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Results on Fourth Hankel Determinant of a Certain Subclass of Analytic Functions

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ABSTRACT

In this paper, The property of subordination is used to define the subclass $\mathcal{F}(\delta, z, t), |a_n|$ bounds on the coefficients. The upper bound for the functions in this class is provided for $n = 2,3,4,5,6,7$. The fourth Hankel determinant for a class $\mathcal{F}(\delta, z, t)$ is obtained. Some new results also are obtained.

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1. Introduction

Let A denote the class of all functions $f(z)$ of the form:

$$f(z) = \sum_{j=1}^{\infty} a_j z^j \qquad (a_1 = 1, j \in N = \{1,2, \dots\}), \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. Although function theory was started in 1851, it emerged as a good area of new research in 1916, due to the conjecture $|a_n| \leq n$, which was proved by De-Branges in 1985 and many scholars attempted to prove or disprove this conjecture as a result they discovered

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multiple subfamilies of a class S of univalent functions that are associated with different image domains. The most basic of these families are th families of starlike and convex functions which are defined by:

$$\mathcal{S}^*(\alpha) = \left\{ f \in S: Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in U \right\},$$

$$\mathcal{C}(\alpha) = \left\{ f \in S: Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in U \right\}.$$

The family $\mathcal{S}_e{}^* = \mathcal{S}^*(\mathrm{e}^{\mathrm{z}})$ was introduced by Mendiratta et al.[6] such that:

$$\mathcal{S}_e{}^* = \left\{ f \in A: \frac{zf'(z)}{f(z)} \prec e^z, z \in U \right\}. \quad (2)$$

Similarly, by using an Alexander type relation the following class was also introduced in [2,12,15]

$$\mathcal{C}_e^* = \left\{ f \in A: 1 + \frac{zf''(z)}{f'(z)} \prec e^z, z \in U \right\}. \quad (3)$$

Each of the functions classes described above has a distinct symmetry. We denote by P, the class of analytic functions p normalized by:

$$p(z) = 1 + c_1 z + c_2 z^2 + c_2 z^2 + \cdots \quad (4)$$

and

$$Re\big(p(z)\big) > 0, \quad z \in U \ . \quad (5)$$

Assume that f *and* g are two analytic functions in U. Then, we say that the function f is subordinate to the function g, and we can write $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ with the following conditions:

$$w(0) = 0 \ and \ |w(z)| < 1, (z \in U),$$

such that

$$f(z) = g\big(w(z)\big), \quad (z \in U).$$

The problem of determining coefficient bounds offers information on a complex valued function's geometry. In particular, the second coefficient provides information about the growth and distortion theorems for functions in class S. Similarly, in the study of singularities and power series with integral coefficients, the Hankel determinants are

particularly useful. In 1976, Noonan and Thomas [15] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ of function f as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \cdots & a_{n+q} \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} \cdots & a_{n+2q-2} \end{vmatrix}$$

The estimates of $H_q(n)$ have been investigated for different subfamilies of univalent functions. For $q = 2$ and $n = 1$, function $H_2(1) = a_3 - a_2^2$ is the well – known Fekete-Szegő functional. The second Hankel determinant $H_2(2)$ defined as $H_2(2) = a_2a_4 - a_3^2$ was studied for the classes of bi–starlike and bi-convex functions [1,6,7,8,9,11,19,20,21] ,also see([2,4,5,6,7,14,15,16,17,18]. For the third Hankel determinant is given as

$$H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = -a_5a_2^2 + 2a_2a_3a_4 - a_3^3 + a_3a_5 - a_4^2 .$$

The third Hankel determinant was studied in [10] for a certain subclass of starlike functions. The fourth Hankel determinant was studied in [3,14,16,17,19] for a certain subclass of starlike functions, convex functions and a new class of analytic functions is also introduced by means of subordination and estimates are given for the upper bound of the fourth Hankel determinant for functions in this class. For $q = 4$ and $n = 1$, fourth Hankel determinant is given as see [3,16,17]:

$$H_4(1) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix}, \qquad a_1 = 1 ,$$

such that:

$$H_4(1) = a_7\{a_3I_1 - a_4I_2 + a_5I_3\} - a_6\{a_3I_4 - a_4I_5 + a_6I_3\} + a_5\{a_3I_6 - a_5I_5 + a_6I_2\} - a_4\{a_4I_6 - a_5I_4 + a_6I_2\} , \qquad (6)$$

where

$$I_1 = a_2a_4 - {a_3}^2, \qquad I_2 = a_4 - a_2a_3, \qquad I_3 = a_3 - {a_2}^2$$

$$I_4 = a_2a_5 - a_3a_4, \qquad I_5 = a_5 - a_2a_4, \qquad I_6 = a_3a_5 - {a_4}^2. \qquad (7)$$

