrightarrow R$ a right R-homomorphism, and $i: K \longrightarrow R$ the inclusion map. Then by hypothesis, we have $\mathrm{Hom}_R(K, R) = Ri$ and hence $f = ci$ for some $c \in R$. Thus there exists $c \in R$ such that $f(a) = ca$ for all $a \in K$. Then R is a right JS-injective ring, by [9, Proposition 2.7]. □

Theorem 2.2. Let R be a right JS-injective ring, then the following statements hold:

(1) $l_R r_R(m) = Rm$, for all $m \in \mathrm{JS}(R_R)$.
(2) If $r_R(m) \subseteq r_R(n)$, where $m \in \mathrm{JS}(R_R)$ and $n \in R$, then $Rn \subseteq Rm$.
(3) $l_R(mR \cap r_R(a)) = l_R(m) + Ra$, for all $m, a \in R$ with $am \in \mathrm{JS}(R_R)$.

Proof. (1) Let $m \in \mathrm{JS}(R_R)$ and let $n \in l_R r_R(m)$, By [1, Proposition 2.15, p. 37], $r_R(m) = r_R l_R r_R(m) \subseteq r_R(n)$. Define $f: mR \longrightarrow R$ by $f(mr) = nr$ for any $r \in R$, thus f is a well-defined right R-homomorphism. By hypothesis, there exists an endomorphism g of R such that $g(x) = f(x)$, for all $x \in mR$. Then $n = n \cdot 1 = f(m \cdot 1) = f(m) = g(m) = g(1)m \in Rm$. Hence $l_R r_R(m) \subseteq Rm$. Conversely, let $rm \in Rm$, where $r \in R$. Thus $rmk = 0$ for all $k \in r_R(m)$ and hence $rm \in l_R r_R(m)$. Therefore, $l_R r_R(m) = Rm$.
(2) Let $n \in R$ and $m \in \mathrm{JS}(R_R)$ such that $r_R(m) \subseteq r_R(n)$. Thus $n \in l_R r_R(n)$. Since $r_R(m) \subseteq r_R(n)$ (by hypothesis), $l_R r_R(n) \subseteq l_R r_R(m)$ (by [1, Proposition 2.15, p. 37]). So, $n \in l_R r_R(m)$. By (1), $n \in Rm$ and this implies that $Rn \subseteq Rm$.
(3) Let $a, m \in R$ such that $am \in \mathrm{JS}(R_R)$. If $x \in l_R(m) + Ra$, then $x = x_1 + x_2$ such that $x_1 m = 0$ and $x_2 = sa$ for some $s \in R$. For all $b \in mR \cap r_R(a)$, we have $b = mr$ and $ab = 0$ for some $r \in R$. Since $x_1 b = x_1(mr) = (x_1 m)r = 0$ and $x_2 b = (sa)b = s(ab) = 0$, it is follows that $x \in l_R(mR \cap r_R(a))$ and this implies that $l_R(m) + Ra \subseteq l_R(mR \cap r_R(a))$. Let $y \in l_R(mR \cap r_R(a))$. If $r \in r_R(am)$, then $(am)r = 0$ and hence $a(mr) = 0$. Thus $mr \in mR \cap r_R(a)$ and hence $(ym)r = y(mr) = 0$ and so $ym \in l_R(r_R(am))$. Thus $r_R l_R(r_R(am)) \subseteq r_R(ym)$. By [1, Proposition 2.15, p. 37], $r_R(am) \subseteq r_R(ym)$. By hypothesis, $am \in \mathrm{JS}(R_R)$. By (2), $Rym \subseteq Ram$. Thus $ym = sam$, for some $s \in R$ and hence $(y - sa)m = 0$ and this implies that $y - sa \in l_R(m)$. Thus $y \in l_R(m) + Ra$ and hence $l_R(mR \cap r_R(a)) = l_R(m) + Ra$. □

Proposition 2.3. If R is a right JS-injective ring, then $l_R(A_1 \cap A_2) = l_R(A_1) + l_R(A_2)$, for all submodules A_1 and A_2 of $\mathrm{JS}(R_R)$.

