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Generalized Hom Γ-Derivation of n- BiHom Γ-Lie algebra

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ABSTRACT

The purpose of this paper, is to introduce a new concepts which are induced n-Bi-Hom Γ-Lie algebra, Γ-Center, $(\theta_1{}^s,\theta_2{}^r)$ Γ-Center, $(\theta_1{}^s,\theta_2{}^r)$ Hom Γ- derivation, $(\theta_1{}^s,\theta_2{}^r)$ Q-Hom Der_Γ, $(\theta_1{}^s,\theta_2{}^r)$ Central Hom Γ- derivation, $(\theta_1{}^s,\theta_2{}^r)$ Hom Γ- Centroid and give the condition to construct induced n- Bi-Hom Γ-Lie algebra, studied Generalized Hom Γ-derivations on direct sum of ideals and we studied the relation between Hom $Der_\lambda\,(g)$, Hom $Cen_\lambda(g)$ and Q Hom $Der_\lambda\,(g)$, Q Hom $Cen_\lambda(g)$, G Hom $Der_\lambda\,(g)$.

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1- INTRODUCTION

Amine, in [1], introduce n- Bi-Hom Lie algebra and custom to studying a *Generalized derivation* on an n-Bi-Hom *Lie algebras*. For several years an algebras of *derivations* and *Generalized derivation* has been topic about the study by many *researchers*. Leger and Luks, in [2], introduced research is more important on the algebras of *Generalized derivation* of *Lie algebras* and those sub algebras, where a writers studied the *structure* and features on an algebras on *Generalized derivation*, Q Cen of limited dimensional *Lie algebras*. The result of Leger and Luks where Generalized by more other researchers on algebras. For instance, Chen and Li, in [3], lesson the *Generalized derivation* of color-*Lie algebras*. Zhou and Fan, in [4,5], cases are considered on Hom Lie Color algebras and n-Hom Lie super algebras. Zhou, Niu and Chen, in [6], investigated *Generalized derivation* on Hom-*Lie algebras*. Kygorodov and Popov, in [7], find they out *Generalized derivation* of *color* n-ary ***Ω***- algebras. For more of a *Generalized derivation* algebras, which is going to be find in [8, 9, 10, 11, 12]. Rezaei and Davvaz, in [13], define Γ- algebra. A. Al-Zaiadi and R. Shaheen, in [14] studied more result on Γ-Lie algebra. The purpose of this paper, is to define n-Bi-Hom Γ-Lie algebra, $(\theta_1{}^s,\theta_2{}^r)$ Hom Γ-derivation and generalized Hom Γ-derivation on n-BiHom Γ-Lie algebra, $(\theta_1{}^s,\theta_2{}^r)$ Q Hom Γ-derivation, $(\theta_1{}^s,\theta_2{}^r)$ Central Hom Γ-derivation and $(\theta_1{}^s,\theta_2{}^r)$ Centroid Hom Γ-derivation on n-Bi-Hom,.We also reached some results, $[Q\ Hom\ Der_\Gamma\ (g)\ ,Q\ Hom\ Cen_\Gamma\ (g)\ ]_\lambda \subseteq$ Q Hom Cen_Γ (g), Studied Generalized derivations on direct sum of ideals.

Now, we will recall the followings concepts which are necessary in this paper.

Definition 1.1:- [1] (n-BiHom Lie-algebra)

An n-Bi-Hom- ***Lie algebra*** be a ***vector space*** g equipped a linear-function [.,…,.] linear-functions and such that

(1) $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$

(2) $[\theta_2(x_1), \dots, \theta_2(x_{n-1}), \theta_1(x_n)] =$

$S_{gn}(\sigma)\ [\theta_2(x_{\sigma(1)}), \dots, \theta_2(x_{\sigma(n-1)}), \theta_1(x_{\sigma(n)})]$

for all $x_1, x_2, x_3 \in g$ and $\sigma \in S_3$

(3) $[\theta_2{}^2(x_1), \dots, \theta_2{}^2(x_{n-1}), [\theta_2(y_1), \dots, \theta_2(y_{n-1}), \theta_1(y_n)]]$

$$= \sum_{k=1}^{n} (-1)^{n-k} \left[\theta_2{}^2(y_1), \dots, \widehat{\theta_2{}^2}(y_k), \dots, \theta_2{}^2(y_n) [\theta_2(x_1), \theta_2(x_{n-1}), \theta_1(y_k)]\right]$$

for all $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in g$ *. If n = 3, then g is called 3- Bi-Hom Lie algebra*

Definition 1.2:- [1]

A subset $S \subseteq g$ is a called sub algebra of $(g, [.,.,.], \theta_1, \theta_2)$ if $\theta_1(S) \subseteq S$ and $\theta_2(S) \subseteq S$ and $[S, S, \cdots, S] \subseteq S$, and *S* is an ideal if $\theta_1(S) \subseteq S$, $\theta_2(S) \subseteq S$ and $[S, S, \cdots, g] \subseteq S$

Definition 1.3:-[1]

The center of $(g, [., \dots, .], \theta_1, \theta_2)$ is the set of $u \in g$ such that

$[u, x_1, x_2, \cdots, x_{n-1}] = 0$. For all $x_1, x_2, \cdots, x_{n-1} \in g$. A center is ideal on g which symbolize by $Z(g)$.

Definition 1.4:- [1]

The (θ_1, θ_2) center of $(g, [., \dots, .], \theta_1, \theta_2)$ is the set $Z_{(\theta_1, \theta_2)}(g) = \{U \in g, [U, \theta_1\theta_2(x_1), \cdots, \theta_1\theta_2(x_{n-1})] = 0\}$.