Chebyshev polynomials of the first and second kind are defined in the case of a real variable x on (-1, 1) by:

$$T_n(x) = cos(n\, arccos\, x),$$

and

$$U_n(x) = \frac{sin[(n+1)\, arccos\, x]}{sin(arccos\, x)} = \frac{sin[(n+1)\, arccos\, x]}{\sqrt{1-x^2}} .$$

Respectively.

We consider the function

$$\mathcal{N}(\mathrm{t},\mathrm{z}) = \frac{1}{1-2\mathrm{tz}+\mathrm{z}^2}, \qquad \mathrm{t} \in \left(\frac{1}{2},1\right), \qquad \mathrm{z} \in \mathrm{U}.$$

It is well-known that if $\mathrm{t} = \cos\alpha$, $\alpha \in \left(0,\frac{\pi}{3}\right)$, then

$$\mathcal{N}(\mathrm{t},\mathrm{z}) = 1 + \sum_{n=1}^{\infty} \frac{\sin[(n+1)\alpha]}{\sin\alpha} \mathrm{z}^n = 1 + 2\cos\alpha\,\mathrm{z} + (3\cos^2\alpha - \sin^2\alpha)\mathrm{z}^2 + (8\cos^3\alpha - 4\cos\alpha)\mathrm{z}^3 + \cdots, \mathrm{z} \in \mathrm{U}.$$

That is,

$$\mathcal{N}(t,z) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + \cdots \quad t \in \left(\frac{1}{2},1\right), z \in U,$$

where $\mathrm{U_n(t)} = \frac{\sin[(n+1)\arccos t]}{\sqrt{1-t^2}}$, $n \in \mathbb{N}$ are the second kind Chebyshev polynomials. From

the definition of the second kind Chebyshev polynomials, we easily obtain that $\mathrm{U_1(t)} = 2\mathrm{t}$.

Also, it is well-known that

$$U_{n+1}(t) = 2tU_n(t) - U_{n-2}(t).$$

For all $n \in \mathbb{N}$. From here, we can easily obtain

$$U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t.$$

In this idea, we introduced a new subclasses of analytic functions by means of subordination and estimates are given for the upper bound of the fourth Hankel determinant for functions in this class.

2. Lemmas and definition:

Lemma(1) [13]: If a function $f \in \mathcal{S}_1{}^*$ is of the form (2) then:

$$|a_2| \le 1, \quad |a_3| \le \frac{3}{4}, \quad |a_4| \le \frac{17}{36}, \quad |a_5| \le 1,$$

where $\mathcal{S}_1{}^*$ denote the class of analytic functions to third Hankel determinant.

Lemma(2) [16]: If the function $f \in \mathcal{S}_e{}^*$ and of the form (1) then:

$$|a_2| \le 1, \quad |a_3| \le \frac{3}{4}, \quad |a_4| \le \frac{1}{18}, \quad |a_5| \le \frac{1}{96}, \quad |a_6| \le \frac{1}{600}, \quad |a_7| \le \frac{2401}{3600}.$$

Lemma(3) [16]: If the function $f \in \mathcal{C}_e^*$ and of the form (1), then:

$$|a_2| \le \frac{1}{2}, \quad |a_3| \le \frac{1}{4}, \quad |a_4| \le \frac{1}{72}, \quad |a_5| \le \frac{1}{480}, \quad |a_6| \le \frac{1}{3600}, |a_7| \le \frac{343}{3600}.$$

Now; we discuss a new subclass of analytic functions using the subordination.