Proof. Let A_1 and A_2 be any two submodules of $\mathrm{JS}(R_R)$. Let $r \in l_R(A_1 \cap A_2)$, thus $r.(A_1 \cap A_2) = 0$. Consider the mapping $f: A_1 + A_2 \longrightarrow R$ is given by $f(a_1 + a_2) = r.a_1$, for all $a_1 \in A_1, a_2 \in A_2$. Thus f is a well-defined right R-homomorphism, since if $a_1 + a_2 = b_1 + b_2$, where $a_1, b_1 \in A_1$, $a_2, b_2 \in A_2$, then $a_1 - b_1 = b_2 - a_2 \in A_1 \cap A_2$. Since $r(A_1 \cap A_2) = 0$, we have that $r(a_1 - b_1) = 0$ and hence $ra_1 = rb_1$, so $f(a_1 + a_2) = f(b_1 + b_2)$ and this implies that f is a well-defined. Also, for every $a_1 + a_2, b_1 + b_2 \in A_1 + A_2$ where $a_1, b_1 \in A_1$, $a_2, b_2 \in A_2$ and $t \in R$, we have $f((a_1 + a_2) + (b_1 + b_2)) = f((a_1 + b_1) + (a_2 + b_2)) = r(a_1 + b_1) = ra_1 + rb_1 = f(a_1 + a_2) + f(b_1 + b_2)$ and $f((a_1 + a_2)t) = f(a_1 t + a_2 t) = r(a_1 t) = (ra_1)t = (f(a_1 + a_2))t$. Thus, f is a well-defined right R-homomorphism. By JS-injectivity of R_R, there is a right R-homomorphism $g: R \longrightarrow R$ such that $g(a_1 + a_2) = f(a_1 + a_2)$, for all $a_1 \in A_1$, $a_2 \in A_2$. Thus $g(a_1 + a_2) = ra_1$, so $ra_1 - g(a_1) = g(a_2) = g(0 + a_2) = r.0 = 0$ and hence $(r - g(1))a_1 = 0$, for all $a_1 \in A_1$. So $r - g(1) \in l_R(A_1)$. Since $g(1) \in l_R(A_2)$ (because $g(1)A_2 = g(A_2) = 0$), we have that $r \in l_R(A_1) + l_R(A_2)$ and hence $l_R(A_1 \cap A_2) \subseteq l_R(A_1) + l_R(A_2)$. The other inclusion is obtained from [1, Proposition 2.16, p. 38]. □

Proposition 2.4. If R is a right JS-injective ring, then $\mathrm{JS}(R_R) \subseteq Z(R_R)$.

Proof. Let $a \in \mathrm{JS}(R_R) = \mathrm{J}(R_R)\mathrm{J}(R_R)$ and $bR \cap r_R(a) = 0$ for any $b \in R$. By Theorem 2.2(3), we have that $l_R(b) + Ra = l_R(bR \cap r_R(a)) = l_R(0) = R$, it follows that $l_R(b) + Ra = R$. Since $a \in \mathrm{JS}(R_R) \subseteq J(R_R)$, it follows from [3, Corollary 9.1.3, p. 214] that $l_R(b) = R$ and hence that $b = 0$. So, $r_R(a)$ is an essential in R_R and hence $a \in Z(R_R)$. Therefore, $\mathrm{JS}(R_R) \subseteq Z(R_R)$. □

A ring R is called reduced if R has no nonzero nilpotent elements [4, p.249].

Corollary 2.5. If R is a JS-injective reduced ring, then every right R-module is JS-injective.

Proof. Let R be a JS-injective reduced ring. By [4, Lemma 7.8, p. 249], $Z(R_R) = 0$. Since R is a right JS-injective ring, it follows from Proposition 2.4 that $\mathrm{JS}(R_R) \subseteq Z(R_R)$ and hence $\mathrm{JS}(R_R) = 0$. By [9, Corollary 2.9], every right R-module is JS-injective. □

A subset K of a ring R is said to be right t-nilpotent if for each sequence $a_1, a_2, a_3, \ldots$ of elements of K, $a_n \ldots a_2 a_1 = 0$, for some $n \in \mathbb{N}$ [2, p.239].

Proposition 2.6. Let R be a right JS-injective ring. If the ascending chain $r_R(a_1) \subseteq r_R(a_2 a_1) \subseteq \cdots \subseteq r_R(a_n \ldots a_2 a_1) \subseteq \cdots$ terminates for any sequence $a_1, a_2, \ldots$ in $\mathrm{J}(R_R) \cap Z(R_R)$, then $\mathrm{J}(R_R) \cap Z(R_R)$ is a right t-nilpotent and $\mathrm{JS}(R_R) \subseteq \mathrm{J}(R_R) \cap Z(R_R)$.

