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Decomposition Matrix for the spin Characters, S_{25} if the field characteristic is $p = 7$

Ahmed Hussein Jassim[a]*, Saeed Abdul-Ameer Taban[b]

[a]*College of Science, University of Basrah, Basrah, Iraq.Email: ahmed_h.jassim@uobasrah.edu.iq*

[b]*College of Science, University of Basrah, Basrah, Iraq.Email:saeed.taban@uobasrah.edu.iq*

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A B S T R A C T

In this work, we compute decomposition matrix for the spin characters S_{25}, connected between irreducible spin characters and irreducible modular spin characters, if the field characteristic is 7. The method used in this work is $(r, \bar{r})$-inducing in a way to generate projective character for S_{25} by projective character of S_{24} and used maple program to see all the possible of columns and then choose the possible the right columns of them. The aim of this research is to pave the way for finding general relationships and theorems to study irreducible modular spin characters.

MSC..

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∗Corresponding author

Email addresses: *ahmed_h.jassim@uobasrah.edu.iq*

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1. Introduction

Symmetric group S_n has a representation group $\bar{S}_n$ with a central $Z = \{-1,1\}$ such that $\bar{S}_n/Z \cong S_n$. The representations which do not have Z in their kernel are called the spin representations of S_n for more information, see[1]. The spin characters of S_n are labeled by the distinct parts of the partitions of n. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is partition of n and $n - m$ is even, then there is one irreducible spin character denoted by $\langle\alpha\rangle^*$ which is self-associate(double), and if $n - m$ is odd, then there are two associate spin characters denoted by $\langle\alpha\rangle$ and $\langle\alpha\rangle'$ see[2]. The number of rows and columns in the decomposition matrix for the spin characters represent the quantity of projective characters and (p,α)-regular classes, respectively.In this work, we identified the spin character decomposition matrix for S_{25} modulo $p = 7$. The $(r, \bar{r})$-inducing (restricting) method is used to distribute the spin characters into p -blocks[3][4]. Several people have contributed to this field of study and perform research on this topic. [5, 6, 7, 8]. Before we declare any results, let's define certain notations and terminologies. "p.s." is the principal spin character ("p.i.s." indecomposable), "m.s." is means modular spin character ("i.m.s." irreducible), "d_i" is p.i.s. of S_{25}, "D_i" is p.i.s. of S_{24}, and "$\langle\quad\rangle^{no}$" is the number of i.m.s.

2. Preliminaries

Several significant findings were required for the investigation.

Theorem 2.1.[9] Any modular (ordinary) character of group G can be written as a linear combination, with non-negative integer coefficients, of the irreducible modular (ordinary) characters of G .

Theorem 2.2.[1] The degree of the spin character $\langle\alpha\rangle = \langle\alpha_1, \dots, \alpha_m\rangle$ is $\langle\alpha\rangle(1) = 2^{[(n-m)/2]}(n!/\prod_{i=1}^{m}(\alpha_i!))\left(\prod_{1\le i<j\le m}(\alpha_i - \alpha_j)/(\alpha_i + \alpha_j)\right)$.

Theorem 2.3.[10] Given that b is the number of p-conjugate characters to the irreducible ordinary character χ of G and that B is an ablock of defect one, then:

1- There exists a positive integer number N such that the irreducible ordinary characters lying in the block B can be partitioned into two disjoint classes: $B_1 = \{\chi \in B \ |b \deg x \equiv N mod p^a\}$, $B_2 = \{\chi \in B \ |b \deg x \equiv -N mod p^a\}$

2- The block B's decomposition matrix has coefficients that are either 1 or 0.

Theorem 2.4.[11] Let G be a group of order $|G| = m_o p^a$, where $(p, m_o) = 0$. If c is a principal character of sub group H of G, then $\deg c \equiv 0 \ mod p^a$.

3. Matrix of Decomposition for the Spin Characters

Decomposition matrix for $\bar{S}_{25}$ of degree **(213,141)** it is decomposed in to blocks it consists of **12** blocks which B_1, B_2, are defect three, B_3, B_4, B_5 , B_6 are defect two, B_7 , B_8, B_9 , B_{10} are defect one, and B_{11} , B_{12} are defects zero.

Lemma 3.1. Decomposition matrix for the block B_1 of type double as shown in the Tables 1.

Table 1- Block B_1

Spin characters	Decomposition matrix																					
$\langle 25\rangle^*$	1																					
$\langle 21,4\rangle$	1	1																				
$\langle 20,4,1\rangle^*$		1	1																			
$\langle 19,4,2\rangle^*$			1	1																		
$\langle 18,7\rangle$	1	1			1																	
$\langle 18,6,1\rangle^*$	2	1	1		1	1																
$\langle 18,5,2\rangle^*$			1	1		1	1															
$\langle 18,4,3\rangle^*$				1			1															
$\langle 16,5,4\rangle^*$						1	1	1														
$\langle 15,6,4\rangle^*$	2				1	1		1	1													
$\langle 14,11\rangle$	1				1					1												
$\langle 14,7,4\rangle^*$	4	2			2				2	2	2											
$\langle 14,6,4,1\rangle$								1	1		1	1										
$\langle 14,5,4,2\rangle$								1				1										

$\langle 13,11,1\rangle^*$	2				1	1				1			1									
$\langle 13,8,4\rangle^*$	4	2	1		1	1		1	1	2	2		1	1								
$\langle 13,7,4,1\rangle$	2	1						1	1	1	2	1		1	1							
$\langle 13,5,4,2,1\rangle^*$												1			1							
$\langle 12,11,2\rangle^*$	2					1	1						1			1						
$\langle 12,9,4\rangle^*$	2		1	1		1	1	1					1	1		1	1					
$\langle 12,7,4,2\rangle$								1			1	1		1	1		1	1				
$\langle 12,6,4,2,1\rangle^*$											1	1			1			1				
$\langle 11,10,4\rangle^*$				1			1									1	1					
$\langle 11,9,5\rangle^*$			1	1										1			1		1			
$\langle 11,8,6\rangle^*$	2	2	1							2	2			1					1	2		
$\langle 11,7,6,1\rangle$	2	1								1	1		1	1	1				1	1	1	
$\langle 11,7,5,2\rangle$	2									2	1		1	1	1	1	1	1	1	1	1	1
$\langle 11,7,4,3\rangle$																1	1	1				1
$\langle 11,6,5,2,1\rangle^*$										2	1				1			1		1		1
$\langle 11,6,4,3,1\rangle^*$																		1			1	1
$\langle 11,5,4,3,2\rangle^*$															1						1	
$\langle 9,7,5,4\rangle$	2									2			1			1			1	2	1	1
$\langle 9,6,5,4,1\rangle^*$										2									2	2	1	1
$\langle 8,7,6,4\rangle$													1						1		1	
$\langle 8,6,5,4,2\rangle^*$																			2	2	1	
$\langle 7,6,5,4,2,1\rangle$																				1		
	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}	d_{21}	d_{22}

