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Trigonometric Approximation and $2\pi - P$eriodic Neural Network Approximation

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ABSTRACT

Many articles studied best trigonometric approximation and many researchers worked on the neural network approximation, but no one related the best trigonometric approximation to neural network approximation. We define trigonometric activation function, then we use it to obtain neural network, which we use it as a best approximation for functions in L_p spaces for $0 < p < 1$. That what we shall introduce in our work have.

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1. Introduction

In [6, 7, 10, 13] studied best trigonometric approximation using continuous function. In [8, 9, 11] studied the approximation using many types of neural networks. No one relate the neural network to trigonometric polynomial approximation. That is what we do in our work have.

Firstly, let us introduce some basic notations and defines that we need in our work.

Begin with T^* is the best approximation of f, where $f: \mathbb{R}^n \to \mathbb{R}$

I_n^* the degree of approximation of f from Y_n is defined as

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$$I_n^* = \inf_{T\in Y_n} \|f - T\|^* \ , n = 1,2,\ldots$$

The norm of f define as

$$\|f\|_p = \left(\int_{-\pi}^{\pi} |f(x)|^p \, dx\right)^{\frac{1}{p}} = \left(\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f(x_1, x_2, \cdots, x_m)|^p \, dx_1 dx_2 \cdots dx_m\right)^{\frac{1}{p}}.$$

And

$L_p^*[-\pi,\pi]$ is the space of all $2\pi - p$eriodic functions in $L_p[-\pi,\pi]$.

The class of all trigonometric polynomials of order at most n denoted by Y_n.

And

$$G_r = \{f : f^{(r)} \in L_p\,[-\pi,\pi]\}, \qquad r > 0$$

$$\|f\|_\rho^* = \frac{\left\|f^{(r)}\,(x + h) - f^{(r)}\,(x)\right\|_\rho}{h^\alpha}$$

Where:

$$\rho = r + \alpha$$

$$r > 0\,, \alpha \in (0,1)$$

$f \in G_r$.

The class of all trigonometric polynomials in s variables is denoted by $Y_{n,s}$, and

$$I_{n,s}^*(f) = \inf_{T\in Y_{n,s}} \|f - T\|_s^*.$$

And

$$C_1^*\,(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\, dt.$$

And

$$C_1^*\,(\emptyset) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \emptyset(t)e^{i(x-t)}\, dt\,.$$

$$w_n(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\ \ dt\ \ W_n^*(t)dt.$$

$$W_n^*(t) = \frac{\sin(nt/2)\ \sin(3nt/2)}{n\,\sin^2(t/2)}$$

And

$L_s^*[-\pi,\pi]^s$ *is the space of all* $2\pi - p$eriodic functions in $L_s[-\pi,\pi]$, when $s \geq 1$.

For $f \in L_s^*[-\pi,\pi]$ and $j \in \mathbb{Z}^s$,

$$C_j^*(f) = \frac{1}{(2\pi)^s}\int_{[-\pi,\pi]^s} f(x)e^{-ij.x}\, dx,$$

$$\pi_{N,n,s}(\emptyset, f, x) = \frac{1}{(2N+1)C_1^*(\emptyset)}\sum_{k=0}^{2N}\sum_{-n\le j\le n} C_j^*(f)\, exp\left(\frac{2ik\pi}{2N+1}\right)\emptyset\left(j.x - \frac{2\pi k}{2N+1}\right).$$

The function $\pi_{N,n,s}(\emptyset, f) \in S_\emptyset(2N+1)(2n+1)^s,\ for\ N, n \ge 1$.

The neural network here has 3-layers: input layer, hidden layer and output layer.