For any $x_1, x_2, \cdots, x_{n-1} \in g$

Definition 1.5:-[15] Gamma Algebra

Assume Γ is a groupoid and V is a *vector space* on a field F. Therefore, V be named a Γ-algebra on the field F if there exist a functioning $V \times \Gamma \times V \to V$ (an image be symbolize by $x\alpha y$ for all $x, y \in V$ and $\alpha \in \Gamma$) such the following *conditions* hold:

(1) $(x+y)\alpha z = x\alpha z + y\alpha z,\ x\alpha(y+z) = x\alpha y + x\alpha z,$
(2) $x(\alpha+\beta)y = x\alpha y + x\beta y,$
(3) $(cx)\alpha y = c(x\alpha y) = x\alpha(cy),$
(4) $0\alpha y = y\alpha 0 = 0$, for all x, y, z $\in V$, c $\in F$ and $\alpha \in \Gamma$. Furthermore it, a Γ- algebra is named associative if
(5) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

Definition 1.6 :- [14] (Γ-Lie algebra)

Assume V is the associative $\Gamma - algebra$ on a field F.Therefore, for all $\lambda \in \Gamma$ one can create the $\Gamma - Lie\ algebra$ $L_\lambda (V)$. Like a $vector\ space$, $L_\lambda (V)$ be a same V. A Lie Γ- arch of 2-elements on $L_\lambda(V)$ be defined to be them reflector in V, $[x,y]_\lambda = x \cdot_\lambda y - y \cdot_\lambda x$. Note that $[x,y]_\lambda = -[y,x]_\lambda$.

2- Main Results

In this section, we will define n-Bi-Hom Γ-Lie algebra, Hom Γ –derivations, $({\theta_1}^s, {\theta_2}^r)$ Hom Γ – derivations and (θ_1 , θ_2) Γ-center, $({\theta_1}^s , {\theta_2}^r)$ Q-Hom Der_λ, Generalized $({\theta_1}^s , {\theta_2}^r)$ -Hom Der_λ, $({\theta_1}^s , {\theta_2}^r)$ Central Hom Der_λ and $({\theta_1}^s , {\theta_2}^r)$ Hom Γ- Centroid. We will use the notation Hom Γ-derivation (Hom Der_Γ), Quasi Hom Γ-derivation (Q Hom Der_Γ), Generalized Hom Γ-derivation (Gen Hom Der_Γ), Hom Γ- centroid (Hom Cen_Γ), Quasi Hom Γ- centroid (Q Hom Cen_Γ) and Generalized Hom Γ-derivation (Gen Hom Der_Γ).

Definition 2.1.:- (n-Bi-Hom Γ-Iie algebra)

An n-Bi-Hom ***Γ- Lie algebra*** be a vector-space g equipped n-linear function

[., …,.] and 2-linear functions $\boldsymbol{\theta_1}$and $\boldsymbol{\theta_2}$ such

(1)$\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$

$$(2)\ [\theta_2(x_1) ,\dots , \theta_2(x_{n-1}) , \theta_1(x_n)]_\lambda =$$

$$S_{gn}(\sigma)\ [\theta_2(x_{\sigma(1)}) ,\dots , \theta_2(x_{\sigma(n-1)}) , \theta_1(x_{\sigma(n)})]_\lambda$$

for all $x_1 , x_2 , x_3 \in g$ and $\sigma \in S_3$

$$(3)\ [{\theta_2}^2(x_1) ,\dots , {\theta_2}^2(x_{n-1}) , [\theta_2(y_1) ,\dots , \theta_2(y_{n-1}) , \theta_1(y_n)]]_\lambda$$

$$= \sum_{k=1}^{n} (-1)^{n-k} \left[{\theta_2}^2(y_1) ,\dots , \widehat{{\theta_2}^2}(y_k) , \dots ,{\theta_2}^2(y_n) [\theta_2(x_1) , \theta_2(x_{n-1}) , \theta_1(y_k)]\right]_\lambda$$

for all $x_1 , \dots , x_{n-1} , y_1 ,\dots , y_n \in g$.

If n=3, then called (3- Bi-Hom **Γ-**Lie algebra)

Proposition 2.2 :-

Assume $(g , [.,\dots,.]_\lambda)$ is n-Γ- $Lie\ algebra$ and let θ_1 , θ_2 maps on g that commute with every other. For $x_1 , \cdots , x_n \in g$. Define $[x_1 , \cdots , x_n]_{\lambda\ \theta_1\theta_2} = [\theta_1(x_1) , \cdots , \theta_1(x_{n-1}) , \theta_2(x_n)]_\lambda$ Then $(g , [.,\dots,.]_{\lambda\ \theta_1\theta_2} , \theta_1 , \theta_2)$ is an n-Bi-Hom Γ-$Lie\ algebra$, which is called induced n-Bi-Hom Γ-$Lie\ algebra$.

Proof:-

The functions θ_1 , θ_2 commute , by hypothesis, we have a prove θ_1 , θ_2 are algebra morphisms, for every $x_1 , \cdots , x_n \in g$, we have:

$\theta_1([x_1,\dots,x_n]_{\lambda\theta_1\theta_2}) = \theta_1([\theta_1(x_1),\dots,\theta_1(x_{n-1}),\theta_2(x_n))]_\lambda$

$=[{\theta_1}^2(x_1),\dots,{\theta_1}^2(x_{n-1}),\theta_1\circ\theta_2(x_n)]_\lambda=[{\theta_1}^2(x_1),\dots,{\theta_1}^2(x_{n-1}),\theta_2\circ\theta_1(x_n)]_\lambda$

$=[\theta_1(x_1),\dots,\theta_1(x_n)]_{\lambda\theta_1\theta_2}$.

Can a prove that, in a similar way, we can a prove that θ_2 like that a morphism. Also we have

$[\theta_2(x_{\sigma(1)}),\dots,\theta_2(x_{\sigma(n-1)}),\theta_1(x_{\sigma(n)})]_{\lambda\theta_1\theta_2}$
$=[\theta_1\circ\theta_2(x_{\sigma(1)}),\dots,\theta_1\circ\theta_2(x_{\sigma(n-1)}),\theta_2\circ\theta_1(x_{\sigma(n)})]_\lambda$

$=\theta_1\circ\theta_2([(x_{\sigma(1)}),\dots,(x_{\sigma(n)})]_\lambda)=S_{gn}(\sigma)\,\theta_1\circ\theta_2\,([x_1,\dots,x_n])$

$=S_{gn}(\sigma)[\theta_1\circ\theta_2(x_{(1)}),\dots,\theta_1\circ\theta_2(x_{(n-1)}),\theta_2\circ\theta_1(x_{(n)})]_\lambda$

