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Bounds on the Wave Speed

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Abstract

In this paper, we will investigate the structure ofbounds for the wave speed $c$presented in [1]. By constructing appropriate sub- and super-solutions to this system

$-cu' = u'' + f(u, v),$
$-cv' = \epsilon^2 v'' + g(u, v),$
$(u, v)(-\infty) = S_-,$
$(u, v)(\infty) = S_+$ (1)

Where, we are interested in component-wise monotone travelling wave solutions of the system of equations

$u_t = u_{xx} + f(u, v),$
$$v_t = \epsilon^2 v_{xx} + g(u, v), \quad (2)$$
for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ for which the asymptotic conditions
$$(u, v)(-\infty, t) = S_-, \ (u, v)(\infty, t) = S_+, \ t > 0 \quad (3)$$
are satisfied. Similar to those introduced in [3] and using essentially identical arguments, itcan be shown that
$$-K \le c \le L\epsilon, (4)$$
where$K$ and L are positive constants independent of ϵ. One immediate consequenceof this result is that in the limit $\epsilon \to 0$ only left travelling waves exist.We investigate the sharpness of these bounds in the special case of CLV kinetics.We show that: the bounds of the wave speed given in [4] are optimal for the given leftand right solutions (sub-solutions and super-solutions).

Mathematics Subject Classification :35A18

Introduction

The method of sub-solutions and super-solutions and its associated monotone iteration is a powerful tool in establishing existence results for differential equations. This method can be applied to systems of coupled equations and to equations with nonlinear boundary conditions. The basic idea of this method is to use a sub-solution or super-solution as the initial iteration in a suitable iterative process, so that the resulting sequence of iterations is monotone and converges in some suitable function space to a solution of the problem.

The underlying monotone iterative scheme can also be used for the computation of numerical solutions when these equations are replaced by corresponding finite difference equations, see [7]. Note that in some literature sub-solutions and super-solutions are sometimes referred to as lower and upper solutions or sub-functions and super-functions, respectively, see again [6].

Another use of sub-solutions and super-solutions is to obtain bounds for the wave speed of travelling waves. It is this subject that we concentrate on in this paper. In [4], Heinze*et al.* stated the following theorem and proved it by considering upper and lower solutions (or left and right solutions as we called them below) of a particular form, as we will discuss below.

Theorem 1.[2]. *For each fixed* $\epsilon > 0$*, let* $(u_\epsilon, v_\epsilon, c_\epsilon)$ *be the unique monotone solution of equation (1).*Then

$$-2\sqrt{L} \le c \in \le 2\sqrt{K\varepsilon}, \qquad (5)$$

where

$$K := \sup_{0<v<1} \frac{g(0,v)}{v} \; and \, L := \sup_{0<u<1} \frac{f(u,0)}{u}.$$

Note that the upper bound implies that as $\epsilon \to 0$ the only type of travelling waves that can exist are left travelling waves.

Heinze*et al.* [3] choose a specific structure for left and right solutions (which will be discussed below) to obtain these bounds. This structure was introduced with no motivation and also it is not clear whether the wave speed bounds obtained are sharp for the given form. In this paper we investigate these bounds further for the special case of the CLV kinetics

$$f(u, v) = u(1 - u - \alpha v),$$
$$g(u, v) = \delta v(1 - v - \beta u). \qquad (6)$$

We are interested in systems that satisfy this Assumption below:

Assumption 1.*The non-linearities* $f, g \in C^2([0, 1]^2, R)$ *satisfy:*

(1) $f(0, v) = 0 = g(u, 0)$.

(2) (1) has exactly two stable, uniform equilibria $S_- = (0, 1)$ and $S_+ = (1, 0)$ and two unstable, uniform equilibria $(0, 0)$ and (u_s, v_s).

(3) $f_v(u, v) < 0$, $g_u(u, v) < 0$ for $(u, v) \in (0, 1)^2$.