In general, we can define the neural network mathematically as

$S_{\emptyset,N,s}(x) = \sum_{k=1}^{N} a_k\, \emptyset\,(w_k.x + b_k), with\ a_k, b_k \in \mathbb{R}, w_k \in \mathbb{R}^s, 1 \le k \le N$

Where $\emptyset$ is the activation function and $\emptyset : \mathbb{R} \longrightarrow \mathbb{R}$

Let us now recall example of activation function

$$\emptyset(x) = (1 + e^{-x})^{-1}\ [the\ squashing\ function]$$

We can define the sigmodal functions as

$$\emptyset(x) = 1, \quad if\ x \ge 0$$

$$\emptyset(x) = 0, \quad other\ wise.$$

We write $e_k(x) = \exp(ik.x), k \in \mathbb{Z}^s$

The Parseval's identity is

$$\|f\|^2_{L_{2(-\pi,\pi)}} = \int_{[-\pi,\pi]} |f(x)|^2\, dx = 2\pi \sum_{n=-\infty}^{\infty} |C_n|.$$

Where C_n is the Fourier coefficients of f are given by

$$C_n = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(x)e^{-inx}\, dx.$$

And a Whitney extension theorem in L_p is define as:

If k is non-negative integer and $k < \alpha < k+1, f \in L_p(\mathbb{R}^n)$ for which the norm

$$\|f\|_p = \sum_{|j|<k} \left\|D^j f(x)\right\|_p \le C \sum_{|j|<k} \left\|D^j f(x)\right\|_p .$$

Where $D^{(j)}$ is the differentiable of functions.

Lemma 1.1

For $f \in L_s^*[-\pi,\pi]$, w_n^* is bounded operator

$$w_n^*(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\ dt\ W_n^*(t)dt.$$

Proof

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\ \frac{\sin(nt/2)\ \sin(3nt/2)}{n\sin^2(t/2)}\,dt$$

$$\|w_n^*(f,x)\|_p \ = \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\ \frac{sin(nt/2)\ sin(3nt/2)}{n\,sin^2(t/2)}\,dt,\right.$$

$$= \left(\int_{-\pi}^{\pi}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\ \frac{sin(nt/2)\ sin(3nt/2)}{n\,sin^2(t/2)}\,dt\right|^p dx\right)^{\frac{1}{p}}.$$

Since $\left|\sin\frac{nt}{2}\right|$ and $\left|\sin\frac{3nt}{2}\right|$ are bounded and $\sin\frac{t}{2}$ bounded below by $\frac{2}{\pi}.\frac{t}{2}$.

Thus

$$\|w_n^*(f,x)\|_p \le \left(\int_{-\pi}^{\pi}\left(\frac{1}{2n}\int_{-\pi}^{\pi}|f(x-t)|\frac{1}{n\frac{2}{\pi}\frac{|t|}{2}}dt\right)^p dx\right)^{\frac{1}{p}}$$

$$= \ (\int_{-\pi}^{\pi}(\int_{-\pi}^{\pi}\frac{1}{2|t|n}|f(x-t)|dt)^p dx)^{\frac{1}{p}}$$

$$\le \ (\int_{-\pi}^{\pi}(\int_{-\pi}^{\pi}\frac{1}{2|t|n}|f(x)|dt)^p dx)^{\frac{1}{p}}$$

$$\le \ (\int_{-\pi}^{\pi}(\frac{2\pi}{n\,ln\,\pi}|f(x)|)^p dx)^{\frac{1}{p}}$$

$$= \frac{\pi}{n\,ln\,\pi}(\int_{-\pi}^{\pi}|f(x)|^p dx)^{\frac{1}{p}}$$

$$= \frac{\pi}{n\,ln\,\pi}\|f\|_p \ \blacksquare$$

Lemma 1.2

$$I_{2n-1}(f)_p \le \|f - w_n^*(f)\|_p \le 2^{p-1} I_n(f)_p \,. \tag{1}$$

Proof:

Using definition of $I_{2n-1}(f)_p$, the first part of the pnality is clear, then the second part of (1) we have

$$\|f - w_n^*(f)\|_p = \ \|(f-T) - w_n^*(f-T)\|_p$$

$$\le 2^{p-1}(\|f-T\|_p + \|w_n^*(f-T)\|_p)$$

$$\le 2^{p-1}(\|f-T\|_p + \frac{\pi}{2\,ln\,\pi}\|(f-T)\|_p)$$

$$\le \pi 2^{p-1}(\|f-T\|)$$

$$= \pi 2^{p-1} I_n(f)_p \ \blacksquare$$

2. The Main Results

In this section we will define trigonometric approximation and $2\pi -$ Periodic neural network approximation in $L_p[-\pi,\pi]$.