$=S_{gn}(\sigma)\,[\theta_2(x_1),\dots,\theta_2(x_{n-1}),\theta_1(x_n))]_{\lambda\theta_1\theta_2}$ for every $x_1,\cdots,x_{n-1}\in g$, $y_1,\dots,y_n\in g$,we have:

$[{\theta_2}^2(x_1),\dots,{\theta_2}^2(x_{n-1}),[\theta_2(y_1),\dots,\theta_2(y_{n-1}),\theta_1(y_n)]_{\lambda\theta_1\theta_2}]_{\lambda\theta_1\theta_2}$

$=[\theta_1\circ{\theta_2}^2(x_1),\dots,\theta_1\circ{\theta_2}^2(x_{n-1}),\theta_2([\theta_1\circ\theta_2(y_1),\dots,\theta_1\circ\theta_2(y_{n-1}),\theta_2\circ\theta_1(y_n)]_\lambda)]_\lambda$

$=\theta_1\circ{\theta_2}^2([x_1,\dots,x_{n-1},[y_1,\dots,y_n]_\lambda]_\lambda)$

$$=\sum_{k=1}^{n}\theta_1\circ{\theta_2}^2([y_1,\dots,[x_1,\dots,x_{n-1},y_k]_\lambda,\dots y_n]_\lambda)$$

$$=\sum_{k=1}^{n}(-1)^{n-k}\;\theta_1\circ{\theta_2}^2([y_1,\dots,\widehat{y_k},\dots,y_n,[x_1,\dots,x_{n-1},y_k]_\lambda]_\lambda$$

$$=\sum_{k=1}^{n}(-1)^{n-k}\;([\theta_1\circ{\theta_2}^2(y_1),\dots,\theta_1\widehat{\circ{\theta_2}^2}(y_k),\dots,\theta_1\circ{\theta_2}^2(y_n),\theta_1\circ{\theta_2}^2([(x_1),\dots,(x_{n-1}),(y_k)]_\lambda)]_\lambda$$

$$=\sum_{k=1}^{n}(-1)^{n-k}\;([\theta_1\circ{\theta_2}^2(y_1),\dots,\theta_1\widehat{\circ{\theta_2}^2}(y_k),\dots,\theta_1\circ{\theta_2}^2(y_n),\theta_2([\theta_1\circ\theta_2(x_1),\dots,\theta_1\circ\theta_2(x_{n-1}),\theta_1\circ\theta_2(y_k)]_\lambda)]_\lambda$$

$$=\sum_{k=1}^{n}(-1)^{n-k}\;([{\theta_2}^2(y_1),\dots,\widehat{{\theta_2}^2(y_k)},\dots,{\theta_2}^2(y_n),([\theta_2(x_1),\dots,\theta_2(x_{n-1}),\theta_1(y_k)]_{\lambda\theta_1\theta_2})]_{\lambda\theta_1\theta_2}$$

Example 2.3:-

Let g is the 4-dimensional vector-space with the *basis* $[e_1, e_2, e_3, e_4]$. Define the next arch:

$$[e_1, e_2, e_3]_\lambda=-e_4\;;\;[e_1, e_2, e_4]_\lambda=e_3\;;$$
$$[e_1, e_3, e_4]_\lambda=-e_2\;;\;[e_2, e_3, e_4]_\lambda=e_1$$

in this bracket, $(g,\;[.,.,.]_\lambda)$ be the 3- Γ-*Lie algebra*. Assume θ_1 and θ_2 be 2- linear functions of g defined:

$\theta_1(e_1) = -e_2$; $\theta_1(e_2) = -e_1$;

$\theta_1(e_3) = -e_4$; $\theta_1(e_4) = -e_3$ and $\theta_2 = -\theta_1$

Let $[x_1 , x_2 , x_3]_{\lambda\, \theta_1\theta_2} = [\theta_1(x_1) , \theta_1(x_2) , \theta_2(x_3)]_{\lambda}$

be the twisted bracket defined on g. Then $(g , [.,.,.]_{\lambda\, \theta_1\theta_2} , \theta_1 , \theta_2)$ is the 3-Bi-Hom **Γ**-lie algebra

Definition 2.4:-

The Γ-center of $(g , [.,\ldots,.]_{\lambda} , \theta_1 , \theta_2)$ is the set of $u \in g$ such that

$[u, x_1 , x_2 , \cdots , x_{n-1}]_{\lambda} = 0$. For all $x_1 , x_2 , \cdots , x_{n-1} \in g$. The Γ-*center* be an ideal of g which we will symbolize by $Z_{\lambda}(g)$.

Definition 2.5 :-

A (θ_1 , θ_2) Γ - *center* on $(g , [.,\ldots,.]_{\lambda} , \theta_1 , \theta_2)$ is the set $Z_{\lambda(\theta_1 , \theta_2)}(g) = \{u \in g , [u , \theta_1\theta_2(x_1) , \cdots , \theta_1\theta_2(x_{n-1})]_{\lambda} = 0\}$,

for any $x_1 , x_2 , \cdots , x_{n-1} \in g$

Definition 2.6:-

Let $(g , [.,\ldots,.]_{\lambda} , \theta_1 , \theta_2)$ is the n-Bi-Hom Γ-*Lie algebra*. The linear function $D: g \longrightarrow g$ $(\theta_1{}^s , \theta_2{}^r)$ be Hom Der_{Γ} if for every $x , y , z \in g$. There exist $\delta: A \longrightarrow A$ is a Homomorphism,

define $(\theta_1{}^s , \theta_2{}^r)$ Hom Der_{λ} on n-Bi-Hom Γ-*Lie algebra*

$$D[x_1 , \cdots , x_n]_{\lambda} = [D(x_1) , \theta_1{}^s\theta_2{}^r(x_2) , \cdots , \theta_1{}^s\theta_2{}^r(x_n)]_{\lambda}$$
$$+\sum_{i=1}^{n} [\theta_1{}^s\theta_2{}^r(x_1) , \cdots , \theta_1{}^s\theta_2{}^r(x_{i-1}), D(x_i) , \theta_1{}^s\theta_2{}^r(x_{i+1}) , \cdots , \theta_1{}^s\theta_2{}^r(x_n)]_{\lambda}$$
$$+\delta[x_1 , \cdots , x_n]_{\lambda}$$