Proof: By using (4,4)-inducing of p.i.s. method on D_{23} in S_{25} we have

$$\begin{aligned} D_1\uparrow^{(4,4)} S_{25} &= \langle 24\rangle + \langle 21,3\rangle^* + \langle 17,7\rangle^* + \langle 17,6,1\rangle + \langle 17,6,1\rangle' + \langle 15,6,3\rangle + \langle 15,6,3\rangle' + \langle 14,10\rangle^* + \\ &\quad 2\langle 14,7,3\rangle + 2\langle 14,7,3\rangle' + \langle 13,10,1\rangle + \langle 13,10,1\rangle' + 2\langle 13,8,3\rangle + 2\langle 13,8,3\rangle' + 2\langle 13,7,3,1\rangle^* + \\ &\quad \langle 12,10,2\rangle + \langle 12,10,2\rangle' + \langle 12,9,3\rangle + \langle 12,9,3\rangle' + \langle 10,8,6\rangle + \langle 10,8,6\rangle' + 2\langle 10,7,6,1\rangle^* + \\ &\quad 2\langle 10,7,5,2\rangle^* + 2\langle 9,7,5,3\rangle^* + \uparrow^{(4,4)} S_{25} \\ &= \langle 25\rangle^* + \langle 21,4\rangle + \langle 21,4\rangle' + \langle 18,7\rangle + \langle 18,7\rangle' + 2\langle 18,6,1\rangle^* + 2\langle 15,6,4\rangle^* + \langle 14,11\rangle + \langle 14,11\rangle' \\ &\quad + 4\langle 14,7,4\rangle^* + 2\langle 13,11,1\rangle^* + 4\langle 13,8,4\rangle^* + 2\langle 13,7,4,1\rangle + 2\langle 13,7,4,1\rangle' + 2\langle 12,11,2\rangle^* + \\ &\quad 2\langle 12,9,4\rangle^* + 2\langle 11,8,6\rangle^* + 2\langle 11,7,6,1\rangle + 2\langle 11,7,6,1\rangle' + 2\langle 11,7,5,2\rangle + 2\langle 11,7,5,2\rangle' + \\ &\quad 2\langle 9,7,5,4\rangle + 2\langle 9,7,5,4\rangle' \\ &= d_1 \end{aligned}$$

similarly, using $(\mathrm{r},\bar{r})$-inducing of p.i.s. $D_3, D_5, D_7, \ldots, D_{43}$ of S_{24} to S_{25} we get on $d_2, d_3, d_4, \ldots, d_{22}$, respectively.

Case 1: $d_5 \not\subset d_1$. To prove this, suppose the opposite.

$(d_1 - d_5)\downarrow_{(1,0)} S_{24} = (\langle 25\rangle^* + \langle 21,4\rangle + \langle 21,4\rangle' + \langle 18,6,1\rangle^* + \langle 15,6,4\rangle^* + 2\langle 14,7,4\rangle^* + \langle 13,11,1\rangle^* + 3\langle 13,8,4\rangle^* + 2\langle 13,7,4,1\rangle + 2\langle 13,7,4,1\rangle' + 2\langle 12,11,2\rangle^* + 2\langle 12,9,4\rangle^* + 2\langle 11,8,6\rangle^* + 2\langle 11,7,6,1\rangle + 2\langle 11,7,6,1\rangle' + 2\langle 11,7,5,2\rangle + 2\langle 11,7,5,2\rangle' + 2\langle 9,7,5,4\rangle + 2\langle 9,7,5,4\rangle')\downarrow_{(1,0)} S_{24} = 2\langle 20,4\rangle^* + \langle 18,6\rangle^* + 3\langle 14,6,4\rangle + 3\langle 14,6,4\rangle' + \langle 13,11\rangle^* + 7\langle 13,7,4\rangle + 7\langle 13,7,4\rangle' + 4\langle 13,6,4,1\rangle^* + 4\langle 11,7,6\rangle + 4\langle 11,7,6\rangle' + 4\langle 11,6,5,2\rangle^* + 4\langle 9,6,5,4\rangle^* = 2D_{67} - D_{68} + 4D_{69} + 2D_{70}$, is not p.s. for S_{24}(controduction with Theorem 2.1) then $d_5 \not\subset d_1$. On $(7,\alpha)$-regular classes we have:

1. $\langle 21,4\rangle = \langle 21,4\rangle'$
2. $\langle 18,7\rangle = \langle 18,7\rangle'$
3. $\langle 14,11\rangle = \langle 14,11\rangle'$
4. $\langle 14,6,4,1\rangle = \langle 14,6,4,1\rangle'$
5. $\langle 14,5,4,2\rangle = \langle 14,5,4,2\rangle'$
6. $\langle 13,7,4,1\rangle = \langle 13,7,4,1\rangle'$
7. $\langle 12,7,4,2\rangle = \langle 12,7,4,2\rangle'$
8. $\langle 11,7,6,1\rangle = \langle 11,7,6,1\rangle'$
9. $\langle 11,7,5,2\rangle = \langle 11,7,5,2\rangle'$
10. $\langle 11,7,4,3\rangle = \langle 11,7,4,3\rangle'$
11. $\langle 9,7,5,4\rangle = \langle 9,7,5,4\rangle'$
12. $\langle 8,7,6,4\rangle = \langle 8,7,6,4\rangle'$
13. $\langle 7,6,5,4,2,1\rangle = \langle 7,6,5,4,2,1\rangle'$
14. $\langle 18,5,2\rangle^* = \langle 18,4,3\rangle^* + \langle 18,6,1\rangle^* - \langle 18,7\rangle + \langle 25\rangle^*$