Definition(1): A function $f \in A$ given by (1) is said to be in the class $\mathcal{F}(\delta, z, t)$ if the following conditions are satisfied holds:

$$\frac{z}{2\delta}[f''(z) + zf'''] \prec \mathfrak{L}(t, z), \tag{8}$$

where $\mathfrak{L}(t,z) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + \cdots \;\; t \in \left(\frac{1}{2}, 1\right), z \in U, \delta \geq 0.$

Theorem (1): Let f be a function given by (1) which belongs to the subclass $\mathcal{F}(\delta, z, t)$. Then:

$$|a_2| \leq 4\delta, \quad |a_3| \leq 2\delta, \quad |a_4| \leq \frac{36\delta}{18}, \quad |a_5| \leq \frac{27}{10}\delta, \quad |a_6| \leq \frac{108\delta}{25}, \quad |a_7| \leq \frac{484}{63}\delta, \tag{9}$$

where $\delta \geq 0$.

Proof: If $f \in \mathcal{F}(\delta, z, t)$, then there exists an analytic function φ in U with $\varphi(z) \leq 1$ and we can write

$$\frac{z}{2\delta}[f''(z) + zf'''] = \mathfrak{L}\big(t, \mathcal{E}(z)\big).$$

Using the definition of subordination, there exists $\mathcal{E}$ a Schwarz function of the form:

$$\mathcal{E}(z) = \sum_{j=1}^{\infty} p_j z^j \quad z \in U$$

and following can be written as:

$$\begin{aligned}\mathfrak{B}(z) = \frac{1+\mathcal{E}(z)}{1-\mathcal{E}(z)} &= 1 + 2p_1 z + 2(p_2 + p_1^2)z^2 + 2[p_3 + p_1(2p_2 + p_1^2)]z^3 + 2[p_4 + p_2^2 + p_1^2(3p_2 + p_1^2) + 2p_1p_3]z^4 \\ &+ 2\big[p_5 + 2p_2(p_3 + 2p_1^3) + 3p_1(p_1p_3 + p_2^2) + \big(p_1(2p_4 + p_1^4)\big)\big]z^5 \\ &+ 2[p_6 + p_1^3(4p_3 + p_1^3) + p_1^2(3p_4 + 5p_2p_1^2) + 2p_1(p_5 + 3p_2p_3) + p_3^2 + p_2^2(p_2 + 6p_1^2) + 2p_2p_4]z^6 \\ &+ \cdots. \end{aligned} \tag{10}$$

Since $f \in S$, then:

$$\frac{z}{2\delta}[f''(z) + zf'''] = \frac{1}{\delta}[a_2 z + 6a_3 z^2 + 18a_4 z^3 + 40a_5 z^4 + 75a_6 z^5 + 126a_7 z^6 + \cdots]. \tag{11}$$

Using (10) with (11) comparing the coefficients of z^j for $j = 1,2,3, \ldots$ we obtain:

$$\frac{1}{\delta}a_2 = 2p_1$$

$$\frac{6}{\delta}a_3 = 2(p_2 + p_1^2),$$

$$\frac{18}{\delta}a_4 = 2[\,p_3 + p_1(2\,p_2 + p_1^2)],$$

$$\frac{40}{\delta}a_5 = 2[\,p_4 + p_2^2 + p_1^2(3\,p_2 + p_1^2) + 2\,p_1\,p_3],$$

$$\frac{75}{\delta}a_6 = 2\big[\,p_5 + 2\,p_2(\,p_3 + 2p_1^3) + 3p_1(p_1p_3 + p_2^2) + \big(p_1(2p_4 + p_1^4)\big)\big],$$

$$\frac{126}{\delta}a_7 = 2[\,p_6 + p_1^3(4\,p_3 + p_1^3) + p_1^2(3p_4 + 5p_2p_1^2) + 2p_1(p_5 + 3p_2p_3) + p_3^2 + p_2^2(\,p_2 + 6p_1^2) + 2\,p_2\,p_4].$$

By Lemma 1, we get relations (9).■

In the following theorem, estimates on $H_4(1)$ are determined for $f \in \mathcal{F}(\delta, z, t)$

Theorem(2): If function f of the form (1) belong to the subclass $\mathcal{F}(\delta, z, t)$, then:

$$H_4(1) \le \frac{1230227}{1875}\delta^4 + \frac{1067929}{45000}\delta^3,$$

where $\delta \ge 0$.