Let Hom Der_{λ} $(\theta_1{}^s , \theta_2{}^r)$ (g) be the set of $(\theta_1{}^s , \theta_2{}^r)$ -Hom Γ -derivation of g and set Hom $Der_{\lambda}(g) = \bigoplus_{S\geqslant 0} \bigoplus_{r\geqslant 0}$ Hom Der_{λ} $(\theta_1{}^s , \theta_2{}^r)$ (g). We show it Hom $Der_{\lambda}(g)$ be equipped with a Γ-*lie algebra* structure. In effect, for all $D \in$ Hom Der $_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g)$ and $D' \in$ Hom Der_{λ} $(\theta_1{}^s , \theta_2{}^r)(g)$ we have $[D , D']_{\lambda} \in$ Hom $Der_{\lambda}(\theta_1{}^{s+s'} , \theta_2{}^{r+r'})(g)$, where $[D , D']_{\lambda}$ us the standard commutation defined by $[D , D']_{\lambda} = D \circ D' - D' \circ D$.

Note that if $(g\ ,\ [.,\ldots,.]_{\lambda})$ be the n- Γ-*Lie algebra* and $(g\ ,\ [.,\ldots,.]_{\lambda\ \theta_1\theta_2}\ ,\ \theta_1\ ,\ \theta_2)$ the induced n- Bi-Hom Γ-*Lie algebra* where $\theta_1\ ,\ \theta_2$ are 2-morphism used to this induction.

Definition 2.7:-

The endo-morphism D on the n-Bi-Hom Γ-*Lie algebra* g be called $(\theta_1{}^s\ ,\ \theta_2{}^r)$ Q-Hom Der_λ if there exist an endomorphism D' of g such that

$$D\circ\theta_1=\theta_1\circ D\ ;\ D\circ\theta_2=\theta_2\circ D\ ,D'\circ\theta_1=\theta_1\circ D'\ ;\ D'\circ\theta_2=\theta_2\circ D'$$

, There exist $\delta{:}A\longrightarrow A$ is a homomorphism

$$D'[x_1\ ,\ \cdots\ ,\ x_n]_\lambda=$$

$$\sum_{i=1}^{n}[\theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_{i-1}),\ D(x_i)\ ,\ \theta_1{}^s\theta_2{}^r(x_{i+1})\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda$$

$+\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$. For any $x_1\ ,\ \cdots\ ,\ x_n\in g$ .Then we define

$$\text{Q Hom } Der_\lambda(g)=\bigoplus_{s\geqslant 0}\bigoplus_{r\geqslant 0}\text{Q Hom } Der_\lambda(\theta_1{}^s\ ,\ \theta_2{}^r)$$

Definition 2.8:-

Let $(g\ ,\ [.,\ldots,.]_{\lambda}\ ,\ \theta_1\ ,\ \theta_2)$ is the n-Bi-Hom Γ-*Lie algebra* and suppose D is endo morphism on g .A linear function D be named the Gen $(\theta_1{}^s\ ,\ \theta_2{}^r)$ -Hom Der_λ on g if there exists $D^{(i)}, i\in\{1,\ldots,.n\}$ family of endomorphism of g such that $D\circ\theta_1=\theta_1\circ D\ ;\ D\circ\theta_2=\theta_2\circ D$

$$D^{(i)}\circ\theta_1=\theta_1\circ D^{(i)}\ ;\ D^{(i)}\circ\theta_2=\theta_2\circ D^{(i)}$$

For any, where $\delta{:}A\longrightarrow A$ is a Homomorphism and

$$D^{(n)}[x_1\ ,\ \cdots\ ,\ x_n]_\lambda=[D(x_1)\ ,\ \theta_1{}^s\theta_2{}^r(x_2)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda$$

$$+\sum_{i=2}^{n}[\theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_{i-1}),\ D^{(i-1)}(x_i)\ ,\ \theta_1{}^s\theta_2{}^r(x_{i+1})\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda$$

$$+\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda\ ,\ for\ all\ x_1,\ldots,x_n\ \epsilon\ g$$

The set of generalized $(\theta_1{}^s\ ,\ \theta_2{}^r)$ - Hom Der_λ of g is Gen Hom Der_λ $(\theta_1{}^s\ ,\ \theta_2{}^r)(g)$ and as for Gen Hom Der_λ (g), we denote

$$\text{Gen Hom } Der_\lambda(g)=\bigoplus_{S\geqslant 0}\bigoplus_{r\geqslant 0}\text{Gen Hom } Der_\lambda(\theta_1{}^s\ ,\ \theta_2{}^r)(\text{g})$$

Proposition 2.9:-

Let $(g\ ,\ [.,.,.]_{\lambda}\ ,\ \theta_1\ ,\ \theta_2)$ is the regular n-Bi-Hom Γ - *Lie algebra* in trivial Hom Γ - *Center* . Assume $g = I \oplus J$; such *I* and *J* are ideals on *g*, then

Gen Hom $Der_{\lambda}(g)$ = Gen Hom $Der_{\lambda}(I) \oplus$ Gen Hom $Der_{\lambda}(J)$, such that there exist $\delta: A \longrightarrow A$ is an isomorphism.

Proof:-

We will prove this for any $D \in$ Gen Hom $Der_{\lambda}(g)$, we have $D(I) \subset I$ and $D(J) \subset J$, therefore it follows a restriction of *D* to *I* (resp. *J*) be the Gen Hom Der_{λ} of *I* (resp. *J*).