15. $2\langle 14,6,4,1\rangle = 2\langle 14,5,4,2\rangle + \langle 14,7,4\rangle^* - 2\langle 14,11\rangle - \langle 21,4\rangle$
16. $\langle 13,7,4,1\rangle = \langle 13,5,4,2,1\rangle^* + \langle 13,8,4\rangle^* - \langle 13,11,1\rangle^* - \langle 20,4,1\rangle^*$
17. $\langle 12,7,4,2\rangle = \langle 12,6,4,2,1\rangle^* + \langle 12,9,4\rangle^* - \langle 12,11,2\rangle^* - \langle 19,4,2\rangle^*$
18. $\langle 12,9,4\rangle^* = \langle 11,10,4\rangle^* + \langle 13,8,4\rangle^* - \langle 14,7,4\rangle^* + \langle 15,6,4\rangle^* - \langle 16,5,4\rangle^* + \langle 18,4,3\rangle^* - \langle 19,4,2\rangle^* + \langle 20,4,1\rangle^* - \langle 21,4\rangle^* + \langle 25\rangle^*$
19. $\langle 11,8,6\rangle^* = 2\langle 7,6,5,4,2,1\rangle + \langle 11,9,5\rangle^* - \langle 11,10,4\rangle^* + \langle 12,11,2\rangle^* - \langle 13,11,1\rangle^* + \langle 14,11\rangle + 2\langle 21,4\rangle - \langle 25\rangle^*$
20. $2\langle 11,7,6,1\rangle = 2\langle 11,5,4,3,2\rangle^* + \langle 11,8,6\rangle^* + \langle 11,9,5\rangle^* - \langle 12,9,4\rangle^* + \langle 12,11,2\rangle^* + 2\langle 13,8,4\rangle^* - 2\langle 14,7,4\rangle^* - 2\langle 15,6,4\rangle^* + \langle 16,5,4\rangle^* - \langle 18,5,2\rangle^* - 2\langle 18,6,1\rangle^* + \langle 18,7\rangle + \langle 19,4,2\rangle^* - 5\langle 25\rangle^*$
21. $\langle 11,5,4,3,2\rangle^* = \langle 8,7,6,4\rangle - \langle 11,9,5\rangle^* + \langle 12,9,4\rangle^* - \langle 13,11,1\rangle^* - \langle 14,5,4,2\rangle + \langle 14,11\rangle - \langle 18,4,3\rangle^* + \langle 18,5,2\rangle^* - \langle 20,4,1\rangle^* + \langle 21,4\rangle$
22. $\langle 8,6,5,4,2\rangle^* = 2\langle 7,6,5,4,2,1\rangle + \langle 8,7,6,4\rangle + \langle 11,9,5\rangle^* - \langle 12,9,4\rangle^* + \langle 12,11,2\rangle^* - \langle 13,11,1\rangle^* + \langle 14,11\rangle + \langle 16,5,4\rangle^* - \langle 18,4,3\rangle^* + \langle 19,4,2\rangle^* - \langle 20,4,1\rangle^* + \langle 21,4\rangle$
23. $\langle 11,7,5,2\rangle = \langle 11,7,4,3\rangle + \langle 11,7,6,1\rangle + \langle 14,11\rangle - \langle 18,7\rangle$
24. $\langle 11,7,4,3\rangle = \langle 11,6,5,2,1\rangle^* - \langle 7,6,5,4,2,1\rangle + \langle 12,9,4\rangle^* - \langle 13,7,4,1\rangle - \langle 13,11,1\rangle^* + \langle 14,6,4,1\rangle + \langle 14,11\rangle - \langle 16,5,4\rangle^* + \langle 18,6,1\rangle^* - \langle 18,7\rangle - \langle 19,4,2\rangle^* - \langle 20,4,1\rangle^* + \langle 21,4\rangle - \langle 25\rangle^*$
25. $\langle 11,6,5,2,1\rangle^* = \langle 11,6,4,3,1\rangle^* + \langle 7,6,5,4,2,1\rangle - \langle 8,7,6,4\rangle + \langle 11,9,5\rangle^* - \langle 12,9,4\rangle^* + \langle 12,11,2\rangle^* + \langle 13,7,4,1\rangle - \langle 13,8,4\rangle^* + 2\langle 13,11,1\rangle^* - \langle 14,6,4,1\rangle + \langle 14,11\rangle + \langle 15,6,4\rangle^* - \langle 16,5,4\rangle + \langle 18,4,3\rangle^* - \langle 18,6,1\rangle^* - \langle 18,7\rangle - \langle 19,4,2\rangle^* + 2\langle 20,4,1\rangle^* - \langle 21,4\rangle - \langle 25\rangle^*$
26. $\langle 9,7,5,4\rangle = \langle 9,6,5,4,1\rangle^* + \langle 12,9,4\rangle^* - \langle 11,9,5\rangle^* - \langle 16,5,4\rangle^*$
27. $\langle 11,6,4,3,1\rangle^* = \langle 9,6,5,4,1,\rangle^* - \langle 11,8,6\rangle^* - \langle 11,9,5\rangle^* + \langle 12,6,4,2,1\rangle^* + \langle 12,9,4\rangle^* - \langle 12,11,2\rangle^* - \langle 13,5,4,2,1\rangle^* + 3\langle 13,8,4\rangle^* - \langle 13,11,1\rangle^* - \langle 14,7,4\rangle^* - 3\langle 16,5,4\rangle^* + 3\langle 18,4,3\rangle^* + 2\langle 18,6,1\rangle^* - \langle 18,7\rangle - 4\langle 19,4,2\rangle^* - \langle 21,4\rangle - 2\langle 25\rangle^*$

Since the number of i.m.s is equal or fewer than the number of spin characters, Table 1 can only have a maximum of 49 columns. However, Table 1 only has a maximum of 22 columns because it has 27 equations that correspond to the spin characters of S_{25} in B_1. And because $d_i - d_j$ is not p.s. to $S_{25} \forall 1 \le i < j \le 22$, and $d_1, d_2, \cdots d_{22}$ are linearly independent. So we get Table 1.

Lemma 3.2. The decomposition matrix for the block B_3 of type associate as shown in Table 2.

Table 2- Block B_3

spin characters	Decomposition matrix																	
$\langle 23,2\rangle$	1																	
$\langle 23,2\rangle'$		1																
$\langle 16,9\rangle$	1		1															
$\langle 16,9\rangle'$		1		1														
$\langle 16,7,2\rangle^*$	2	2	1	1	1	1												
$\langle 16,6,2,1\rangle$					1		1											
$\langle 16,6,2,1\rangle'$						1		1										
$\langle 16,4,3,2\rangle$							1											
$\langle 16,4,3,2\rangle'$								1										
$\langle 14,9,2\rangle^*$	2	2	1	1	1	1			1	1								
$\langle 13,9,2,1\rangle$	1	1			1		1		1		1							
$\langle 13,9,2,1\rangle'$	1	1				1		1		1		1						
$\langle 11,9,3,2\rangle$							1				1		1					
$\langle 11,9,3,2\rangle'$								1				1		1				
$\langle 10,9,4,2\rangle$											1		1		1			
$\langle 10,9,4,2\rangle'$												1		1		1		
$\langle 9,8,6,2\rangle$	1	1							1	1	1				1		1	1
$\langle 9,8,6,2\rangle'$	1	1							1	1		1				1	1	1
$\langle 9,7,6,2,1\rangle^*$									1	1					1	1	1	1
$\langle 9,7,4,3,2\rangle^*$															1	1	1	1
$\langle 9,6,4,3,2,1\rangle$																	1	
$\langle 9,6,4,3,2,1\rangle'$																		1
	d_{67}	d_{68}	d_{69}	d_{70}	d_{71}	d_{72}	d_{73}	d_{74}	d_{75}	d_{76}	d_{77}	d_{78}	d_{79}	d_{80}	d_{81}	d_{82}	d_{83}	d_{84}

Proof: By used $(r, \bar{r})$-inducing of p.i.s. $D_{46}, D_{50}, D_{103}, D_{104}, D_{49}, D_{105}, D_{106}, D_{60}, D_{48}, D_{64}, D_{107}, D_{108}$ of S_{24} to S_{25}, then we get on k_1,k_2,d_{71},d_{72},k_3, d_{75}, d_{76}, k_4, k_5, k_6,d_{83},d_{84}, respectively.

Case 2: $k_2 \not\subset k_1$. To prove this, suppose the opposite.

$$(k_1 - k_2) \downarrow_{(2,6)} S_{24} = (\langle 23,2\rangle + \langle 23,2\rangle' + 2\langle 16,7,2\rangle^* + 2\langle 14,9,2\rangle^* + 2\langle 13,9,2,1\rangle + 2\langle 13,9,2,1\rangle' + 2\langle 9,8,6,2\rangle + 2\langle 9,8,6,2\rangle') \downarrow_{(2,6)} S_{24} = 2D_{45} - 2D_{50}$$

is not p.s. for S_{24} (controduction with Theorem 2.1) then $k_2 \not\subset k_1$The similar approach, we demonstrate that $k_5 \not\subset k_4$, since $(k_4 - k_5) \downarrow_{(6,2)} S_{24} = 2D_{59} - 2D_{61}$ is not p.s. Since $\langle 16,9\rangle \neq \langle 16,9\rangle'$ thenk_1 is split to d_{67}, d_{68} or k_2 is split to d_{69}, d_{70} is split. If k_2 split. But $\langle 23,2\rangle \neq \langle 23,2\rangle'$ then k_1 also split. If k_1 is split and from $(7,\alpha)$-regular classes,

$$\langle 16,9\rangle - \langle 23,2\rangle \neq \langle 16,9\rangle' - \langle 23,2\rangle' \quad (1)$$