Proof. Let $a_1, a_2, \ldots$ be any sequence in $\mathrm{J}(R_R) \cap Z(R_R)$, then we have $r_R(a_1) \subseteq r_R(a_2 a_1) \subseteq \cdots$. By hypothesis, there exists $m \in \mathbb{N}$ such that $r_R(a_m \ldots a_2 a_1) = r_R(a_{m+1} a_m \ldots a_2 a_1)$. Assume that $a_m \ldots a_2 a_1 \neq 0$. Since $r_R(a_{m+1}) \subseteq^{ess} R_R$, then $(a_m \ldots a_2 a_1)R \cap r_R(a_{m+1}) \neq 0$ and hence $0 \neq a_m \ldots a_2 a_1 r \in r_R(a_{m+1})$ for some $r \in R$. Then $a_{m+1} a_m \ldots a_2 a_1 r = 0$ and this means that $a_m \ldots a_2 a_1 r = 0$ and this is a contradiction. Hence $\mathrm{J}(R_R) \cap \mathrm{Z}(R_R)$ is a right t-nilpotent. Since $\mathrm{JS}(R_R) \subseteq Z(R_R)$ by Proposition 2.4 and $\mathrm{JS}(R_R) \subseteq \mathrm{J}(R_R)$, we have that $\mathrm{JS}(R_R) \subseteq \mathrm{J}(R_R) \cap Z(R_R)$. □

Proposition 2.7. If Ra is a simple left ideal of a right JS-injective ring R, then $\mathrm{JS}(aR) \cap soc(aR)$ is zero or simple, for any $a \in R$.

Proof. Suppose that $\mathrm{JS}(aR) \cap soc(aR)$ is a nonzero. Assume that $\mathrm{JS}(aR) \cap soc(aR)$ is not simple. Thus there exist simple submodules $x_1 R$ and $x_2 R$ of $\mathrm{JS}(aR)$ with $x_i \in aR$, $i = 1, 2$. Thus $x_1 R \cap x_2 R = 0$. By Proposition 2.3, $l_R(x_1 R \cap x_2 R) = l_R(x_1 R) + l_R(x_2 R)$. Since $l_R(0) = R$, it implies $l_R(x_1 R) + l_R(x_2 R) = R$. Since $x_1, x_2 \in aR$, we have $x_i = a r_i$ for some $r_i \in R$, $i = 1, 2$ and hence $l_R(a) \subseteq l_R(a r_i) = l_R(x_i)$, $i = 1, 2$. Since Ra is a simple (by assumption), $l_R(a)$ is a maximal left ideal in R, that is $l_R(x_1 R) = l_R(x_2 R) = l_R(a)$ (because $l_R(x_i) \subsetneq R$) and hence $l_R(a) = R$. Therefore, $a = 0$ and this is a contradiction with minimality of Ra. Hence $\mathrm{JS}(aR) \cap soc(aR)$ is simple. □

Proposition 2.8. Let R be a right JS-injective ring with $\mathrm{JS}(R_R)$ is a semisimple module. Then $r_R l_R(\mathrm{JS}(R_R)) = \mathrm{JS}(R_R)$ if and only if $r_R l_R(K) = K$ for all submodule K of $\mathrm{JS}(R_R)$.

Proof. (⇒) Suppose that $r_R l_R(\mathrm{JS}(R_R)) = \mathrm{JS}(R_R)$ and let K be a submodule of $\mathrm{JS}(R_R)$. First, we have $K \subseteq r_R l_R(K)$ by [1, Proposition 2.15, p.37]. We will prove that K is essential in $r_R l_R(K)$. If $K \cap xR = 0$ for some $x \in r_R l_R(K)$, then by Proposition 2.3, $l_R(K \cap xR) = l_R(K) + l_R(xR) = l_R(0) = R$, since $x \in r_R l_R(K) \subseteq r_R l_R(\mathrm{JS}(R_R)) = \mathrm{JS}(R_R)$. Now, let $a \in l_R(K)$, then $ax = 0$. Thus $a(xr) = 0$ for any $r \in R$ and so $a \in l_R(xR)$. Hence $l_R(K) \subseteq l_R(xR)$. Thus $l_R(xR) = l_R(0) = R$ and hence $x = 0$ and this implies that K is essential in $r_R l_R(K)$. Since $r_R l_R(K) \subseteq r_R l_R(\mathrm{JS}(R_R)) = \mathrm{JS}(R_R)$ and $\mathrm{JS}(R_R)$ is semisimple (by hypothesis), we have $r_R l_R(K)$ is semisimple and hence $K = r_R l_R(K)$.