Assume $u \in I$ and suppose $D(u) = a + b\ ,\ a \in I\ ,\ b \in J$ be the decomposition of $D(u)$. For any $y_1\ ,\ \cdots\ ,\ y_{n-1} \in g$, we have $[b\ ,\ y_1\ ,\ \cdots\ ,\ y_{n-1}]_{\lambda} \in J$. On the other hand,

$$[b\ ,\ y_1\ ,\ \cdots\ ,\ y_{n-1}]_{\lambda} = [D(u) - a\ ,\ y_1\ ,\ \cdots\ ,\ y_{n-1}]_{\lambda}$$
$$= [D(u)\ ,\ y_1\ ,\ \cdots\ ,\ y_{n-1}]_{\lambda} - [a\ ,\ y_1\ ,\ \cdots\ ,\ y_{n-1}]_{\lambda}.$$

Since *I* is an ideal and $a \in I$, so $[a\ ,\ y_1\ ,\ \cdots\ ,\ y_{n-1}]_{\lambda} \in I$. Moreover, for each $1 \leqslant i \leqslant n-1$, let $y_i = \theta_1{}^s \theta_2{}^r (x_i)$, then

$$[D(u)\ ,\ y_1\ ,\ \cdots\ ,\ y_{n-1}]_{\lambda} = [D(u)\ ,\ \theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_{n-1})]_{\lambda}$$
$$= D^{(n)}[u\ ,x_1\ ,\ \cdots\ ,\ x_{n-1}]_{\lambda} -$$
$$\sum_{i=1}^{n-1} [\theta_1{}^s\theta_2{}^r(u)\ ,\ \theta_1{}^s\theta_2{}^r(x_1), \ldots\ ,\ D^{(i)}(x_i)\ ,\ \theta_1{}^s\theta_2{}^r(x_{i+1})\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_{n-1})]_{\lambda}$$

$+\delta[u\ ,x_1\ ,\ \cdots\ ,\ x_n]$. For every *I* , where, $\delta: A \longrightarrow A$ is an isomorphism

$$[\theta_1{}^s\theta_2{}^r(u)\ ,\ \theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ D^{(i)}(x_i)\ ,\ \theta_1{}^s\theta_2{}^r(x_{i+1})\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_{n-1})]_{\lambda} \in I,$$

so

$$\sum_{i=1}^{n-1} [\theta_1{}^s\theta_2{}^r(u)\ ,\ \theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ D^{(i)}(x_i)\ ,\ \theta_1{}^s\theta_2{}^r(x_{i+1})\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_{n-1})]_{\lambda} \in I$$

Now let $x_i = a_i + b_i$ be the decomposition of x_i , i=1,..., n-1

$$[u\ ,\ x_1\ ,\ \cdots\ ,\ x_{n-1}]_{\lambda} = [u\ ,\ a_1 + b_1\ ,\ \cdots\ ,\ a_{n-1} + b_{n-1}]_{\lambda}$$
$$= [u\ ,\ a_1 + b_1\ ,\ \cdots\ ,\ a_{n-1}]_{\lambda} + [u\ ,\ a_1 + b_1\ ,\ \cdots\ ,\ b_{n-1}]_{\lambda}\ ,$$

but $[u\ ,\ a_1 + b_1\ ,\ \cdots\ ,\ b_{n-1}]_{\lambda} \in I \cap J = \{0\}$, so

$$[u\ ,\ x_1\ ,\ \cdots\ ,\ x_{n-1}]_{\lambda} = [u\ ,\ a_1 + b_1\ ,\ \cdots\ ,\ a_{n-2} + b_{n-2}\ ,\ a_{n-1}].$$

Similarly, $[u\ ,\ a_1+b_1\ ,\ldots\ ,\ b_{n-2}\ ,\ a_{n-1}]_{\lambda}=0$

Thus $[u\ ,\ x_1\ ,\ \cdots\ ,\ x_{n-1}]_{\lambda}=[u\ ,\ a_1\ ,\ \cdots\ ,\ a_{n-2}\ ,\ a_{n-1}]_{\lambda}$.

Therefore, $D^{(n)}[u\ ,\ x_1\ ,\ \cdots\ ,\ x_{n-1}]_{\lambda}=D^{n}[u\ ,\ a_1\ ,\ \cdots\ ,\ a_{n-1}]_{\lambda}$

$$=[D(u)\ ,\ \theta_1^{\ s}\theta_2^{\ r}(a_1)\ ,\ \cdots\ ,\ \theta_1^{\ s}\theta_2^{\ r}(a_{n-1})]_{\lambda}$$

$$+\sum_{i=1}^{n-1}[\theta_1^{\ s}\theta_2^{\ r}(u)\ ,\ \theta_1^{\ s}\theta_2^{\ r}(a_1)\ ,\ \cdots\ ,\ D^{(i)}(a_i)\ ,\ \theta_1^{\ s}\theta_2^{\ r}(a_{i+1})\ ,\ \cdots\ ,\ \theta_1^{\ s}\theta_2^{\ r}(a_{n-1})]_{\lambda}\in I$$

Then $[D(u)\ ,\ y_1\ ,\ \cdots\ ,\ y_{n-1}]_{\lambda}\in I$ and so is $[b\ ,\ y_1\ ,\ \cdots\ ,\ y_{n-1}]_{\lambda}$.

Hence $[b\ ,\ y_1\ ,\ \cdots\ ,\ y_{n-1}]_{\lambda}\in I\cap J$. We can conclude that $b\in Z_{\lambda}(g)=\{0\}$ and so $D(I)\subseteq I$

Remark 2.10:-

Since any Hom Der_{λ} and quasi Hom Der_{λ} is a Generalized Hom Der_{λ}.

Hom $Der_{\lambda}(g)\subseteq Q$ Hom $Der_{\lambda}(g)\subseteq Gen$ Hom $Der_{\lambda}(g)$.