(4) The non-trivial solutions (u, v) of $g(u, v) = 0$ are given by $u = \Gamma(v)$ for a monotonically decreasing function Γ. Setting $\Gamma(1) = 0$ and $\Gamma(0) = \hat{u}$, where $0 < \hat{u} < 1$, Γ has an inverse $\hat{y} \in C^1([0, \hat{u}], [0, 1])$, which can be extended trivially to a function $\gamma \in C^0([0, 1], [0, 1])$ where

$$y(u) = \begin{cases} \hat{y} & u \in [0, \hat{u}], \\ 0 & u \in [\hat{u}, 1]. \end{cases}$$

Moreover, $\Gamma'(v) = 0$ implies Γ has a local maximum.

Alternatively, if we make the substitution $z = x - ct$, but do not assume that the solution is a travelling wave, *i.e.* if we assume (u, v) is of the form $(u(z, t), v(z, t))$, then (1) becomes

$$u_t - cu_z - u_{zz} = f(u, v),$$
$$v_t - cv_z - \epsilon^2 v_{zz} = g(u, v). \qquad (7)$$

Any solutions $u(z, t)$, $v(z, t)$ of (2) provides a solution $u(x, t)$, $v(x, t)$ of (1). Moreover, travelling waves are steady states of this system ($ut = vt = 0$). Below, we will apply sub-solution and super-solution techniques (from [7]) to system (2).

In order to employ the comparison solutions mentioned above, the following definitions and results given in [7] are required.

Definition 1. *A function* $Q(u, v) := (f(u, v), g(u, v))$ *is called* quasimonotone nonincreasing*in* $R^+ \times R^+$ *if both* $f(u, v)$ *and* $g(u, v)$ *are* quasimonotone nonincreasing*for* $(u, v) \in R^+ \times R^+$, *i.e.*
$\partial f/\partial v \leq 0$, $\partial g/\partial u \leq 0$ *for* $(u, v) \in R^+ \times R^+$.

Definition 2. *If* $Q(u, v) := (f(u, v), g(u, v))$ *is quasimonotonenonincreasing in* $R^+ \times R^+$, *then a pair of functions* $\widehat{\omega} = (\widehat{u}, \widehat{v})$ *and* $\widetilde{\omega} = (\widetilde{u}, \widetilde{v})$ *are called ordered sub-solution and super-solution of* (2) *if they satisfy the relation* $\widetilde{\omega} > \widetilde{v}$ *and if*

$$\widetilde{u}_t - \widetilde{c}\widetilde{u}_z - \widetilde{u}_{zz} - f(\widetilde{u}, \widetilde{v}) \geq 0 \geq \widetilde{u}_t - \widetilde{c}\widetilde{u}_z - \widetilde{u}_{zz} - f(\widetilde{u}, \widetilde{v})$$
$$\widetilde{v}_t - \widetilde{c}\widetilde{v}_z - \in^2 \widetilde{v}_{zz} - g(\widetilde{u}, \widetilde{v}) \geq 0 \geq \widetilde{v}_t - \widetilde{c}\widetilde{v}_z - \in^2 \widetilde{v}_{zz} - g(\widetilde{u}, \widetilde{v}) \qquad (8)$$

Note that in this section it is actually combinations of sub-solution and supersolutionthat are useful in obtaining our results. We therefore make the followingdefinition.