(⇐) Suppose that $r_R l_R(K) = K$, for all right submodule K of $\mathrm{JS}(R_R)$. Thus $r_R l_R(\mathrm{JS}(R_R)) = \mathrm{JS}(R_R)$. □

Proposition 2.9. Let K be a right ideal of R such that $K = eR \oplus B$ for some right ideal B of R with $B \subseteq \mathrm{JS}(R_R)$ and an idempotent $e^2 = e \in K$. If R is a right JS-injective ring, then each R-homomorphism from K into R is extended to R.

Proof. Let K be a right ideal of R such that $K = eR \oplus B$ for a right ideal B of R with $B \subseteq \mathrm{JS}(R_R)$ and an idempotent $e^2 = e \in K$. Let $f: K \longrightarrow R$ be a homomorphism. We will prove that $K = eR \oplus (1-e)B$. It is clear that $eR + (1-e)B$ is direct sum, since if $x \in eR \cap (1-e)B$, then $x = er$ and $x = (1-e)b$, for some $b \in B$ and hence $b = er + eb \in eR \cap B = 0$. Thus $b = 0$ and hence $x = 0$, so $eR \cap (1-e)B = 0$. Let $x \in K$, then $x = a + b$, for some $a \in eR, b \in B$, we can write $x = a + eb + (1-e)b$ and so $x \in eR \oplus (1-e)B$. The converse, if $x \in eR \oplus (1-e)B$, then $x = a + (1-e)b$, for some $a \in eR$ and $(1-e)b \in (1-e)B$, we obtain $x = a + (1-e)b = a - eb + b \in eR \oplus B$. Hence $K = eR \oplus (1-e)B$. It is obvious that $(1-e)B \subseteq \mathrm{JS}(R_R)$ and $(1-e)B$ is a right ideal of R. Let $f' : (1-e)B \longrightarrow R$ be a right R-homomorphism defined by $f'(x) = f(x)$, for all $x \in (1-e)B$. JS-injectivity of a ring R implies that there exists a right R-homomorphism $g: R \longrightarrow R$ with $g\big((1-e)b\big) = f'((1-e)b)$ for all $(1-e)b \in (1-e)B$. Define $\alpha: R \longrightarrow R$ by $\alpha(y) = f(ey) + g((1-e)y)$, for any $y \in R$. Then α is a well-defined R-homomorphism. If $x \in K$, then $x = a + b$ where $a \in eR$ and $b \in (1-e)B$. So $\alpha(x) = f(ex) + g\big((1-e)x\big) = f(a) + f(b) = f(a) + g(b) = f(a+b) = f(x)$. Then we get the result. □

Corollary 2.10. Let R be a ring such that for any right ideal K of R, we have $K = eR \oplus B$ for some right ideal B of R with $B \subseteq \mathrm{JS}(R_R)$ and an idempotent $e^2 = e \in K$. Then R is a right JS-injective ring if and only if R is a right self-injective ring.

Proof. Let R be a ring in which any right ideal K of R, we have $K = eR \oplus B$ for some right ideal B of R with $B \subseteq \mathrm{JS}(R_R)$ and an idempotent $e^2 = e \in K$.

($\Longrightarrow$) Suppose that R is a right JS-injective ring. Let I be a right ideal of R, $i: I \longrightarrow R$ an inclusion mapping and $f: I \longrightarrow R$ any right R-homomorphism. By hypothesis, $I = eR \oplus B$ for some right ideal B of R with $B \subseteq \mathrm{JS}(R_R)$ and an idempotent $e^2 = e \in K$. By Proposition 2.9, there is a right R-homomorphism $g: R \longrightarrow R$ with $g(a) = f(a)$, for all $a \in I$. Thus R is a right injective ring.

($\Longleftarrow$) It is clear. □

Theorem 2.11. Let R be a right JS-injective ring, and let $a, b \in R$ with $b \in \mathrm{JS}(R_R)$.

(1) If bR embeds in aR, then Rb is an image of Ra.

(2) If aR is an image of bR, then Ra embeds in Rb.