Hence proposition **2.9** holds of Q Hom $Der_{\lambda}(g)$ and Hom $Der_{\lambda}(g)$ as well, that Hom $Der_{\lambda}(g)$ = Hom $Der_{\lambda}(I)$ $\oplus$ Hom $Der_{\lambda}(J)$ and

Q Hom $Der_{\lambda}(g)$ = Q Hom $Der_{\lambda}(I)$ $\oplus$ Q Hom $Der_{\lambda}(J)$

Definition 2.11:-

The linear function D be called $(\theta_1^{\ s}\ ,\ \theta_2^{\ r})$ *central*-Hom Der_{λ} on g if it satisfies

$$D([x_1\ ,\ \cdots\ ,\ x_n]_{\lambda})=[\theta_1^{\ s}\theta_2^{\ r}(x_1)\ ,\ \cdots\ ,\ \theta_1^{\ s}\theta_2^{\ r}(x_{i-1})\ ,\ D(x_i)\ ,\ \theta_1^{\ s}\theta_2^{\ r}(x_{i+1})\ ,\ \cdots\ ,\ \theta_1^{\ s}\theta_2^{\ r}(x_n)]_{\lambda}+\delta[x_1\ ,\ \cdots\ ,\ x_n]_{\lambda}=0,$$

for all $i\in\{1\ ,\ \cdots\ ,\ n\}$

The set of $(\theta_1^{\ s}\ ,\ \theta_2^{\ r})$ central Der_{λ} is denoted by Z Hom $Der_{\lambda}(\theta_1^{\ s}\ ,\ \theta_2^{\ r})$

and we set

$$Z\ \text{Hom}\ Der_{\lambda}(g)=\bigoplus_{s\geqslant 0}\ \bigoplus_{r\geqslant 0} Z\ \text{Hom}\ Der_{\lambda}(\theta_1^{\ s}\ ,\ \theta_2^{\ r})(g).$$

Definition 2.12:-

The $(\theta_1^{\ s}\ ,\ \theta_2^{\ r})$ Hom Γ - *Centroid* of $(g\ ,\ [.,.,.]_{\lambda}\ ,\ \theta_1\ ,\ \theta_2)$ denoted by Hom Cen_{Γ} $(\theta_1^{\ s}\ ,\ \theta_2^{\ r})(g)$ be a set of linear functions D satisfying

$$D([x_1\ ,\ \cdots\ ,\ x_n]_{\lambda})=[\theta_1^{\ s}\theta_2^{\ r}(x_1)\ ,\ \cdots\ ,\ \theta_1^{\ s}\theta_2^{\ r}(x_{i-1})\ ,\ D(x_i)\ ,\ \theta_1^{\ s}\theta_2^{\ r}(x_{i+1})\ ,\ \cdots\ ,\ \theta_1^{\ s}\theta_2^{\ r}(x_n)]_{\lambda}+\delta[x_1\ ,\ \cdots\ ,\ x_n]_{\lambda},$$

there exist $\delta:A\longrightarrow A$ is a Homomorphism for all $i\in\{1,\ldots,n\}$. We set

$$\text{Hom } Cen_{\Gamma}(g) = \underset{s\geqslant 0}{\oplus}\ \underset{r\geqslant 0}{\oplus}\ \text{Hom } Cen_{\Gamma}(\theta_1{}^s , \theta_2{}^r)(g).$$

Proposition 2.13:-

For any s , $r.$ we have

$$Z\text{ Hom } Der_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g) = \text{Hom } Der_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g) \cap \text{Hom} Cen_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g).$$

Proof:-

It is clear that $Z\text{ Hom } Der_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g) \subseteq \text{Hom } Der_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g)$ and

$Z\text{ Hom } Der_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g) \subseteq \text{Hom } Cen_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g)$

Conversely,

Let $D \in \text{Hom } Der_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g) \cap \text{Hom } Cen_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g)$,

so for each I, there exist , $\delta{:}A \longrightarrow A$ is a homomorphism we have

$$D([x_1 , \cdots , x_n]_{\lambda}) = [\theta_1{}^s\theta_2{}^r(x_1) , \cdots , \theta_1{}^s\theta_2{}^r(x_{i-1}) , D(x_i) , \theta_1{}^s\theta_2{}^r(x_{i+1}) , \cdots , \theta_1{}^s\theta_2{}^r(x_n)]_{\lambda} + \delta[x_1 , \cdots , x_n]_{\lambda}$$

In addition,

$$D([x_1 , \cdots , x_n]_{\lambda}) = \sum_{i=1}^{n}[\theta_1{}^s\theta_2{}^r(x_1) , \cdots , \theta_1{}^s\theta_2{}^r(x_{i-1}) , D(x_i) , \theta_1{}^s\theta_2{}^r(x_{i+1}) , \cdots , \theta_1{}^s\theta_2{}^r(x_n)]_{\lambda} + \delta[x_1 , \cdots , x_n]_{\lambda}$$

Then $D([x_1 , \cdots , x_n]_{\lambda}) = nD([x_1 , \cdots , x_n]_{\lambda})$.

Thus $D([x_1 , \cdots , x_n]_{\lambda}) = 0$ and $D \in Z\text{ Hom } Der_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g)$.

Definition 2.14:-

$Q\text{ Hom } Cen_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g)$ be a set of linear functions D such

$$[D(x_1) , \theta_1{}^s\theta_2{}^r(x_2) , \cdots , \theta_1{}^s\theta_2{}^r(x_n)]_{\lambda} = [\theta_1{}^s\theta_2{}^r(x_1) , \cdots , \theta_1{}^s\theta_2{}^r(x_{i-1}) , D(x_i) , \theta_1{}^s\theta_2{}^r(x_{i+1}) , \cdots , \theta_1{}^s\theta_2{}^r(x_n)]_{\lambda} + \delta[x_1 , \cdots , x_n]_{\lambda}$$

For all $i \in \{1 , \cdots , n\}$, there exist , $\delta{:}A \longrightarrow A$ is a Homomorphism. We set

$$Q\text{ Hom } Cen_{\lambda}(g) = \underset{s\geqslant 0}{\oplus}\ \underset{r\geqslant 0}{\oplus}\ Q\text{ Hom } Cen_{\lambda}(\theta_1{}^s , \theta_2{}^r)(g)$$

Lemma 2.15:-

Let $(g , [.,.,.]_{\lambda} , \theta_1 , \theta_2)$ is n-Bi-Hom Γ-*Lie algebra*.