Proposition 2.1.

Let $\emptyset \in L_p^*[-\pi,\pi]$ and $C_1^*(\emptyset) \neq 0$ for any integer $N \geq 1$,

$$\left\| e^{i} - \frac{1}{(2N+1)C_1^*(\emptyset)} \sum_{k=0}^{2N} exp(\frac{2ik\pi}{2N+1})\emptyset(.-\frac{2\pi k}{2N+1}) \right\|_p \leq \frac{\pi 2^{p-1}}{|C_1^*(\emptyset)|} I_n^*(\emptyset).$$

Proof:

By the definition of $C_1^*(\emptyset)$, we get for $x \in [-\pi,\pi]$,

$$e^{ix} = \frac{1}{2\pi C_1^*(\emptyset)} \int_{-\pi}^{\pi} \emptyset(t) e^{i(x-t)} dt$$

$$= \frac{1}{2\pi C_1^*(\emptyset)} \int_{-\pi}^{\pi} \emptyset(x-t) e^{it} dt \,.$$

Now, for any $N \geq 1$,

$$\int_{-\pi}^{\pi} \emptyset(x-t) e^{it} dt = \int_{-\pi}^{\pi} w_n^*(\emptyset, x-t) e^{it} dt \,.$$

As a function of t, $w_n^*(\emptyset, x-t)e^{it} \in Y_{2n}$, we evaluate the last integral by using

$$\frac{1}{n+1} \sum_{k=0}^{2N} T(\frac{2\pi}{n+1}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T(t) dt \,, \quad T \in Y_n$$

We get

$$C_1^*(\emptyset) e^{ix} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \emptyset(x-t) e^{it} dt$$
$$= \frac{1}{2N+1} \sum_{k=0}^{2N} exp(\frac{2i\pi k}{2N+}) \, w_n^*((\emptyset, (x-\frac{2\pi k}{2N+1})).$$

Now, by using Lemma 1.1 and Lemma 1.2, we get

$$\|w^*(f)\|_p \leq C\|f\|_p, I_{2N-1}^*(f) \leq \|f - w^*(f)\|_p \leq \pi 2^{p-1} I_n^*(f)$$

We obtain for all $x \in [-\pi,\pi]$

$$\left\| \frac{1}{2N+1} \sum_{k=0}^{2N} exp(\frac{2ik\pi}{2N+1}) w_n^*(\emptyset, (x-\frac{2k\pi}{2N+1})) - \frac{1}{2N+1} \sum_{k=0}^{2N} exp(\frac{2ik\pi}{2N+1}) \emptyset(x-\frac{2k\pi}{2N+1}) \right\|_p \leq \pi 2^{p-1} I_n^*(\emptyset) \; \blacksquare$$

Theorem 2.2

Let $s, n, N \geq 1$ be integers and $T \in Y_{n,s}$, then

$$\|T - \prod_{N,n,s}(\emptyset, T)\|_s^* \leq \frac{(2n+1)^{S/2} I_N^*(\emptyset)}{|C_1^*(\emptyset)|} \|T\|_s^* \,.$$

Proof:

By proposition 2.1, we get for $-n \le k \le n$,

$$\left\| e_k - \prod_{N,n,s}(\emptyset, e_k) \right\|_s^* \le \frac{\pi 2^{p-1} I_N^*(\emptyset)}{|C_1^*(\emptyset)|} \tag{2}$$

We note that

$$\prod_{N,n,s}(\emptyset, T) \; = \sum_{-n \le k \le n} C_k^*(T) \prod_{N,n,s}(\emptyset, e_k)$$

Hence, by (2) we get

$$\left\| T - \prod_{N,n,s}(\emptyset, T) \right\|_s^* \le \frac{\pi 2^{p-1} I_N^*(\emptyset)}{|C_1^*(\emptyset)|} \sum_{-n \le k \le n} |C_k^*(T)| \,.$$