(3) If $bR \cong aR$, then $Ra \cong Rb$.

Proof. Let $a, b \in R$ with $b \in \mathrm{JS}(R_R)$ and let $f \in \mathrm{Hom}_R(bR, aR)$. Since $b \in \mathrm{JS}(R_R)$ (by hypothesis), it follows from JS-injectivity of R that there is a right R-homomorphism $g: R \longrightarrow R$ with $gi_1 = i_2 f$, where $i_1: bR \to R$ and $i_2: aR \to R$ are the inclusion maps. Thus $f(b) = g(b) = g(1)b = vb$, where $v = g(1)$. Since $f(b) \in aR$, it follows that $vb \in aR$ and hence there is $u \in R$ with $vb = au$. Define $\theta: Ra \longrightarrow Rb$ by $\theta(ra) = (ra)u = r(vb)$, for all $r \in R$. Thus θ is a well-defined left R-homomorphism.

(1) If f is a right monomorphism, we have $r_R(vb) \subseteq r_R(b)$. By Theorem 2.2(2), $Rb \subseteq Rvb$. Thus $b = r(vb) = \theta(ra)$ (for some $r \in R$). Hence θ is a left R-epimorphism.

(2) If f is an epimorphism, then there is $s \in R$ with $f(bs) = a$ and hence $a = f(b)s = vbs$. We will prove $\ker(\theta) = 0$. Let $x \in \ker(\theta)$, thus $\theta(x) = 0$. Since $x \in Ra$, we have $x = ra$, for some $r \in R$. Thus $\theta(ra) = 0$ and hence $r(vb) = 0$. So, $x = ra = r(bvs) = (rvb)s = 0$ and hence $\ker(\theta) = 0$. Therefore, θ is a left R-monomorphism.

(3) If f is an isomorphism, then by the proofs of (1) and (2), we have that θ is a left R-isomorphism. □

The class of JS-injective right R-modules is denoted by JSI_R.

Proposition 2.12. The following two statements are equivalent for a ring R:

(1) Mod-$R = JSI_R$.
(2)(i) R is a JS-injective ring;
(ii) every cyclic submodule of $\mathrm{JS}(R_R)$ is projective.

Proof. (1) $\Rightarrow$ (2). Suppose that every right R-module is JS-injective. Thus R_R is a JS-injective module and every epimorphic image of JS-injective module is JS-injective. By [9, Corollary 2.19], every submodule of $\mathrm{JS}(R_R)$ is projective.

(2) $\Longrightarrow$ (1). Let aR be a cyclic submodule of $\mathrm{JS}(R_R)$. By (2)(ii), aR is projective. Define $h: R \longrightarrow aR$ by $h(r) = ar$, for any $r \in R$. It is clear that h is a right epimorohism. By projectivity of aR, there is a homomorphism $f: aR \longrightarrow R$ such that $(hf)(x) = x$, for all $x \in aR$. Thus $(hf)(a) = a$ and hence $af(a) = a$. Since R_R is JS-injective (by hypothesis), there is a homomorphism $g: R \longrightarrow R$ such that $g(x) = f(x)$, for all $x \in aR$. Thus $a = af(a) = ag(a) = ag(1)a = aba$, where $b = g(1)$. Put $e = ab$. Thus $e^2 = abab = ab = e$ and $ea = aba = a$. Let $x \in aR$, then $x = ar$, for some $r \in R$. Thus $x = ar = ear \in eR$ and hence $aR \subseteq eR$. Let $y \in eR$, thus $y = et$ for some $t \in R$ and hence $y = abt \in aR$.

Thus $eR = aR$. Since $R = eR \oplus (1-e)R$, it follows that aR is a direct summand of R_R. Since R_R is JS-injective module, we have from [9, Corollary 2.4] that aR is a JS-injective module. By [9, Theorem 2.15], every right R-module is JS-injective. □

Lemma 2.13. Let R be a ring, then $D(R) = \{a \in R \,|\, r_R(a) \cap mR \neq 0 \text{ for each } 0 \neq m \in JS(R_R)\}$ is a left ideal of R.