then k_2 must split, so in both cases k_1, k_2 are splits. Since $\langle 11,9,3,2\rangle \neq \langle 11,9,3,2\rangle'$ then k_3 is split to d_{77}, d_{78} or k_4 is split to d_{73}, d_{74}. If k_4 is split. But $\langle 16,4,3,2\rangle \neq \langle 16,4,3,2\rangle'$ so k_3 also split. If k_3 is split and from $(7,\alpha)$-regular classes

$$\langle 11,9,3,2\rangle - \langle 16,4,3,2\rangle \neq \langle 11,9,3,2\rangle' - \langle 16,4,3,2\rangle' \quad (2)$$

then k_4 split to d_{77}, d_{78}, so, in both cases we get k_3, k_4 are splits.k_5 splits to d_{79}, d_{80} because $\langle 11,9,3,2\rangle \neq \langle 11,9,3,2\rangle'$, or there are two columns, Y_1, Y_2. Since $\langle 11,9,3,2\rangle \downarrow S_{24} = \langle 10,9,3,2\rangle^{*1} + \langle 11,8,3,2\rangle^{*8} + \langle 11,9,3,1\rangle^{*7}$ has 16 of i.m.s then we have $x_1 \in \{0,1,\dots,13\}$, using the same method we got on $x_6 \in \{0,1,\dots,5\}$, $x_5 \in \{0,1,\cdots,7\}$,$x_2, x_4 \in \{0,1,\cdots,12\}$, $x_3 \in \{0,1,\cdots,32\}$. Take $x_1 \in \{1,\dots,13\}$ (if $x_1 = 0$ we have contradiction), so that:

$Y_1 = x_1\langle 11,9,3,2\rangle + x_2\langle 10,9,4,2\rangle + x_3\langle 9,8,6,2\rangle + x_4\langle 9,7,6,2,1\rangle^* + x_5\langle 9,7,4,3,2\rangle^* + x_6\langle 9,6,4,3,2,1\rangle$,
$Y_2 = x_1\langle 11,9,3,2\rangle' + x_2\langle 10,9,4,2\rangle' + x_3\langle 9,8,6,2\rangle' + x_4\langle 9,7,6,2,1\rangle^* + x_5\langle 9,7,4,3,2\rangle^* + x_6\langle 9,6,4,3,2,1\rangle'$.

To find decomposition matrix we must discuss all probabilities such that the degree $Y_1, Y_2 \equiv 0 \, mod 7^3$, that's difficult, so we do the algorithm by maple program in appendix to help us and we find it are equal **50836** probabilities, therefore, we aim to limit the number of possibilities. Since inducing m.s. is m.s. we have:

$$(\langle 8,7,4,3,2\rangle - \langle 8,7,6,2,1\rangle + \langle 14,9,1\rangle^*) \uparrow^{(2,6)} S_{25} hence x_5 \geq x_4 \quad (3)$$
$$(\langle 8,7,6,2,1\rangle - \langle 8,7,4,3,1\rangle + \langle 9,8,4,3\rangle^*) \uparrow^{(2,6)} S_{25} hence x_4 \geq x_5, \therefore x_4 = x_5 \quad (4)$$
$$(\langle 9,6,4,3,2\rangle + \langle 9,6,4,3,2\rangle' - \langle 9,7,6,2\rangle^* + \langle 13,9,2\rangle + \langle 13,9,2\rangle') \uparrow^{(0,1)} S_{25} hence\ 2x_6 + x_5 \geq x_3 \quad (5)$$
$$(\langle 9,7,6,2\rangle^* - \langle 9,6,4,3,2\rangle - \langle 9,6,4,3,2\rangle') \uparrow^{(0,1)} S_{25} hence x_3 \geq 2x_6 + x_5, \therefore x_3 = 2x_6 + x_5 \quad (6)$$
$$(\langle 10,8,4,2\rangle^* - \langle 11,8,3,2\rangle^* + \langle 15,4,3,2\rangle^*) \uparrow^{(2,6)} S_{25} hence x_2 \geq x_1 \quad (7)$$
$$(\langle 8,7,4,3,2\rangle - \langle 8,6,4,3,2,1\rangle^* + \langle 9,8,7\rangle) \uparrow^{(2,6)} S_{25} hence x_5 \geq x_6 \quad (8)$$
$$(\langle 9,7,4,3,1\rangle + \langle 9,7,5,2,1\rangle - \langle 9,8,6,1\rangle^* + \langle 13,8,2,1\rangle^*) \uparrow^{(2,6)} S_{25} hence\ 2x_4 \geq x_3 \quad (9)$$
$$(\langle 9,8,6,1\rangle^* - \langle 8,7,4,3,2\rangle - \langle 8,7,6,2,1\rangle + \langle 9,8,4,3\rangle^*) \uparrow^{(0,1)} S_{25} hence x_3 \geq 2x_4, \therefore x_3 = 2x_4 \quad (10)$$

then we get degree $Y_1, Y_2 \equiv 0 \, mod 7^3$only when $Y_1 + Y_2 = \mathrm{m}(d_{79} + d_{80})$, m $\in \{1,2,\dots,12\}$ Which is originally a division of k_5 to d_{79} and d_{80}. k_6 splits to d_{81}, d_{82} because $\langle 10,9,4,2\rangle \neq \langle 10,9,4,2\rangle'$, or there are two columns, Y_1, Y_2, by using the same method we got:

$Y_1 = x_2\langle 10,9,4,2\rangle + x_3\langle 9,8,6,2\rangle + x_4\langle 9,7,6,2,1\rangle^* + x_5\langle 9,7,4,3,2\rangle^* + x_6\langle 9,6,4,3,2,1\rangle$,
$Y_2 = x_2\langle 10,9,4,2\rangle' + x_3\langle 9,8,6,2\rangle' + x_4\langle 9,7,6,2,1\rangle^* + x_5\langle 9,7,4,3,2\rangle^* + x_6\langle 9,6,4,3,2,1\rangle'$.

then all probabilities such that the degree $Y_1, Y_2 \equiv 0 \, mod 7^3$equal to **5042**, so we aim to limit the number of possibilities. Since inducing m.s. is m.s. we have:

$$(\langle 9,8,5,2\rangle^* - \langle 10,8,4,2\rangle^* + \langle 11,8,3,2\rangle^*) \uparrow^{(2,6)} S_{25} hence x_3 \geq x_2 \quad (11)$$
$$(\langle 8,7,4,3,2\rangle - \langle 8,7,6,2,1\rangle + \langle 14,9,1\rangle) \uparrow^{(2,6)} S_{25} hence x_5 \geq x_4 \quad (12)$$
$$(\langle 8,7,6,2,1\rangle - \langle 8,7,4,3,1\rangle + \langle 9,8,4,3\rangle^*) \uparrow^{(2,6)} S_{25} hence x_4 \geq x_5 \therefore x_4 = x_5 \quad (13)$$
$$(\langle 9,7,5,2,1\rangle - \langle 10,9,4,1\rangle^* + \langle 11,8,3,2\rangle) \uparrow^{(2,6)} S_{25} hence x_4 \geq x_2 \quad (14)$$
$$(\langle 9,6,4,3,2\rangle + \langle 9,6,4,3,2\rangle' - \langle 9,7,6,2\rangle^* + \langle 13,9,2\rangle + \langle 13,9,2\rangle') \uparrow^{(0,1)} S_{25} hence\ 2x_6 + x_5 \geq x_3 \quad (15)$$
$$(\langle 9,7,6,2\rangle^* - \langle 9,6,4,3,2\rangle - \langle 9,6,4,3,2\rangle') \uparrow^{(0,1)} S_{25} hence x_3 \geq 2x_6 + x_5 \therefore x_3 = 2x_6 + x_5 \quad (16)$$

then we get degree $Y_1, Y_2 \equiv 0 \, mod 7^3$only when $Y_1 + Y_2 = \mathrm{m}(d_{81} + d_{82})$, m $\in \{1,2,\dots,7\}$ Which is originally a division of k_6 to d_{81} and d_{82}. Based on the above, we get Table 2.