(1) $[\text{Hom Der}_{\lambda}(\theta_1{}^s,\theta_2{}^r)(g) , \text{Hom } Cen_{\lambda}(\theta_1{}^s,\theta_2{}^r)(g)]_{\lambda} \subseteq Hom\ Cen_{\lambda}(\theta_1{}^s,\theta_2{}^r)(g);$

(2) $\text{Hom } Cen_{\lambda}(\theta_1{}^s,\theta_2{}^r)(g) \oplus \text{Hom Der}_{\lambda}(\theta_1{}^s,\theta_2{}^r)(g) \subseteq \text{Hom Der}_{\lambda}(\theta_1{}^s,\theta_2{}^r)(g).$

Proof:-

Let $D \in \text{Hom Der}_\lambda(\theta_1^{s}, \theta_2^{r})(g)$ and $D' \in \text{Hom } Cen_\lambda(\theta_1^{s'}, \theta_2^{r'})(g)$ for some s, s', r, r'. Let $x_1, \cdots, x_n \in g$.

There exist $\delta: A \longrightarrow A$ is a homomorphism

(1) Compute

$$[DD'(x_1), \theta_1^{s+s'}\theta_2^{r+r'}(x_2), \cdots, \theta_1^{s+s'}\theta_2^{r+r'}(x_n)]_\lambda$$

$$= D([D'(x_1), \theta_1^{s'}\theta_2^{r'}(x_2), \cdots, \theta_1^{s'}\theta_2^{r'}(x_n)]_\lambda$$

$$-\sum_{i=2}^{n}[\theta_1^{s}\theta_2^{r}D'(x_1), \cdots, D(x_i), \ldots, \theta_1^{s}\theta_2^{r}(x_n)]_\lambda \qquad -\delta[x_1, \cdots, x_n]_\lambda$$

$$= DD'([x_1, \cdots, x_n]_\lambda) - \sum_{i=2}^{n}[\theta_1^{s}\theta_2^{r}(x_1), \cdots, D'D(x_i), \ldots, \theta_1^{s}\theta_2^{r}(x_n)]_\lambda$$

$$-\delta[x_1, \cdots, x_n]_\lambda.$$

On the other hand,

$$[D'D(x_1), \theta_1^{s+s'}\theta_2^{r+r'}(x_2), \cdots, \theta_1^{s+s'}\theta_2^{r+r'}(x_n)]_\lambda$$

$$= D'([D(x_1), \theta_1^{s}\theta_2^{r}(x_2), \cdots, \theta_1^{s}\theta_2^{r}(x_n)]_\lambda) - \delta[x_1, \cdots, x_n]_\lambda$$

$$= DD'([x_1, \cdots, x_n]_\lambda)$$

$$-D'\left(\sum_{i=2}^{n}[\theta_1^{s}\theta_2^{r}(x_1), \cdots D(x_i), \cdots, \theta_1^{s}\theta_2^{r}(x_n)]_\lambda - \delta[x_1, \cdots, x_n]_\lambda,\right.$$

but since for each i,

$$D'([\theta_1^{s}\theta_2^{r}(x_1), \cdots, D(x_i), \cdots, \theta_1^{s}\theta_2^{r}(x_n)]_\lambda) - \delta[x_1, \cdots, x_n]_\lambda$$

$$= [\theta_1^{s}\theta_2^{r}(x_1), \cdots, D'D(x_i), \cdots, \theta_1^{s}\theta_2^{r}(x_n)]_\lambda - \delta[x_1, \cdots, x_n]_\lambda,$$

so

$$D'\left(\sum_{i=2}^{n}[\theta_1^{s}\theta_2^{r}(x_1), \cdots, D(x_i), \cdots, \theta_1^{s}\theta_2^{r}(x_n)]_\lambda - \delta[x_1, \cdots, x_n]_\lambda\right.$$

$$= \sum_{i=2}^{n}[\theta_1^{s}\theta_2^{r}(x_1), \cdots, D'D(x_i), \cdots, \theta_1^{s}\theta_2^{r}(x_n)]_\lambda - \delta[x_1, \cdots, x_n]_\lambda$$

Hence

$$[[D, D']_\lambda(x_1), \theta_1^{s+s'}\theta_2^{r+r'}(x_2), \cdots, \theta_1^{s+s'}\theta_2^{r+r'}(x_n)]_\lambda$$

$$= [D, D']_\lambda([x_1, \cdots, x_n]_\lambda)$$

The same proof holds for any $i \in \{1, \cdots, n\}$

Thus, $[D, D']_\lambda \in Hom\ C_\lambda(\theta_1^{s+s'}, \theta_2^{r+r'})(g)$

(2) Now

$$D'D([x_1, \cdots, x_n]_\lambda) = D'([D(x_1), \theta_1^{s}\theta_2^{r}(x_2), \cdots, \theta_1^{s}\theta_2^{r}(x_n)]_\lambda)$$

$$+D'\left(\sum_{i=2}^{n}[\theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ D(x_i)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda + \delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda\right.$$

$$=[D'D(x_1),\theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_2),\ \ldots\ ,\theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda$$

$$+\sum_{i=2}^{n}[\theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_1)\ ,\ \cdots\ ,\ D'D(x_i)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda$$

$$+[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

Thus $D'D \in Hom\,\mathrm{Der}_\lambda(\theta_1{}^{s+s'}\ ,\ \theta_2{}^{r+r'})\ (g)$

Theorem 2.16:-

Let $(g\ ,\ [.,.,.]_\lambda\ ,\ \theta_1\ ,\ \theta_2)$ is the multiplicative n-Bi-Hom Γ- *Lie algebra*.

(1) $[Q\ \mathrm{Hom\ Der}_\lambda(g)\ ,\ Q\ \mathrm{Hom}\ Cen_\lambda(g)]_\lambda \subseteq Q\ \mathrm{Hom}\ Cen_\lambda(g)$

(2) $\mathrm{Hom}\ Cen_\lambda(g) \subseteq Q\ \mathrm{Hom\ Der}_\lambda(g)$;

(3) $[Q\ \mathrm{Hom}\ Cen_\lambda(g)\ ,\ Q\ \mathrm{Hom}\ Cen_\lambda(g)]_\lambda \subseteq Q\ \mathrm{Hom\ Der}_\lambda(g)$

(4) $Q\ \mathrm{Hom\ Der}_\lambda\ (g) + Q\ \mathrm{Hom}\ Cen_\lambda(g) \subseteq Gen\ \mathrm{Hom\ Der}_\lambda(g)$

Proof:-

Let $D \in Q\ \mathrm{Hom\ Der}_\lambda\ (\theta_1{}^s\ ,\ \theta_2{}^r)\ (g)$ and $D' \in Q\ \mathrm{Hom}\ Cen_\lambda\ (\theta_1{}^{s'}\ ,\ \theta_2{}^{r'})\ (g)$ for some $s\ ,\ s'\ ,\ r\ ,\ r'$. Let $\delta: A \longrightarrow A$ is a homomorphism.