Definition 3. *We say that* $(\underline{u}, \bar{v})$ *is a* right solution *of* (2) *iff* $\underline{u}$ *is a sub- solutionand* $\bar{v}$ *is a super-solution. Similarly, we define a* left solution *to be apair* $(\bar{u}, \underline{v})$ *where* $\bar{u}$ *a super-solution and* $\underline{v}$ *is is a sub-solution. Hence, a directconsequence from the Definition 2 in* [7] *we have that that* $(\underline{u}, \bar{v})$ *is a right solutionof (*2) *iff it satisfies*

$$\widetilde{u}_t - \widetilde{c}_r\widetilde{u}_z - \widetilde{u}_{zz} - f(\widetilde{u}, \widetilde{v}) \leq 0$$
$$\widetilde{v}_t - \widetilde{c}_r\widetilde{v}_z - \in^2 \widetilde{v}_{zz} - g(\widetilde{u}, \widetilde{v}) \geq 0 \qquad (9)$$

and $(\bar{u}, \underline{v})$ *is a left solution of* (2) *iff it satisfies*

$$\widetilde{u}_t - \widetilde{c}_l\widetilde{u}_z - \widetilde{u}_{zz} - f(\widetilde{u}, \widetilde{v}) \geq 0$$
$$\widetilde{v}_t - \widetilde{c}_l\widetilde{v}_z - \in^2 \widetilde{v}_{zz} - g(\widetilde{u}, \widetilde{v}) \leq 0 \qquad (10)$$

We know that if we have a sub solutionand super-solution, then a solution must lie between that sub-solutionand super-solution. In the CLV case, then in the bound (1), we have

$$K := \sup_{0<v<1} \delta(1-v) = \delta \quad and \quad L := \sup_{0<u<1} (1-u) = 1$$

In this chapter, we will show that these values are optimal for the form of leftand right solutions introduced in [4].

In order to do this we reformulate the left and right solutions.

i) ***Right solution***:

Following [3] we define the right solution:

$$\overline{c} = c_r\,,$$

$$\underline{u} = u_m \max\{1 - e^{Fz}, 0\},$$

$$\overline{v} = \begin{cases} \min\{v_m e^{T_z}, 1\}, & z \le \dfrac{M}{\in}, \\ \dfrac{1}{2}\overline{v}(\dfrac{M}{\varepsilon})(1 + e^{S(z-\frac{M}{\varepsilon})}), & z > \dfrac{M}{\in} \end{cases}$$

where $z \in \mathbb{R}$, $F<0$, $T<0$, $S<0$, $M>0$ and $c_r \ge 0$ are constants to be defined later and $\epsilon \ge 0$, (see Figure 1).

We seek values of F, T, S and M, that give the smallest value of c_r, *i.e.* the sharpest right bound on the wave speed. First, note that $\underline{u}_t = \overline{v}_t = 0$. Hence we must choose u_m and v_m so that $f(\underline{u}, \overline{v}) \ge 0$.

First consider the equation for u. For $z<0$, $\underline{u} \equiv 0$. Therefore,

$$c_r \underline{u}_z + \underline{u}_{zz} + f(\underline{u}, \overline{v}) = 0 = \underline{u}_t.$$