Lemma 3.3. Decomposition matrix for the block B_4 of type double as shown in the Tables 3.

Table 3- Block B_4

Spin characters	Decomposition matrix								
$\langle 22,2,1\rangle^*$	1								
$\langle 16,8,1\rangle^*$	1	1							
$\langle 15,9,1\rangle^*$		1	1						

$\langle 15,8,2\rangle^*$	1	1	1	1					
$\langle 15,7,2,1\rangle$				1	1				
$\langle 15,4,3,2,1\rangle^*$					1				
$\langle 14,8,2,1\rangle$	1		1	1	1	1			
$\langle 11,8,3,2,1\rangle^*$					1	1	1		
$\langle 10,8,4,2,1\rangle^*$						1	1	1	
$\langle 9,8,5,2,1\rangle^*$			2			1		1	2
$\langle 9,8,7,1\rangle$	1		1			1			1
$\langle 9,8,4,3,1\rangle^*$								1	1
$\langle 8,7,4,3,2,1\rangle$									1
	d_{85}	d_{86}	d_{87}	d_{88}	d_{89}	d_{90}	d_{91}	d_{92}	d_{93}

Proof: By using (0,1)-inducing of p.i.s. $D_{47}, D_{112}, D_{56}, D_{54}, D_{52}, D_{59}, D_{61}, D_{60}, D_{65}$ of S_{24} to S_{25} we get on $d_{85}, d_{86}, \ldots, d_{93}$, respectively.

Case 3: $d_{91} \not\subset d_{90}$. To prove this, suppose the opposite.

$(d_{90} - d_{91}) \downarrow_{(0,1)} S_{24} = D_{57} - D_{62}$ is not p.s. for S_{24}(controduction with Theorem 2.1) then $d_{91} \not\subset d_{90}$, also there are 8 of $(7, \alpha)$-regular classes then Table 3 only has a maximum of 9 columns because it has 8 equations that correspond to the spin characters of S_{25} in B_4. And because $d_i - d_j$ is not p.s. to $S_{25} \forall 1 \le i < j \le 9$, and $d_{85}, d_{86}, \cdots d_{93}$ are linearly independent.

Lemma 3.4. The decomposition matrix for the block B_5 of type associate as shown in the Tables 4.

Table 4- Block B_5

Spin characters	Decomposition matrix																	
$\langle 20,5\rangle$	1																	
$\langle 20,5\rangle'$		1																
$\langle 19,6\rangle$	1		1															
$\langle 19,6\rangle'$		1		1														
$\langle 14,6,5\rangle^*$			1	1	1	1												
$\langle 13,12\rangle$			1				1											
$\langle 13,12\rangle'$				1				1										
$\langle 13,7,5\rangle^*$	1	1	1	1	1	1	1	1	1	1								
$\langle 13,6,5,1\rangle$					1				1		1							
$\langle 13,6,5,1\rangle'$						1				1		1						
$\langle 13,5,4,3\rangle$											1							
$\langle 13,5,4,3\rangle'$												1						
$\langle 12,7,6\rangle^*$	1	1					1	1	1	1			1	1				
$\langle 12,6,5,2\rangle$							1	1	1		1		1		1			
$\langle 12,6,5,2\rangle'$							1	1		1		1		1		1		
$\langle 12,6,4,3\rangle$											1				1			
$\langle 12,6,4,3\rangle'$												1				1		
$\langle 11,6,5,3\rangle$							1	1					1		1		1	
$\langle 11,6,5,3\rangle'$							1	1						1		1		1
$\langle 10,6,5,4\rangle$							1	1					1	1			1	
$\langle 10,6,5,4\rangle'$							1	1					1	1				1
$\langle 7,6,5,4,3\rangle^*$													1	1				
	d_{94}	d_{95}	d_{96}	d_{97}	d_{98}	d_{99}	d_{100}	d_{101}	d_{102}	d_{103}	d_{104}	d_{105}	d_{106}	d_{107}	d_{108}	d_{109}	d_{110}	d_{111}

Proof: By using $(\mathrm{r}, \bar{\mathrm{r}})$-inducing $D_{67}, D_{68}, D_{114}, D_{115}, D_{70}, D_{95}, D_{72}, D_{74}, D_{73}, D_{121}$ of p.i.s. of S_{24} to S_{25} we get on $k_1, k_2, d_{98}, d_{99}, k_3, k_4, \ldots, k_8$, respectively.

Case 4: $k_8 \not\subset k_3$. To prove this, suppose the opposite.

$(k_3 - k_8) \downarrow_{(2,6)} S_{24} = (2\langle 12,7,6\rangle^* + \langle 12,6,5,2\rangle + \langle 12,6,5,2\rangle' + \langle 10,6,5,4\rangle + \langle 10,6,5,4\rangle' + 2\langle 7,6,5,4,3\rangle^*) \downarrow_{(2,6)} S_{24} = 2\langle 11,7,6\rangle + 2\langle 11,7,6\rangle' + 2\langle 11,6,5,2\rangle^* + 2\langle 12,6,4,2\rangle^* + 2\langle 9,6,5,4\rangle^* + 2\langle 7,6,5,4,2\rangle + 2\langle 7,6,5,4,2\rangle' = 2D_{73} - 2D_{74} + 2D_{76}$, is not p.s. for S_{24}(controduction with Theorem 2.1) then $k_8 \not\subset k_3$The similar approach, we demonstrate that $k_8 \not\subset k_3$, since $(k_3 - k_8) \downarrow_{(4,4)} S_{24} = 2D_{70} + 4D_{73} - 2D_{75}$ is not p.s. Since $\langle 10,6,5,4\rangle \neq \langle 10,6,5,4\rangle'$ so k_3 split to,

d_{100}, d_{101} or k_8 is split to d_{110}, d_{111}. If k_8 is split. But $\langle 13,12\rangle \neq \langle 13,12\rangle'$ thenk_3 split. If k_3 is split and from $(7,\alpha)$-regular classes,

$$\langle 10,6,5,4\rangle - 2\langle 13,12\rangle + 2\langle 19,6\rangle \neq \langle 10,6,5,4\rangle' - 2\langle 13,12\rangle' + 2\langle 19,6\rangle' \quad (17)$$

then k_8 split, so we get k_3 and k_8 are splits. Since $\langle 12,6,5,2\rangle \neq \langle 12,6,5,2\rangle'$ so k_4 is split to d_{102}, d_{103}or k_6 split to, d_{106}, d_{107}. If k_4 is split. But $\langle 11,6,5,2\rangle \neq \langle 11,6,5,2\rangle'$ then k_6 split. If k_6 is split and from $(7,\alpha)$-regular classes,

$$\langle 12,6,5,2\rangle - \langle 11,6,5,3\rangle + \langle 10,6,5,4\rangle \neq \langle 12,6,5,2\rangle' - \langle 11,6,5,3\rangle' + \langle 10,6,5,4\rangle' \quad (18)$$

then k_4 split, so we get k_4 and k_6 are splits. Since $\langle 12,6,4,3\rangle \neq \langle 12,6,4,3\rangle'$ so k_5 split to, d_{104}, d_{105} or k_7 is split to d_{108}, d_{109}. If k_7 split. But $\langle 13,5,4,3\rangle \neq \langle 13,5,4,3\rangle'$ then k_5 also split. If k_5 split and from $(7,\alpha)$-regular classes,

$$\langle 12,6,4,3\rangle - \langle 13,5,4,1\rangle \neq \langle 12,6,4,3\rangle' - \langle 13,5,4,1\rangle' \quad (19)$$

then k_7 must split So in both cases we get k_5 and k_7 are splits. Since $\langle 19,6\rangle \neq \langle 19,6\rangle'$ so k_1 is split to, d_{94}, d_{95} or k_2 is split to d_{96}, d_{97} is split. If k_2 split. But $\langle 20,5\rangle \neq \langle 20,5\rangle'$ then k_1 split. If k_1 split and from $(7,\alpha)$-regular classes,

$$\langle 19,6\rangle - \langle 20,5\rangle \neq \langle 19,6\rangle' - \langle 20,5\rangle' \quad (20)$$

Then k_2 must split, so in both cases we get k_1 and k_2 are splits. Based on the above, we get Table 4.