Proof: The fourth Hankel determinant can be written as (6) and (7), and we can write it by:

$$|H_4(1)| = |a_7 n_1 - a_6 n_2 + a_5 n_3 - a_4 n_4| \,, \tag{12}$$

where

$$|n_1| = |a_3 I_1 - a_4 I_2 + a_5 I_3|, \qquad |n_2| = |a_3 I_4 - a_4 I_5 + a_6 I_3|,$$

$$|n_3| = |a_3 I_6 - a_5 I_5 + a_6 I_2|, \qquad |n_4| = |a_4 I_6 - a_5 I_4 + a_6 I_2|,$$

where I_1, I_2, I_3, I_4, I_5 and I_6 as (7).

If we use (9) in n_1, n_2, n_3 and n_4, then

$$|n_1| \le \frac{336}{5}\delta^3 + \frac{7}{5}\delta^2, \tag{13}$$

$$|n_2| \le \frac{-988}{25}\delta^3 + \frac{-54}{25}\delta^2, \tag{14}$$

$$|n_3| \le \frac{-254}{25}\delta^3 + \frac{27}{20}\delta^2 \,, \tag{15}$$

$$|n_4| \le \frac{43}{25}\delta^3 \,. \tag{16}$$

Using (13), (14), (15) and (16) in (12), we get:

$$H_4(1) \leq \frac{1230227}{1875}\delta^4 + \frac{1067929}{45000}\delta^3.$$

In case $\delta = 1$, we get the following corollary:

Corollary (1): If f function of the form (1) belongs to the subclass $\mathcal{F}(\delta, z, t)$, then:

$$H_4(1) \leq\approx 679.8528222.$$

Theorem (3): If $f \in \mathcal{F}(\delta, z, t)$, with $\delta \geq 0$, then,

$$|H_4(1)| = \mathcal{R}_1(\delta) - \mathcal{R}_2(\delta) + \mathcal{R}_3(\delta) - \mathcal{R}_4(\delta). \tag{17}$$

Proof: We can rewrite fourth Hankel determinant as (12) and by applying Lemma (1)

$$|\mathcal{R}_1(\delta)| = |a_7|\left(\frac{336}{5}\delta^3 + \frac{7}{5}\delta^2\right),$$

$$|\mathcal{R}_2(\delta)| = |a_6|\left(\frac{-988}{25}\delta^3 + \frac{-54}{25}\delta^2\right),$$

$$|\mathcal{R}_3(\delta)| = |a_5|\left(\frac{-254}{25}\delta^3 + \frac{27}{20}\delta^2\right),$$

$$|\mathcal{R}_4(\delta)| = |a_4|\left(\frac{43}{25}\delta^3\right).$$

By using $|a_7|, |a_6|, |a_5|$ *and* $|a_4|$ from Theorem (1) in fourth Hankel determinant (12), we get:

$$|\mathcal{R}_1(\delta)| \leq \frac{7744}{15}\delta^4 + \frac{484}{45}\delta^3, \tag{18}$$

$$|\mathcal{R}_2(\delta)| \leq \frac{-106704}{625}\delta^4 + \frac{-5832}{625}\delta^3, \tag{19}$$

$$|\mathcal{R}_3(\delta)| \leq \frac{3429}{125}\delta^4 + \frac{729}{200}\delta^3\ , \tag{20}$$

$$|\mathcal{R}_4(\delta)| \leq \frac{86}{25}\delta^4\ . \tag{21}$$

Inserting (18), (19), (20) and (21) in (17), we get:

$$|H_4(1)| \leq \mathcal{R}_1(\delta) - \mathcal{R}_2(\delta) + \mathcal{R}_3(\delta) - \mathcal{R}_4(\delta),$$

where

$$|\mathcal{R}_1(\delta)| \leq \frac{7744}{15}\delta^4 + \frac{484}{45}\delta^3,$$

$$|\mathcal{R}_2(\delta)| \leq \frac{-106704}{625}\delta^4 + \frac{-5832}{625}\delta^3,$$

$$|\mathcal{R}_3(\delta)| \leq \frac{3429}{125}\delta^4 + \frac{729}{200}\delta^3,$$

$$|\mathcal{R}_4(\delta)| \leq \frac{86}{25}\delta^4 .$$

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