In the case $z>0$,

$$\underline{u} = u_m(1 - e^{Fz})$$

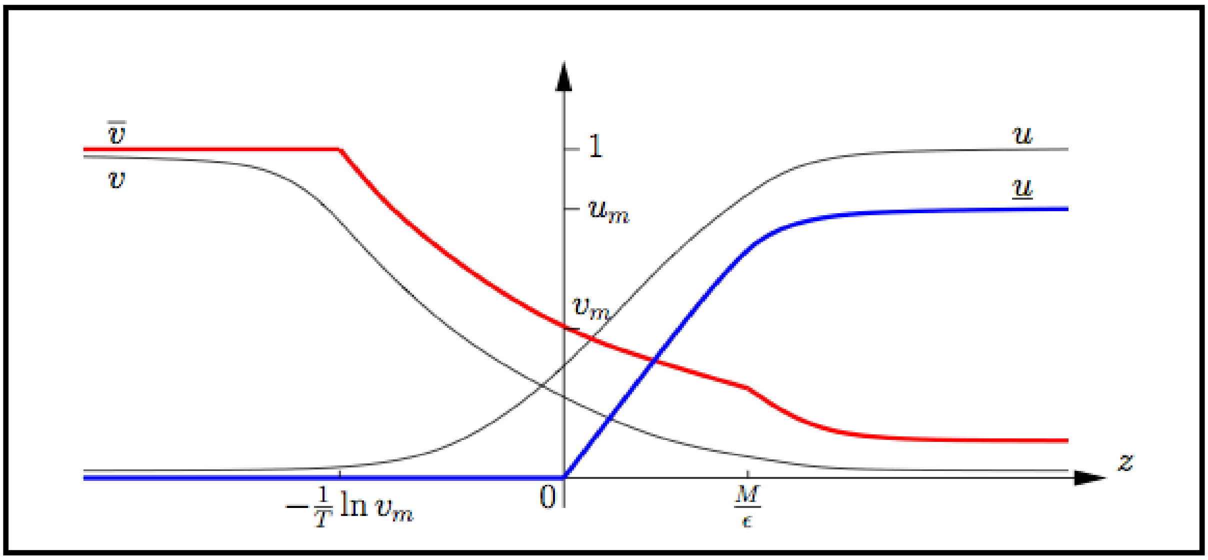

Figure 1: The right solution ($\underline{u}$, $\overline{v}$), shown in (blue, red) and the solution(u, v) of system (2) shown in black.

By

$$f(\underline{u}, \overline{v}) = u_m(1 - u_m - \frac{\alpha}{2} v_m e^{\frac{TM}{\epsilon}}),$$

it follows that $f(\underline{u}, \overline{v}) \ge 0$ because $v_m \in (0, \dfrac{1}{\alpha}(1 - u_m))$. Therefore,

$$c_r\underline{u}z + \underline{u}_{zz} + f(\underline{u}, \bar{v}) \geq c_r\underline{u}_z + \underline{u}_{zz}$$

$= cr - u_m F e^{Fz}] + [-u_m F^2 e^{Fz}]$
$= -u_m F e^{Fz}[c_r + F]$
$\geq 0 = \underline{u}_t,$
iff

$$c_r \leq -F. \quad (11)$$

Thus, we ensured that the first relation in (4) is satisfied. We can obtain $f(\underline{u}, \bar{v}) \geq 0$ by taking $u_m \in (1/\beta,\ 1)$ and $v_m \in (0,\ \frac{1}{\alpha}(1 - u_m))$.

Next we check that upon setting $\bar{v} = \frac{1}{2} v_m (1 + e^{\frac{TM}{\varepsilon}})$ for large positive z, that

$$\lim_{z\to\infty} \bar{v}(z) \in \left(0, \frac{1}{\alpha}(1-u_m)\right) \geq \frac{1}{\beta},$$

Fix F and choose $M > 0$ sufficiently large, such that

$$\underline{u}\left(\frac{M}{\varepsilon}\right) = u_m\left(1 - e^{\frac{FM}{\in}}\right) \geq \frac{1}{\beta}. \quad (12)$$

Hence, for an appropriately chosen u_m and an arbitrarily chosen F, we require M to be chosen such that

$$M \leq \frac{\in}{F}\ln\left(\frac{\beta u_m - 1}{\beta u_m}\right) \quad (13)$$

We now consider the v-equation which ensures the second relation in (4) holds. For $z < -\frac{1}{T}\ln v_m$, $\underline{u} \equiv 0$ and $v \equiv 1$. Therefore,

$$c_r \bar{v}_z + \epsilon^2 v_{zz} + g(\underline{u}, \bar{v}_z) = 0\ \bar{v}_t.$$

For $z > \frac{M}{\in}$

$$\bar{v} = \frac{1}{2}\bar{v}\left(\frac{M}{\in}\right)\left(1 + e^{S(z - \frac{M}{\varepsilon}}\right) < v_m.$$

Since from equation (7), $u(z) \geq \underline{u}\left(\frac{M}{\varepsilon}\right) \geq \frac{1}{\beta}$ for $z \geq \frac{M}{\in}$.