Lemma 3.5. Decomposition matrix for the block B_6 of type double as shown in the Tables 5.

Table 7- Block B_6

Spin characters	Decomposition matrix								
$\langle 20,3,2\rangle^*$	1								
$\langle 17,6,2\rangle^*$	1	1							
$\langle 16,6,3\rangle^*$		1	1						
$\langle 14,6,3,2\rangle$			1	1					
$\langle 13,10,2\rangle^*$		1			1				
$\langle 13,9,3\rangle^*$	1	1	1		1	1			
$\langle 13,7,3,2\rangle$			1	1		1	1		
$\langle 13,6,3,2,1\rangle^*$				1			1		
$\langle 10,9,6\rangle^*$	1					1		1	
$\langle 10,7,6,2\rangle$					1	1	1	1	1
$\langle 10,6,4,3,2\rangle^*$							1		1
$\langle 9,7,6,3\rangle$					1			1	1
$\langle 9,6,5,3,2\rangle^*$								2	1
	d_{112}	d_{113}	d_{114}	d_{115}	d_{116}	d_{117}	d_{118}	d_{119}	d_{120}

Proof: Using $(\mathrm{r},\bar{\mathrm{r}})$-inducing of p.i.s. $D_7, D_9, D_{17}, D_{23}, D_{25}, D_{27}, D_{29}, D_{73}, D_{43}$ of S_{24} to S_{25} we get on $d_{112}, d_{113},\ldots, d_{120}$, respectively, and since there are 8 of $(7,\alpha)$-regular classes then Table 5 only has a maximum of 9 columns because it has 8 equations that correspond to the spin characters of S_{25} in B_6, and $d_i - d_j$ is not p.s. to $S_{25} \forall 1 \leq i < j \leq 9$. and $d_{112}, d_{113}, \cdots d_{120}$ are linearly independent.

Lemma 3.6. The decomposition matrix for the block B_7 is

Table 6- Block B_7

Spin characters	Decomposition matrix		
$\langle 19,5,1\rangle^*$	1		
$\langle 12,8,5\rangle^*$	1	1	
$\langle 12,7,5,1\rangle$		1	1
$\langle 12,5,4,3,1\rangle^*$			1
	d_{121}	d_{122}	d_{123}

Proof: Since

- degree $\{\langle 12,8,5\rangle^*, \langle 12,5,4,3,1\rangle^*\} \equiv 294\ mod 7^3$

- degree $\{\langle 19,5,1\rangle^{*}, \langle 12,7,5,1\rangle + \langle 12,7,5,1\rangle'\} \equiv -294\ mod 7^{3}$

Using (3,5)-inducing of p.i.s. D_{78}, D_{80}, D_{82} for S_{24} to S_{25}, also there 2 $(7,\alpha)$-regular classes, then the matrix contains at most 3 columns since there are two equations corresponding the spin characters of S_{25} in $\boldsymbol{B}_{7}$, we get Table 6.

Lemma 3.7. The decomposition matrix for the block B_8 is Table 7.

Table 7- Block $\boldsymbol{B}_{8}$

Spin characters	Decomposition matrix					
$\langle 18,4,2,1\rangle$	1					
$\langle 18,4,2,1\rangle'$		1				
$\langle 11,9,4,1\rangle$	1		1			
$\langle 11,9,4,1\rangle'$		1		1		
$\langle 11,8,4,2\rangle$			1		1	
$\langle 11,8,4,2\rangle'$				1		1
$\langle 11,7,4,2,1\rangle^{*}$					1	1
	d_{124}	d_{125}	d_{126}	d_{127}	d_{128}	d_{129}

Proof: When we induce $D_{97}, D_{98}, D_{99}, D_{100}, D_{55}$ for S_{24} through S_{25}, we get on $d_{124}, d_{125}, k_1, k_2, k_3$ respectively. Since B_8 of defect one and $\langle 11,8,4,2\rangle \neq \langle 11,8,4,2\rangle'$ then frome (Theorem 2.3) k_3 must splite to d_{128}, d_{129}

Case 5:$d_{128} \subset k_2$. To prove this, suppose the opposite, , in this state we have $(\langle 11,8,4,2\rangle - \langle 11,9,4,1\rangle' + \langle 18,4,2,1\rangle')$ is m.s to S_{25}, but $(\langle 11,8,4,2\rangle - \langle 11,9,4,1\rangle' + \langle 18,4,2,1\rangle') \downarrow_{(0,1)} S_{24} = \langle 11,7,4,2\rangle$ $-\langle 11,9,4\rangle' + \langle 18,4,2\rangle'$ in not i.m.s to S_{24}, then $d_{128} \subset k_2$. When d_{128} is subtracted from k_2, then d_{129} is subtracted from k_1 Since d_{128} and d_{129} are associate columns. Put $k_2 - d_{128} = d_{127}, k_1 - d_{129} = d_{126}$. then we get Table 7.

Lemma 3.8. The decomposition matrix for the block B_9 is

Table 8- Block $\boldsymbol{B}_{9}$

Spin characters	Decomposition matrix		
$\langle 17,5,3\rangle^{*}$	1		
$\langle 12,10,3\rangle^{*}$	1	1	
$\langle 10,7,5,3\rangle$		1	1
$\langle 10,6,5,3,1\rangle^{*}$			1
	d_{130}	d_{131}	d_{132}

Proof: Since

- degree $\{\langle 17,5,3\rangle^{*}, \langle 10,7,5,3\rangle + \langle 10,7,5,3\rangle'\} \equiv 294\ mod 7^{3}$
- degree $\{\langle 12,10,3\rangle^{*}, \langle 10,6,5,3,1\rangle^{*}\} \equiv -294\ mod 7^{3}$

Using (3,5)-inducing of p.i.s. D_5, D_1, D_{21} for S_{24} to S_{25}, also there $(7,\alpha)$-regular classes then the matrix contains at most 5 columns since there are two equations corresponding to the spin characters of S_{25} in $\boldsymbol{B}_{9}$, we get Table 8.

Lemma 3.9. The decomposition matrix for the block B_{10} is Table 9.