Now, we recall the personal identity, which states that

$$\left(\sum_{-n \le k \le n} |C_k^*(T)|^2 \right)^{1/2} = \left(\frac{1}{(2\pi)^s} \int_{[-\pi,\pi]^s} |T(x)|^2 \, dx \right)^{1/2}$$

Since T is polynomial, so

$$\sum_{-n \le k \le n} |C_k^*(T)|^2 = \left(\frac{1}{(2\pi)^s} \right)^{1/p} \left(\int_{[-\pi,\pi]^s} (|T|^p dx \right)^{1/p}, p < q$$

And

$$\sum_{[-\pi,\pi]^s} (|T|^q)^{1/q} < \sum_{[-\pi,\pi]^s} (|T|^p)^{1/p}$$

So, we obtain

$$\sum_{-n \le k \le n} |C_k^*(T)| \le (2n+1)^{s/p} \left\{ \sum_{-n \le k \le n} \left| C_k^*(T) \right|^p \right\}^{1/p}$$

$$= \; (2n+1)^{s/p} \left\{ \frac{1}{(2\pi)^s} \int_{[-\pi,\pi]} |T(x)|^p \, dx \right\}^{1/p}$$

$$\le (2n+1)^{s/p} \|T\|_s^* \; \blacksquare$$

Theorem 2.3

Let C_o be a set of distinct points in $[-\pi, \pi]^s$ and $n \ge 1$ be an integer such that

$$S_{C_o} < \pi / (2 - 3^{s+4} n) \,.$$

And there exist numbers $\{h_\xi\}_{\xi \in C_o}$

where $|h_\xi| \le cn^{-s}, \xi \in C_o$ then $T_{n,s}^*$ be defined by

$$T_{n,s}^*(f, x) = \sum_{\xi \in C_o} h_\xi \, f(\xi) W_{n,s}^*(x - \xi), \qquad f \in L_s^* \tag{3}$$

And $T^*_{n,s}(T) = T$ for every $T \in Y_{n,s}$.

Also, for $f \in C^*_s$, $T^*_{n,s}(f) \in Y_{2n-1,s}$ and we have

$$\left\|T^*_{n,s}(f)\right\|^*_s \leq C\|f\|^*_s \tag{4}$$

$$I^*_{2n-1,s}(f) \leq \left\|f - T^*_{n,s}(f)\right\|^*_s \leq C I^*_{n,s}(f) \tag{5}$$

Proof:

$$T^*_{n,s}(f) = \sum_{\xi \in C_o} h_\xi\, f(\xi) w^*_{n,s}(x-\xi)$$

$$\left\|T^*_{n,s}(f)\right\|^p_p \leq \sum_{\xi \in C_o} \left\|h_\xi f(\xi) w^*_{n,s}(x-\xi)\right\|_p \ .$$

Let $f(\xi) = t$

$$= \sum_{\xi \in C_o} \left\| \frac{sin(\frac{nx-1}{2})\, sin(\frac{3nx-\xi}{2})}{n sin^2(\frac{1}{2})} h_\xi f(\xi) \right\|_p \ .$$

By using Lemma 1.1, we obtain

$$\left\|T^*_{n,s}(f)\right\|^p_p \leq \sum_{\xi \in C_o} \frac{\pi}{n\, ln\, \pi} \left\| h_\xi f(\xi)\right\|_p \ .$$

Let

$$x - \xi = y$$

$$\xi = x - y$$

$$d\xi = dx$$

Since $h_\xi \leq \frac{1}{n^s}, s > 1$, so

$$\left\|T^*_{n,s}(f)\right\|_p \leq C \frac{\pi}{n\, ln\, \pi} \|f\|_p \ .$$

Where C is a positive constant.