By

$$g(\underline{u}, \bar{v}) = \frac{\delta}{2} v_m e^{\frac{TM}{\varepsilon}}\left(1 - \frac{1}{2} v_m e^{\frac{TM}{\varepsilon}} - \beta u_m\right),$$

it follows that $g(\underline{u}, \bar{v}) \leq 0$ for any $\bar{v} \geq 0$ because $u_m \in (\frac{1}{\beta},\ 1)$. Therefore,

$$c_r\bar{v}_x+\varepsilon^2\bar{v}_{zz}+g(\underline{u},\bar{v})\le c_r\bar{v}_x+\in^2\bar{v}_{zz}$$
$$=c_r\left[\frac{1}{2}\bar{v}\left(\frac{M}{\varepsilon}\right)Se^{S\left(z-\frac{M}{\varepsilon}\right)}\right]+\varepsilon^2\left[\frac{1}{2}\bar{v}\left(\frac{M}{\varepsilon}\right)S^2e^{S\left(z-\frac{M}{\varepsilon}\right)}\right]$$
$$=\frac{1}{2}\bar{v}\left(\frac{M}{\varepsilon}\right)Se^{S\left(z-\frac{M}{\varepsilon}\right)}\left[c_r+\varepsilon^2S\right]$$
$$\le 0=\bar{v}_t,$$

iff

$$c_r\le -\epsilon^2 S. \quad (14)$$

In the case $z<\left(\frac{M}{\varepsilon}\right)$ and $\bar{v}<1$, *i.e.* $-\frac{1}{T}\ln v_m<z<\left(\frac{M}{\varepsilon}\right)$, we have that

$$\bar{v}=v_m e^{Tz}.$$

Note also that for all $0\le u,\ v\le 1$, $g(u,v)\le \delta v$. So, in this case,

$$c_r\bar{v}_x+\varepsilon^2\bar{v}_{zz}+g(\underline{u},\bar{v})\le c_r\bar{v}_x+\in^2\bar{v}_{zz}+\delta\bar{v}$$
$$=c_r\left[v_mTe^{Tz}\right]+\varepsilon^2\left[v_mT^2e^{Tz}\right]+\delta\left[v_mTe^{Tz}\right]$$
$$=v_mTe^{Tz}\left[\varepsilon^2T^2+c_rT+\delta\right]$$
$$\le 0=\bar{v}_t,$$

iff $\epsilon^2T^2+c_rT+\delta\le 0$. By solving this quadratic equation, we get $T_-\le T\le T_+$ where

$$T_\pm=\frac{1}{2\varepsilon^2}\left[-c_r\pm\sqrt{c_r^2-4\varepsilon^2\delta}\right]$$

We need $T_\pm$ real so the smallest c_r can be is $2\in\sqrt{\delta}$ at which point

$$T_+=T_-=-\frac{c_r}{2c^2}=-\frac{\sqrt{\delta}}{\varepsilon}.$$

Also, as from (6) and (9) we require $c_r\le\min\{-F,-\epsilon^2S\}$, we may set $-F=-\epsilon^2S=c_r$, *i.e.* $F=-2\epsilon\sqrt{\delta}$ and $S=-\frac{2\sqrt{\delta}}{\varepsilon}$. Finally, M is given by (8) and hence we find that the choice of u_m and v_m and thus M does not affect thewave speed. Hence, we have shown that the right solution

$$c_r=2\in\sqrt{\delta},$$
$$\underline{u}=u_m\max\left\{1-e^{-2\varepsilon\sqrt{\delta}z},0\right\},$$
$$\bar{v}=\begin{cases}\min\left\{v_me^{-\frac{\sqrt{\delta}}{\varepsilon}z},1\right\}, & z\le\frac{M}{\varepsilon},\\ \frac{1}{2}\bar{v}\left(\frac{M}{\varepsilon}\right)\left(1+e^{-2\frac{\sqrt{\delta}}{\varepsilon}(z-\frac{M}{\varepsilon})}\right), & z>\frac{M}{\varepsilon},\end{cases}$$

is in some sense optimal: $c_r$is an upper bound for the wave speed of travellingwaves of (1) and $2 \in \sqrt{\delta}$ is the lowest upper bound in this case. Note that for anyϵ >0 the profile $(\underline{u}, \overline{v})$ remains in both components at a positive distance from$S_+ = (1, 0)$ for all $z > 0$.