Table 9- Block $\boldsymbol{B}_{10}$

Spin characters	Decomposition matrix					
$\langle 15,5,4,1\rangle$	1					
$\langle 15,5,4,1\rangle'$		1				
$\langle 12,8,4,1\rangle$	1		1			
$\langle 12,8,4,1\rangle'$		1		1		
$\langle 11,8,5,1\rangle$			1		1	
$\langle 11,8,5,1\rangle'$				1		1
$\langle 8,7,5,4,1\rangle^{*}$					1	1
	d_{133}	d_{134}	d_{135}	d_{136}	d_{137}	d_{138}

Proof: When we induce $D_{78}, D_{79}, D_{76}, D_{77}, D_{92}$ of p.i.s. for S_{24} through S_{25}, we get on $d_{133}, d_{134}, d_{135}, d_{136}$,and $k = \langle 11,8,5,1\rangle + \langle 11,8,5,1\rangle' + 2\langle 8,7,5,4,1\rangle^{*}$ respectively. Since B_{10} of defect one so that frome (Theorem 2.3) k must splite to d_{137}, d_{138}. Then we get Table 9.

Theorem 3.10. The decomposition matrix for the block B_2 of type associate in Table 10.

Table 10- Block B_2

Spin characters	Decomposition matrix																																										
⟨24,1⟩	1																																											
⟨24,1⟩′		1																																										
⟨22,3⟩	1		1																																								
⟨22,3⟩′		1		1																																							
⟨21,3,1⟩*			1	1	1	1																																					
⟨19,3,2,1⟩					1		1																																				
⟨19,3,2,1⟩′						1		1																																			
⟨17,8⟩	1		1						1																																		
⟨17,8⟩′		1		1						1																																	
⟨17,7,1⟩*	2	2	1	1	1	1			1	1	1	1																																
⟨17,5,2,1⟩					1		1				1		1																														
⟨17,5,2,1⟩′						1		1			1		1																													
⟨17,4,3,1⟩							1					1																														
⟨17,4,3,1⟩′								1					1																													
⟨16,5,3,1⟩											1		1		1																												
⟨16,5,3,1⟩′												1		1		1																											
⟨15,10⟩	1								1									1																										
⟨15,10⟩′		1								1									1																									
⟨15,7,3⟩*	2	2	1	1					1	1	1	1					1	1	1	1																									
⟨15,6,3,1⟩											1				1				1		1																							
⟨15,6,3,1⟩′												1				1				1		1																						
⟨15,5,3,2⟩																1						1																							
⟨15,5,3,2⟩′																	1						1																						
⟨14,10,1⟩*	2	2							1	1	1	1					1	1					1	1																					
⟨14,8,3⟩*	2	2	2	2	1	1			1	1	1	1					2	2	1	1			1	1	1	1																			
⟨14,7,3,1⟩			1	1														1	1	1	1	1	1			2		1	1																
⟨14,7,3,1⟩′			1	1														1	1	1	1	1	1				2	1	1																
⟨14,5,3,2,1⟩*																					1	1					1	1																
⟨13,8,3,1⟩	1	1	1	1	1						1				1		1	1	1		1		1		1	1	1	1	1																
⟨13,8,3,1⟩′	1	1	1	1		1						1				1	1	1		1		1		1	1	1	1	1		1															
⟨12,10,2,1⟩	1	1									1		1										1								1													
⟨12,10,2,1⟩′	1	1										1		1										1								1												
⟨12,9,3,1⟩	1	1					1				1		1		1									1						1		1		1												
⟨12,9,3,1⟩′	1	1						1				1		1		1									1						1		1		1											
⟨12,8,3,2⟩															1						1					1		1	1	1				1			1									
⟨12,8,3,2⟩′																1						1				1	1	1		1				1			1								
⟨12,7,3,2,1⟩*																					1	1			1	1	2	2								1	1							
⟨11,10,3,1⟩							1						1																			1		1												
⟨11,10,3,1⟩′								1						1																			1		1											
⟨10,9,5,1⟩					1		1																							1				1				1						
⟨10,9,5,1⟩′						1		1																							1				1				1					
⟨10,8,7⟩*			1	1	1	1											1	1							1	1												1	1				
⟨10,8,6,1⟩	1	1	1	1	1												1	1					1	1	1	1	1	1	1									1		1	1	1	1		
⟨10,8,6,1⟩′	1	1	1	1		1											1	1					1	1	1	1	1	1		1									1	1	1	1	1		
⟨10,8,5,2⟩	1	1															1	1					1	1	1		1	1	1		1		1		1		1		1		1	1	1	
⟨10,8,5,2⟩′	1	1															1	1					1	1		1	1	1		1		1		1		1		1		1	1	1		1
⟨10,8,4,3⟩																															1		1		1								1	
⟨10,8,4,3⟩′																																1		1		1								1
⟨10,7,5,2,1⟩*																		2	2					1	1	1	1	1	1								1	1			1	1	1	1	1	1
⟨10,7,4,3,1⟩*																											1	1								1	1					1	1	1	1
⟨10,5,4,3,2,1⟩																											1														1			
⟨10,5,4,3,2,1⟩′																												1														1		
⟨9,8,5,3⟩	1	1															1	1					1	1							1		1						1	1	1	1	1	
⟨9,8,5,3⟩′	1	1															1	1					1	1								1		1					1	1	1	1	1	1
⟨9,7,5,3,1⟩*																		2	2					1	1									2	2			1	1	2	2	2	2	1	1
⟨8,7,6,3,1⟩*																								1	1									1	1			1	1			1	1		
⟨8,7,5,3,2⟩*																																			1	1			1	1	1	1	1	1		
⟨8,6,5,3,2,1⟩																																									1	1	1		
⟨8,6,5,3,2,1⟩′																																									1	1		1	
	d_{23}	d_{24}	d_{26}	d_{27}	d_{27}	d_{28}	d_{29}	d_{30}	d_{31}	d_{32}	d_{33}	d_{34}	d_{35}	d_{36}	d_{37}	d_{38}	d_{39}	d_{40}	d_{41}	d_{42}	d_{43}	d_{44}	d_{45}	d_{46}	d_{47}	d_{48}	d_{49}	d_{50}	d_{51}	d_{52}	d_{53}	d_{54}	d_{55}	d_{56}	d_{57}	d_{58}	d_{59}	d_{60}	d_{61}	d_{62}	d_{63}	d_{64}	d_{65}	d_{66}

Proof: Using $(\mathrm{r},\bar{\mathrm{r}})$-inducing of p.i.s. $D_{45}, D_3, D_4, D_5, D_7, D_8, D_{50}, D_{11}, D_{12}, D_{13}, D_{14}, D_{109}, D_{19}, D_{54}, D_{23}, D_{25}, D_{26}, D_{57}, D_{29}$, $D_{30}, D_{110}, D_{60}, D_{63}, D_{35}, D_{111}, D_{39}, D_{41}, D_{42}, D_{43}$,of S_{23} to S_{24} we get on k_1, k_2,d_{27},d_{28},d_{29}, d_{30}, k_3,d_{33}, d_{34}, d_{35}, d_{36}, k_4, k_5, k_6, k_7, k_8,d_{45}, d_{46}, k_8,d_{49}, d_{50}, k_9, k_{10}, k_{11}, k_{12}, k_{13}, k_{14}, d_{63}, d_{64}, k_{15}, respectively.