(1) Compute

$$[DD'(x_1)\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_2)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda$$

$$=D([D'\ (x_1),\theta_1{}^{s'}\theta_2{}^{r'}(x_2),\ \ldots\ ,\theta_1{}^{s'}\theta_2{}^{r'}(x_n)]_\lambda) -$$

$$\sum_{i=2}^{n}[\theta_1{}^s\theta_2{}^r D'(x_1)\ ,\ \cdots\ ,\ D(x_i)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda - \delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

$$=DD'([x_1,\ldots,x_n]_\lambda) - \sum_{i=2}^{n}[\theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ \theta_1{}^{s'}\theta_2{}^{r'}(x_{i-1})\ ,\ D'D(x_i)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda$$

$$-\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda.$$

On the other hand,

$$[D'D(x_1)\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_2)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda$$

$$=D'([D(x_1)\ ,\ \theta_1{}^s\theta_2{}^r(x_2)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda) - \delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

$$=DD'([x_1,\ldots,x_n]_\lambda) - D'\left(\sum_{i=2}^{n}[\theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ D(x_i)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda\right) - \delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda,$$

but since for each i, $D'([\theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ D(x_i)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)\ ]_\lambda)$

$$=[\theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ D'D(x_i)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda + \delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

so

$$D'\left(\sum_{i=2}^{n}[\theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ D(x_i)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda\right)+\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

$$=\sum_{i=2}^{n}[\theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ D'D(x_i)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda+\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

Hence

$$[[D,D']_\lambda(x_1)\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_2)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda-\delta[x_1,\dots,x_n]_\lambda$$

$$=[D,D']_\lambda([x_1\ ,\ \cdots\ ,\ x_n]_\lambda),\ i\in\{1,\cdots,n\}$$

Thus $[D,D']_\lambda\in Q\ \text{Hom}\ C_\lambda(\theta_1{}^{s+s'}\ ,\ \theta_2{}^{r+r'})\ (g)$

(2) It be the immediate consequence on a definition on the Q-Hom Der_Γ. If $D\in\text{Hom}\ Cen_\lambda(\theta_1{}^s\ ,\ \theta_2{}^r)$, then

$$\sum_{i=1}^{n}[\theta_1{}^s\theta_2{}^r(x_1)\ ,\ \cdots\ ,\ D(x_i)\ ,\ \cdots\ ,\ \theta_1{}^s\theta_2{}^r(x_n)]_\lambda+n\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

$$=nD([x_1\ ,\ \cdots\ ,\ x_n]_\lambda)+\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

(3)

Let $D\in Q\ \text{Hom}\ Cen_\lambda(\theta_1{}^s\ ,\ \theta_2{}^r)\ (g)$ and $D'\in Q\ \text{Hom}\ Cen_\lambda(\theta_1^{s'}\ ,\ \theta_2^{r'})\ (g)$

For any $x_1\ ,\ \cdots\ ,\ x_n\in g$ we have

$$[DD'(x_1)\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_1)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda$$

$$=[\theta_1{}^s\theta_2{}^rD'(x_1)\ ,\ D\theta_1{}^{s'}\theta_2{}^{r'}(x_2)\ ,\ \cdots\ ,\ \theta_2{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda+\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

$$=[\theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_1)\ ,\ D\theta_1{}^{s'}\theta_2{}^{r'}(x_2)\ ,\ D'\theta_1{}^s\theta_2{}^r(x_3)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda$$

$$+2\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

$$=[D\theta_1{}^{s'}\theta_2{}^{r'}(x_1)\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_2)\ ,\ D'\theta_1{}^s\theta_2{}^r(x_3)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda$$

$$+3\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

$$=[D'D(x_1)\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_2)\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_3)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda$$

$$+n\delta[x_1\ ,\ \cdots\ ,\ x_n]_\lambda$$

Then $[[D,D']_\lambda(x_1)\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_2)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda=0.$

In the same way we have

$$[\theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_1)\ ,\ \cdots\ ,\ [D,D'](x_i)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda=0.$$

For all *i*. Hence

$$\sum_{i=1}^{n}[\theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_1)\ ,\ \cdots\ ,\ [D,D'](x_i)\ ,\ \cdots\ ,\ \theta_1{}^{s+s'}\theta_2{}^{r+r'}(x_n)]_\lambda=0\cdot$$

And so $[D,D']_\lambda\in Q\ \text{Hom Der}_\lambda(\theta_1{}^{s+s'}\ ,\ \theta_2{}^{r+r'})\ (g)$

(4)

By Remark 2.10 we have Q Hom $\boldsymbol{Der_\lambda}(g)\subseteq Gen$ Hom $\boldsymbol{Der_\lambda}(g)$, **by definition** Q Hom $\boldsymbol{Cen_\lambda}(g)\subseteq$ Hom $\boldsymbol{Cen_\lambda}(g)$ and by above **(2)** we have

Hom $\boldsymbol{Cen}_\lambda(g) \subseteq$ **Q Hom** $\boldsymbol{Der}_\lambda(g)$, then Q Hom $\boldsymbol{Cen}_\lambda$(g) ⊆ **Q Hom** $\boldsymbol{Der}_\lambda(g) \subseteq Gen$ **Hom** $\boldsymbol{Der}_\lambda(g)$, thus Q Hom $\boldsymbol{Cen}_\lambda$(g) ⊆ Gen Hom $\boldsymbol{Der}_\lambda(g)$.

Hence $Q\ \text{Hom Der}_\lambda\ (g) + Q\ \text{Hom}\ Cen_\lambda(g) \subseteq Gen\ \text{Hom Der}_\lambda(g)$

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