Proof. It is obviously that $D(R)$ is a non-empty set, since $0 \in D(R)$. If $a \in D(R)$ and $0 \neq m \in JS(R_R)$, thus $mb \in r_R(a) \cap mR$, for some $b \in R$ and so $a(mb) = 0$. Since $(-a)(mb) = -(amb) = 0$, then $mb \in r_R(-a)$ and hence $r_R(-a) \cap mR \neq 0$. Thus $-a \in D(R)$. Now, let $a_1, a_2 \in D(R)$ and $0 \neq m \in JS(R_R)$. We have that $0 \neq mb \in r_R(a_1) \cap mR$ for some $b \in R$. Since $a_2 \in D(R)$, it follows that $-a_2 \in D(R)$ and hence $0 \neq mbc \in r_R(-a_2) \cap mR$ for some $c \in R$. Therefore, $0 \neq mbc \in r_R(a_1) \cap r_R(-a_2) \cap mR$. Since $r_R(a_1) \cap r_R(-a_2) = r_R(a_1 + (-a_2)) = r_R(a_1 - a_2)$ (by [1, Proposition 2.16, p. 38]), we have $r_R(a_1 - a_2) \cap mR \neq 0$ for all $0 \neq m \in JS(R_R)$ and hence $a_1 - a_2 \in D(R)$. Also, let $x \in R$ and $a \in D(R)$. Since $r_R(a) \subseteq r_R(xa)$, it follows that $r_R(xa) \cap mR \neq 0$ for all $0 \neq m \in JS(R_R)$, that is $xa \in D(R)$. Thus $D(R)$ is a left ideal of R. □

Proposition 2.14. Let R be a right JS-injective ring. Then $r_R(a) \subsetneq r_R(a - axa)$, for all $a \notin D(R)$ and for some $x \in R$.

Proof. For all $a \notin D(R)$, we can find $0 \neq m \in JS(R_R)$ such that $r_R(a) \cap mR = 0$. Clearly, $r_R(am) = r_R(m)$, so $Rm = Ram$ by Theorem 2.2(2). Thus $m = xam$ for some $x \in R$ and this implies that $m - xam = 0$ and hence $(1 - xa)(m) = 0$. Thus $a.(1 - xa)(m) = a.0$ and so $(a - axa)m = 0$. Therefore, $m \in r_R(a - axa)$, but $m \notin r_R(a)$ because $r_R(a) \cap mR = 0$ and hence the inclusion is strictly. □

Proposition 2.15. Let R be a right JS-injective ring, then the set $\{a \in R \,|\, r_R(1 - sa) = 0 \text{ for all } s \in R\}$ is contained in $D(R)$.

Proof. We will prove that by contradiction. Assume that there is a such that $r_R(1 - sa) = 0$ for all $s \in R$ with $a \notin D(R)$. Then there exists $0 \neq m \in JS(R_R)$ with $r_R(a) \cap mR = 0$. If $r \in r_R(am)$, then $(am)r = 0$ and hence $a(mr) = 0$ and so $mr \in r_R(a)$. Since $r_R(a) \cap mR = 0$, it follows that $mr = 0$ and so $r \in r_R(m)$. Hence $r_R(am) \subseteq r_R(m)$. By Theorem 2.2(2), $Rm \subseteq Ram$. Thus $m = sam$, for some $s \in R$. Therefore, $(1 - sa)m = 0$ and hence $m \in r_R(1 - sa) = 0$ so $m = 0$ and this is a contradiction. Thus the statement is hold. □

References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules.* New York: Springer-Verlag, (1974).

[2] P. E. Bland, *Rings and Their Modules.* Berlin:Walter de Gruyter & Co., (2011).

[3] F. Kasch, *Modules and Rings.* London: Academic Press, (1982).

[4] T. Y. Lam, *Lectures on Modules and Rings*. New York: Springer-Verlag, (1999).

[5] A. R. Mehdi, "On *L*-injective modules," *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki*, 28(2) (2018), pp. 176–192.

[6] E. A. Naim and A. R. Mehdi, "On gs-pseudo-injective modules," *Journal of Discrete Mathematical Sciences*, vol. 25, no. 5 (2022) , pp. 1535-1545.

[7] L. Shen and J. Chen, "New characterizations of quasi-Frobenius rings," *Comm. Algebra*, vol. 34 (2006), pp. 2157-2165.

[8] A. A. Tuganbaev, "Muitiplication Modules," *J. Mathimatical sciences*, 123 (2004), pp. 3839-3905.

[9] Z. A. Zone and A. R. Mehdi, "On a generalization of small-injective modules," *Iraqi Journal of Science*, to appear.