Furthermore, $(\underline{u}, \overline{v})(z) = (0, 1)$ for all $z < 0$. Thus,any initial data (u_o, v_o) of problem (2) with $(u_o, v_o)(z) \rightarrow (1, 0)$ as $z \rightarrow \infty$

can be shifted to be comparable with $(\underline{u}, \overline{v})$. This implies that no travelling wave solution of (3) can travel at speeds faster than the comparison solution, *i.e.*$c_\epsilon \leq c_r = 2\epsilon\sqrt{\delta}$.

ii) ***Left solution***. Following [4] we define the left solution:

$$\underline{c} = c_l,$$
$$\overline{u} = \begin{cases} \min\{u_m e^{Rz}, 1\}, & z \geq M, \\ \frac{1}{2}\overline{u}(-M)\left(1 + e^{P(z-M)}\right), & z < M, \end{cases} \qquad (15)$$
$$\underline{v} = v_m \max\{1 - e^{Qz}, 0\},$$

as shown in Figure 2, where $R > 0$, $P > 0$, $Q > 0$, $M > 0$ and $c_l \leq 0$ are constants to be determined. We seek values of$R$, P, Q and M, that give thesmallest value ofc_l, *i.e.* the sharpest left bound on the wave speed.

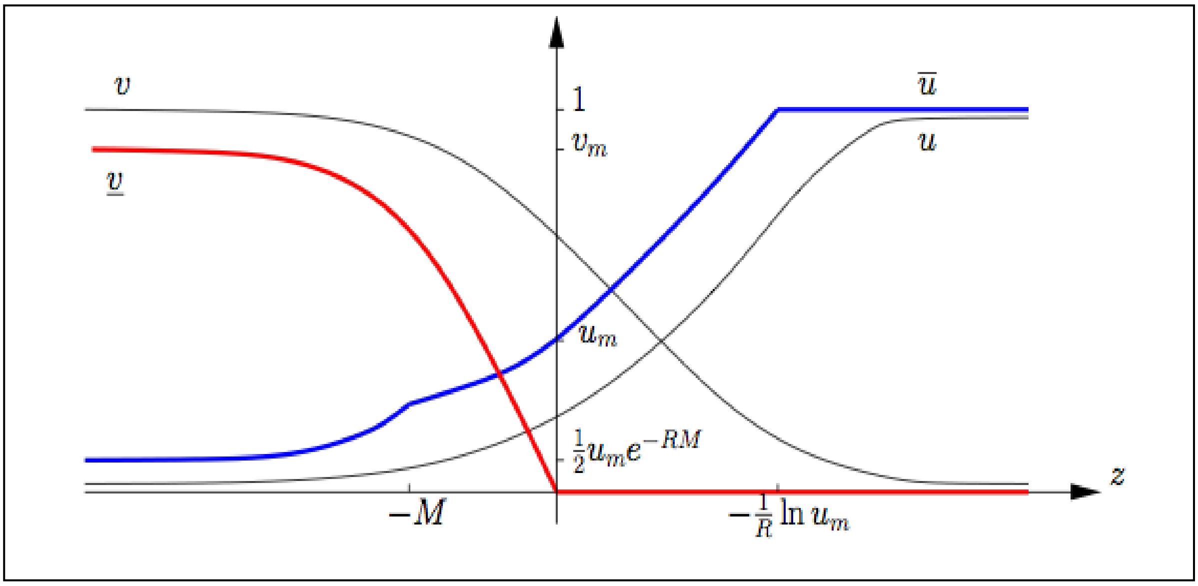

Figure 2: The left solution $(\overline{u}, \underline{v})$, shown in (blue, red) and the solution $(u, v)$of system (2) shown in black.