Using the same lines of Lemma 1.2, we get the Lemma 2.4:

$$I^*_{2n-1,s}(f) \leq \left\|f - T^*_{n,s}(f)\right\|^*_s \leq C I^*_{n,s}(f)\ . \blacksquare$$

Lemma 2.4

For $f \in L^*_p[-\pi, \pi]$, we have $w^*_n \in Y_{2n-1}$ and

$$I^*_{2n-1}(f) \leq \|f - w^*_n(f)\|^*_p$$

$$\leq C I^*_n(f).$$

Proof:

Since $w_n^*(f) \in Y_{2n-1}$, it is clear that

$$I_{2n-1}^*(f) \le \|f - w_n^*(f)\|_p^*$$

If $T \in Y_n$, then $w_n^*(T) = T$ and

$$\|f - w_n^*(f)\|_p^* = \|(f-T) - w_n^*(f-T)\|_p^*$$

$$\le \|f - T^*\|_p^* + \|w_n^*(f-T)\|_p^*$$

$$\le \|C(f-T)\|_p^* \quad ■$$

Theorem 2.5

Let $\emptyset \in L_p^*[-\pi,\pi]$ and $C_1^*(\emptyset) \neq 0$. Let $s, n \ge 1$, then for any $f \in L_p^*[-\pi,\pi]$ and $N \ge 1$, we have

$$\left\|f - \prod_{N,2n-1,s}(\emptyset, T_{n,s}^*(f)\right\|_s^* \le C\left\{I_{n,s}^*(f) + \frac{n^{(S/2)} I_N^*(\emptyset)}{|C_1^*(\emptyset)|}\|f\|_s^*\right\}.$$

Proof:

By using Proposition 2.1 and Theorem 2.3 we get

$$\left\|f - \prod_{N,2n-1,s}(\emptyset, T_{n,s}^*(f)\right\|_s^* \le \left\|f - T_{n,s}^*(f)\right\|_s^* + \left\|T_{n,s}^*(f) - \prod_{N,2n-1,s}(\emptyset, T_{n,s}^*(f)\right\|_s^*$$

By using Lemma 2.4:

$$\left\|f - \prod_{N,2n-1,s}(\emptyset, T_{n,s}^*(f)\right\|_p \le C I_{n,s}^*(f) + \frac{4(4n-1)^{S/2} I_N^*(\emptyset)}{|C_1^*(\emptyset)|}\left\|T_{n,s}^*(f)\right\|_s^*.$$

By using Theorem 2.3, we get

$$\left\|f - \prod_{N,2n-1,s}(\emptyset, T_{n,s}^*(f)\right\|_p \le C\left\{I_{n,s}^*(f) + \frac{n^{(S/2)} I_N^*(\emptyset)}{|C_1^*(\emptyset)|}\|f\|_s^*\right\} \quad ■$$

Lemma 2.6

Let $\emptyset(x) = (1+e^{-x})^{-1}$,

$$\beta(x) = (1+e^{-(x+1)})^{-1} - \frac{1}{1+e^{1-x}}$$

Then

$$\|\beta(x)\|_p \le \frac{1}{2\pi}\int_{-R}^{R} \beta(x)\, e^{ix}\, dx \qquad \neq 0$$

Proof:

$$\|\beta(x)\|_p = \left\|\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\beta(x)\, e^{ix}\, dx\right\|_p$$

$$\leq \int_{C} \left| \frac{1}{2\pi} \int_{-R}^{R} \beta(x)\, e^{ix} dx \right| dx \quad .$$

$$|\beta(x)| = \left| \frac{1}{2\pi} \int_{-R}^{R} \left(\frac{1}{1+e^{-(x+1)}} - \frac{1}{1+e^{1-x}} \right) e^{ix} dx \right|$$

$$\leq \frac{1}{2\pi} \int_{-R}^{R} \left| \frac{1}{1+e^{-(x+1)}} - \frac{1}{1+e^{1-x}} \right| \left| e^{ix} \right| dx$$

$$\int_{-R}^{R} \left| \frac{1}{1+e^{-(x+1)}} - \frac{1}{1+e^{1-x}} \right| \left| e^{ix} \right| dx = \int_{-R}^{-1} \left| \frac{1}{1+e^{-(x+1)}} - \frac{1}{1+e^{1-x}} \right| \left| e^{ix} \right| dx + \int_{-1}^{R} \left| \frac{1}{1+e^{-(x+1)}} - \frac{1}{1+e^{1-x}} \right| \left| e^{ix} \right| dx$$