Case 6:$k_{15} \subset k_4$. To prove this, suppose the opposite, , in this state we have $(\langle 9,7,5,3,1\rangle^* + \langle 10,7,4,3,1\rangle^* - \langle 10,7,5,2,1\rangle^* + \langle 10,9,5,1\rangle + \langle 10,9,5,1\rangle' + \langle 15,7,3\rangle^* - \langle 19,3,2,1\rangle - \langle 19,3,2,1\rangle' - \langle 22,3\rangle - \langle 22,3\rangle' - \langle 24,1\rangle - \langle 24,1\rangle')$ is m.s to S_{25}, but $\big((\langle 9,7,5,3,1\rangle^* + \langle 10,7,4,3,1\rangle^* - \langle 10,7,5,2,1\rangle^* + \langle 10,9,5,1\rangle + \langle 10,9,5,1\rangle' + \langle 15,7,3\rangle^* - \langle 19,3,2,1\rangle - \langle 19,3,2,1\rangle' - \langle 22,3\rangle - \langle 22,3\rangle' - \langle 24,1\rangle - \langle 24,1\rangle')\big) \downarrow_{(2,6)} S_{24}$ is not i.m.s to S_{24}so that $k_{15} \subset k_4$. Put $k_4 - k_{15} = c$. All i.m.s. are associated in block B_1, and since $\langle 15,5,3,2\rangle \neq \langle 15,5,3,2\rangle'$, so k_4 split to, d_{37}, d_{38} or k_7 is split to $d_{43}, d_{44}$is split. If k_7 is split. But $\langle 12,8,3,2\rangle \neq \langle 12,8,3,2\rangle'$ then k_4 also split. If k_4 is split and from $(7,\alpha)$-regular classes,

$$\langle 15,5,3,2\rangle - \langle 16,5,3,1\rangle + \langle 17,5,2,1\rangle - \langle 19,3,2,1\rangle \neq \langle 15,5,3,2\rangle' - \langle 16,5,3,1\rangle' + \langle 17,5,2,1\rangle' - \langle 19,3,2,1\rangle' \qquad (21)$$

Then k_7 split, so we get k_4 and k_7 are splits. Since $\langle 14,7,3,1\rangle \neq \langle 14,7,3,1\rangle'$, then$k_2$ is split to, d_{25}, d_{26} or k_6 is split to d_{41}, d_{42} is split. If k_6 is split. But $\langle 22,3\rangle \neq \langle 22,3\rangle'$ then k_2 also split too. If k_2 is split and from $(7,\alpha)$-regular classes,

$$\langle 14,7,3,1\rangle - \langle 17,8\rangle - \langle 22,3\rangle + \langle 15,10\rangle + \langle 24,1\rangle \neq \langle 14,7,3,1\rangle' - \langle 17,8\rangle' - \langle 22,3\rangle' + \langle 15,10\rangle' + \langle 24,1\rangle' \qquad (22)$$

then k_6 split. we get k_2 and k_6 are splits. Since $\langle 15,10\rangle \neq \langle 15,10\rangle'$, so k_3 split to, d_{31}, d_{32} or k_5 is split to d_{39}, d_{40} is split. If k_5 is split. But $\langle 17,8\rangle \neq \langle 17,8\rangle'$ then k_3 also split. If k_3 is split and from $(7,\alpha)$-regular classes,

$$\langle 15,10\rangle + \langle 22,3\rangle - \langle 17,8\rangle \neq \langle 15,10\rangle' + \langle 22,3\rangle' - \langle 17,8\rangle' \qquad (23)$$

then k_5 split, so we get k_3 and k_5 are splits. Since $\langle 9,8,5,3\rangle \neq \langle 9,8,5,3\rangle'$, so k_{11} split to, d_{55}, d_{56} or k_{14} is split to d_{61}, d_{62} is split. If k_{14} is split. But $\langle 12,8,3,2\rangle \neq \langle 12,8,3,2\rangle'$ then k_{11} also split. If k_{11} is split and from$(7,\alpha)$-regular classes,

$$\langle 9,8,5,3\rangle - \langle 11,10,3,1\rangle + \langle 17,4,3,1\rangle \neq \langle 9,8,5,3\rangle' - \langle 11,10,3,1\rangle' + \langle 17,4,3,1\rangle' \qquad (24)$$

then k_{14} split, so we get k_{11} and k_{14} are splits. Since $\langle 10,8,5,2\rangle \neq \langle 10,8,5,2\rangle'$, so k_9 split to, d_{51}, d_{52} or k_{13} is split to d_{59}, d_{60}. If k_{13} is split. But $\langle 12,8,3,2\rangle \neq \langle 12,8,3,2\rangle'$ then k_9 also split. If k_9 is split and from $(7,\alpha)$-regular classes,

$$\langle 10,8,5,2\rangle - \langle 12,8,3,2\rangle + \langle 15,5,3,2\rangle \neq \langle 10,8,5,2\rangle' - \langle 12,8,3,2\rangle' + \langle 15,5,3,2\rangle' \qquad (25)$$

then k_{13} split, so we get k_9 and k_{13} are splits. Since $\langle 10,8,4,3\rangle \neq \langle 10,8,4,3\rangle'$, so k_{10} is split to, d_{53}, d_{54} or k_{15} is split to d_{65}, d_{66}. If k_{15} is split. But $\langle 9,8,5,3\rangle \neq \langle 9,8,5,3\rangle'$ then k_{10} also split. If k_{10} is split and from $(7,\alpha)$-regular classes,

$$\langle 10,8,4,3\rangle - \langle 11,10,3,1\rangle + \langle 17,4,3,1\rangle \neq \langle 10,8,4,3\rangle' - \langle 11,10,3,1\rangle' + \langle 17,4,3,1\rangle' \qquad (26)$$

then k_{15} must split, so in both cases we get k_{10} and k_{15} are splits. Since $\langle 24,1\rangle \neq \langle 24,1\rangle'$ on $(7,\alpha)$-regular classes, then frome (Theorem 2.5.) k_1 is split to d_{23}, d_{24}. When there are 140 columns and $\langle 14,7,3,1\rangle \neq \langle 14,7,3,1\rangle'$ on $(7,\alpha)$-regular classes, $c = d_{43} + d_{44}$. As a result, Table 10 represents the decomposition matrix for block B_2.

4. Conclusions

There isn't a set method for investigating the subject, especially when we prove the field and the change of groups, therefore we had to carry out several studies to collect enough data to find new characteristics and theorems if the field characteristic is prime. When prior researchers examined the division matrix at the field where the characteristic is 0, they did this. Also, we employed Maple programming to view all valid probability. We also ran across brand-new problems as a result of defect three matrices. This opens the way for a comprehensive analysis that will first look at matrices of the defect type 4, then look at irreducible modular spin characters, and last categorize groups.

5. Appendix (Maple Programming)

Let the approximation matrix for the block to the spin characters of S_{25} for field has characteristic $p = 7$

The degree	Spin characters	Approximation matrix	
A_1	β_1	a_1	
A_1	$\beta_1{}'$		a_1
A_2	$\beta_2{}^*$	a_2	a_2
.	.	.	.

.	.	.	.
A_i	β_i	a_i	
A_i	$\beta_i{}'$		a_i
		Y_1	Y_2

$a_i \in \mathbb{N}, \forall i$. To find decomposition matrix we must discuss all probabilities such that the degree $Y_1, Y_2 \equiv 0\ mod 7^3$.

```
>P:=7^3;
D1:=A_1;
D2:=A_2;
.
.
Di:=A_i;
S:=0;
j:=1;
for a_1 from 0 to n_1 do
for a_2 from 0 to n_2 do
.
.
for a_i from 0 to n_i do
S:= D1*A_1+ D2*A_2+…+ Di*A_i;
G:=modp(S,P);
if G=0 then
print(j,'a_1'=a_1,'a_2'=a_2,…,'a_i'=a_i);
j:=j+1;
fi;
S:=S;
od;
od;       } i index
.
.
od;
```

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