First, note that $\overline{u}_t = \underline{v}_t = 0$. Hence we must choose $u_m$and $v_m$so that $g(\overline{u},\underline{v}) \geq 0,$. We have $g(\overline{u},\underline{v}) = \delta v_m(1 - v_m - \frac{\beta}{2}u_m e^{-RM})$, it follows that $g(\overline{u},\underline{v}) \geq 0,$ by taking $v_m \in (\frac{1}{\alpha}, 1)$ and $um \in (0, \frac{1}{\beta}(1 - v_m))$.

Next we check that upon setting$u = \frac{1}{2}u_m(e^{-RM})$for large negative z, that

$$\lim_{z\to\infty}\overline{u}(z) \in \left[0, \frac{1}{\beta}(1 - v_m)\right].$$

Fix Q and choose $M > 0$ sufficiently large, such that

$$\underline{v}(-M) = v_m\left(1 - e^{-QM}\right) \geq \frac{1}{\alpha}. \qquad (16)$$

Hence, for an appropriately chosen $vm$and an arbitrarily chosen Q, we require$M$ to be chosen such that

$$M \geq \frac{1}{Q}\ln\left(\frac{\alpha v_m}{\alpha v_m - 1}\right) \qquad (17)$$

Equations (10) describe a left solution if we can find the constants$R$, P, Q and$M$ satisfy system (10) and $M > 0$ such that$(\overline{u}, \underline{v}, c_l)$.

Next we ensure that the first relation in (5) is satisfied. First consider the u-equation. For$z > -\frac{1}{R}\ln u_m$, $\overline{u} \equiv 1$ and $\underline{v} \equiv 0$. Therefore,

$$c_l\overline{u}_z + \overline{u}_{zz} + f(\overline{u},\underline{v}). = 0 = \overline{u}_t.$$

If $z < -M$ and M is sufficiently large, we have

$$\overline{u} = \frac{1}{2}\overline{u}(-M)\left(1 + e^{P(z+M)}\right) < u_m. \qquad (18)$$

Since$f(\overline{u},\underline{v}) = \overline{u}(1 - \overline{u} - \alpha\underline{v})$ and $\underline{v}(z) \geq \underline{v}(-M) \geq \frac{1}{\alpha}$ from equation (11) for $z \leq -M$, so $f(\overline{u}, \underline{v}) = \frac{1}{2}u_m e^{-RM}(1 - 12u_m e^{-RM} - \alpha v_m) \leq 0$ because $v_m \in (1/\alpha, 1)$.

We deduce that

$$\begin{aligned} c_l\overline{u}_{zz} + f(\overline{u},\underline{v}) &\leq c_l\overline{u}_z + \overline{u}_{zz} \\ &= c_l\left[\frac{1}{2}\overline{u}(-M)Pe^{P(z+M)}\right] + \left[\frac{1}{2}\overline{u}(-M)P^2e^{P(z+M)}\right] \\ &= \frac{1}{2}\overline{u}(-M)Pe^{P(z+M)}\left[c_l + P\right] \\ &\leq 0 = \overline{u}_t, \end{aligned}$$

iff

$$c_l \le -P. (17)$$

For $z > -M$ and $\bar{u} < 1$, it follows that
$\bar{u} = \min\{u_m e^{Rz}, 1\} = u_m e^{Rz}$.
Also, for all $0 \le u, v \le 1, f(u, v) \le u$. Hence in this case,