$$= I_1 + I_2$$

$$I_1 = \int_{-R}^{-1} \left| \frac{1}{1+e^{-(x+1)}} - \frac{1}{1+e^{1-x}} \right| \left| e^{ix} \right| dx$$

$$\leq \int_{-R}^{-1} \left(\frac{1}{2} + \frac{1}{2} \right) \left| e^{ix} \right| dx$$

$$= \int_{-R}^{-1} \left| e^{ix} \right| dx$$

$$= \int_{-R}^{-1} |cos\, x + i\, sin\, x| dx$$

$$\leq \int_{-R}^{-1} \sqrt{cos^2 x + sin^2 x}\, dx$$

$$= -1 + R = R - 1$$

$$I_2 = \int_{-1}^{R} \left| \frac{1}{1+e^{-(x+1)}} - \frac{1}{1+e^{1-x}} \right| \left| e^{ix} \right| dx$$

$$\leq \int_{-1}^{R} \left(\frac{1}{2} + \frac{1}{2} \right) \left| e^{ix} \right| dx$$

$$= \int_{-1}^{R} \left| e^{ix} \right| dx$$

$$= \int_{-1}^{R} |cos\, x + i\, sin\, x| dx$$

$$\leq \int_{-1}^{R} \sqrt{cos^2 x + sin^2 x}\, dx = R + 1$$

$$\int_{-R}^{R}\left|\frac{1}{1+e^{-(x+1)}}-\frac{1}{1+e^{1-x}}\right|\left|e^{ix}\right|dx = I_1 + I_2$$

$$\leq R-1+R+1=2R \quad \blacksquare$$

Lemma 2.7

Let $\emptyset(x) = (1+e^{-x})^{-1}$,

$$\beta(x) = (1+e^{-(x+1)})^{-1} - \frac{1}{1+e^{1-x}}$$

Then

$$S^{[sq]}(x) = \sum_{K\in\mathbb{Z}} \beta(x-2\pi k)$$

Proof:

Let $\epsilon > 0$, then

$$\left\|\sum_{K\in\mathbb{Z}} \beta(x-2\pi k) - \sum_{|K|\leq H} \beta(x-2\pi k)\right\|_p \leq 2\left\|\sum_{K\in\mathbb{Z}} \beta(x-2\pi k)\right\|_p \leq 2$$

Since $\sum_{K\in Z} \beta(x-2\pi k)$ is uniformly convergent series, so

$$\left\|S^{[sq]} - \sum_{K\in\mathbb{Z}} \beta(x-2\pi k)\right\|_p \leq \epsilon \ .$$

For a given $\epsilon > 0$. ■

Example 2.8

Let $\emptyset(x) = (1+e^{-x})^{-1}$,

$$\beta(x) = (1+e^{-(x+1)})^{-1} - \frac{1}{1+e^{1-x}}$$

Then β is integrable. The Fourier transform of β can be computed by Contour integration, which clear that

$$\beta(1) = \frac{1}{2\pi}\int_{-R}^{R} \beta(x)\, e^{ix} dx, \qquad \neq 0$$

We construct a periodization of β by

$$S^{[sq]}(x) = \sum_{K\in\mathbb{Z}} \beta(x-2\pi k) \tag{6}$$

Since $\|\beta(x)\|_p \leq (e-e^{-1})e^{-|x|}$, for $x \in \mathbb{R}$, the series in (6) converge uniformly in compact subsets of $\mathbb{R}$ and the function $S^{[sq]}$ is clearly 2π −Periodic, one can compute easily that $C_1^*(S^{[sq]}) = \hat{\beta}(1) \neq 0$. There exist $\alpha > 0$ such that:

$$I_N^*(S^{[sq]}) \leq e^{-\alpha N}, \qquad N = 1,2,\ldots\ldots$$

Now, let $f \in L_p[-\pi,\pi]^s, f\colon [-1,1]^s \to \mathbb{R}$.

According to Whitney extension theorem, there exist an extension of $f, g: [-4,4]^s \to \mathbb{R}$, such that

$$\sum_{0\le j\le s} \left\|D^j g(x)\right\|_p \le C \sum_{0\le j\le s} \left\|D^j f(x)\right\|_p.$$

Now, let $\Psi \in L_p[-\pi,\pi]^s$, such that

$$\Psi(x) = \begin{cases} 1, & if\ x \in [-1,1]^s \\ 0, & if\ x\ outside\ \left[-{}^{\pi}\!/_{2}, {}^{\pi}\!/_{2}\ \right]^s \end{cases}$$

Then, the function Ψ_g has the properties that $\Psi(x)g(x) = f(x)$, for $x \in [-1,1]^s$ and

$$\sum_{0\le j\le s} \left\|D^j(\Psi_g\ )(x)\right\|_p \le C \sum_{0\le j\le s} \left\|D^j f(x)\right\|_p$$

Further, since $\Psi(x)g(x) = 0$ outside $\left[-{}^{\pi}\!/_{2}, {}^{\pi}\!/_{2}\ \right]^s$, we may extend Ψ_g as a function on $\mathbb{R}^S$ that is 2π –Periodic in each of its variables. Denoting this extension by f^*, $f^*(x) = f(x)$, $x \in [-1,1]^s$ and

$$\sum_{j=1}^{s} \left\|D_j^s f^*,\right\|_{\acute{q}}^* \le C \sum_{0\le j\le s} \left\|D^j f(x)\right\|_p, \qquad 0 < \acute{q}, p < 1$$

By using Theorem (2.5) and theorem [let $r \ge 1$ be an integer, for integer $n \ge 1$ and $f \in L_S^*$, we have $I_{n,s}^*(f) \le C\sum_{j=1}^{s}\left\|D_j^r f\ \right\|_{\acute{q}}^*$].

And taking

$$E = \frac{{}^{(r+s)}\!/_{2}}{\alpha} \log n$$

We obtain that

$$\left\|f(x) - \beta_{N,2n-1,s}(S^{[sq]}, T_{n,s}^*(f^*)), x\right\|_p \le \left\|f^* - \beta_{N,2n-1,s}(S^{[sq]}, T_{n,s}^*(f^*))\right\|_{\acute{q}}^*$$

$$\le C(\acute{q})\left\{I_{n,s}^*(f^*) + \frac{n^{S/2} I_N^* S^{[sq]}}{|C_1^*(S^{[sq]})|}\|f^*\|_{\acute{q}}^*\right\}$$

$$\le C(\acute{q})(n^{-r} + n^{S/2}e^{-\alpha n}) \sum_{0\le j\le s} \left\|D^j f(x)\right\|_p$$

$$\le Cn^{-s} \sum_{0\le j\le s} \left\|D^j f(x)\right\|_p \qquad (7)$$

We observe that

$$\left\|S^{[sq]}(x) - \sum_{|K|\ge H} \beta(x - 2\pi k)\right\|_p \le Ce^{-2\Pi(H-|x|)}, \qquad x \in \mathbb{R}: f$$

If we choose $H = 2n$ and replace each ocurrence of S[sq] $(j.x - {}^{(2\pi k)}\!/_{(2N+1)})$ in $\prod_{N,2n-1,s}(S^{[sq]}, T_{n,s}^*(f^*), x)$ by its partial sum, we get a network $V(f)$ having cn^{s+1} neurons.
Using the proof of Theorem 2.2, we can proof that

$$\left\|\prod_{N,2n-1,s}(S^{[sq]}, T_{n,s}^*(f^*), x) - V(f,x)\right\|_p \le Cn^{S/2}e^{\acute{c}/n} \le C_2 e^{-C_3 n}$$

Thus, (7) leads to a network $V(f)$ with n^{s+1} neurons such that:

$$\|f(x) - V(f,x)\|_p \leq Cn^{-s} \sum_{0\leq j\leq s} \left\|D^j f(x)\right\|_p .$$

3. Conclusions

We define a new type of neural network such as trigonometric neural network. Then, we using it to approximate functions in L_p quasi normed space for $0 < p < 1$. This paper shows that we relate the neural network to trigonometric polynomial approximation.
This leads to that our approximation in this neural network is strong.

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