$$\begin{aligned} c_l \bar{u}_z + \bar{u}_{zz} + f(\bar{u}, \underline{v}) &\le c_l \bar{u}_z + \bar{u}_{zz} + \bar{u} \\ &= c_l \left[\frac{1}{2} \bar{u}_m (-M) P e^{P(z+M)} \right] + \left[u_m R^2 e^{Rz} \right] + \left[u_m e^{Rz} \right] \\ &= u_m e^{Rz} \left[R^2 + c_l R + 1 \right] \\ &\le 0 = \bar{u}_t, \end{aligned}$$

iff $R^2 + c_l R + 1 \le 0$. Solving this quadratic inequality, yields $R_- \le R \le R_+$
where

$$R_\pm = \frac{1}{2}\left[-c_l \pm \sqrt{c_l^2 - 4} \right].$$

As we are seeking $c_l \le 0$, this requires $c_l \le -2$. The sharpest left solution is therefore $c_l = -2$, and in this case $R = 1$.
Now consider the v-equation to ensure that the second relation (2.5) is satisfied. For $z > 0$, $v \equiv 0$ so

$$c_l \underline{v}_z + \epsilon^2 \underline{v}_{zz} + \delta \underline{v}(1 - \bar{v} - \beta \bar{u}) = 0 = \underline{v}t.$$

For $z < 0$, we have

$$\underline{v} = v_m(1 - e^{Qz}).$$

By the choice of u_m and v_m, $g(\bar{u}, \underline{v}) = \delta v_m (1 - v_m - \frac{\beta}{2} u_m e^{-RM}) \ge 0$, because $u_m \in (0, \frac{1}{\beta}(1 - v_m))$

Hence, we get

$$\begin{aligned} c_l \underline{v}_z + \in^2 \underline{v}_{zz} + f(\bar{u}, \underline{v}) &\ge c_l \underline{v}_z + \in^2 \underline{v}_{zz} \\ &= c_l \left[-v_m Q e^{Qz} \right] + \in^2 \left[-v_m Q^2 e^{Qz} \right] \\ &= -v Q_m e^{Qz} \left[c_l + \varepsilon^2 Q \right] \\ &\le 0 = \bar{v}_t, \end{aligned}$$

iff $c_l \le -\epsilon^2 Q$. (18)

From (14) and (15), we now establish an upper bound for cl, we therefore require

$$c_l \le \min\{-P, -\epsilon 2Q\}$$

Therefore, the sharpest left solution for the wave speed that can be obtained with this form of left solution is $cl = -2$. Hence, we have shown that the left solution

$$c_r = 2,$$

$$\overline{u} = \begin{cases} \min\{u_m e^z, 1\}, & z \le -M, \\ \frac{1}{2}\overline{u}(-M)\left(1 + e^{-2(z+M)}\right), & z > -M, \end{cases}$$

$$\underline{v} = \max\left\{v_m(1 - e^{\frac{2}{\varepsilon^2}z}), 0\right\},$$

is in some sense optimal, where M is given by (12).

Hence, cl is a lower bound for the wave speed of travelling waves of (1) and−2 is the largest lower bound in this case. Since u and v are uniformly boundedaway from 0 and 1, respectively, and $((\overline{u}, \underline{v})(z)) = (1, 0)$ for $z > 0$

we can alwaysshift initial data $(u0, v0)$ of problem (2.2) with $(u0, v0)(z) \to (1, 0)$ for $z \to -\infty$to be comparable with the left solution. We conclude $c_\epsilon \ge c_l = -2$.

2. Conclusion

In this chapter, we demonstrated that, for the CLV case at least, the bounds

of the wave speed given in [8] are optimal for the given left and right solutionpair. We have tried to find other right and left solutions in order to improve thebounds on the wave speed that is stated in Theorem 1. However, we could not find an alternative that would allow for the kind of explicit calculations done above. Notwithstanding this, the fact that the wave speed must be ≤ 0 in the limit as $\epsilon \to 0$ and that$c_\epsilon \le C.\epsilon$ for some positive constant C for $\epsilon > 0$, will be very useful in the results to come.

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