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On The Solution Set of Euler's Systems Using Elzaki Transform

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ABSTRACT

In this work, general formula of set solutions for system of Euler's equation in dimension m are derived by using Elzaki transformation. Also, supported examples are presented utilizing formula derived.

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1. Introduction

2. Differential equation have been used to some degree in every branch of applied mathematics, Physics and engineering [8]. The systems of ordinary differential equations can be solved by integral transformations which are considered effective and accurate ways to find solutions of mathematical systems. There are many integral transformations, for example Laplace, Elzaki, Temem and SEE transformation [1- 6].
3. In 1990 Gamage has introduced new transform namely Sumudu transform which is similar to Laplace transform [7]. He introduced this transform to solve differential equations and control engineering problems [10], its resolved with initial condition, but its failed resolved without any initial condition [3]. Anew integral transform namely Elzaki transform was introduced by Tarig Elzaki in 2010 [11]. Elzaki transform is a modified form of Sumudu and Laplace transforms [12]. Elzaki transformation is a new integral transform, it is define by:
4.
5. $E[\eta(x)] = T(w) = w\int_0^\infty \eta(x) e^{-\frac{x}{w}}\,dx\,,\quad x \geq 0,\quad \varphi_1 \leq w < \varphi_2.$
6. Furthermore, the set μ is define
7. $\mu = \left\{\eta(x): \exists M_1, \varphi_1, \varphi_2 > 0, |\eta(x)| < e^{\frac{|x|}{\varphi_j}}\; if\; x \in (-1)^j \times [0, \infty)\right\}.$

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8. where the constant M_1 must be finite, but φ_1 *and* φ_2 may be finite or infinite.
9. In this work, general solutions of first order system of Euler's equations in dimension m are derived through converted to system of constant coefficients from using some assumption. In addition, some examples are solved by utilizing these formula.

2. Preliminaries

Some basic definitions and theorems are presented in this section

2.1 Theorem [9] : Let T(w) is the Elzaki transformationη(t), then

i. $E[\eta`(t)] = \frac{T(w)}{w} - w\eta(0)$
ii. $E[\eta``(t)] = \frac{T(w)}{w^2} - w\eta`(0) - \eta(0)$
iii. $E\left[\eta^{(n)}(t)\right] = \frac{T(w)}{w^n} - \sum_{k=0}^{n-1} w^{2-n+k}\eta^k(0)$

where

$$E[\eta`(t)] = w\int_0^\infty \eta`(t)e^{-\frac{t}{w}}\,dx = \frac{T(w)}{w} - w\,\eta(0)\ .$$

2.2 The n-dimensional first order Euler's System

Consider the non-homogenous Euler's system of n-th order

$$\begin{pmatrix} x^n \frac{d^n\eta_1(x)}{dx^n} \\ \vdots \\ x^n \frac{d^n\eta_m(x)}{dx^n} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \\ \vdots \\ \eta_m(x) \end{pmatrix} + \begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_m(x) \end{pmatrix} \qquad \ldots(1)$$

where $a_{ij}: i,j = 1,\ldots,m$ are constants coefficient and $\eta_1(x),\ldots,\eta_n(x), b_1(x),\ldots,b_n(x)$ are functions of x.

2.3 Converting the n-order of Euler's system to a system with constant coefficients ."

System (1) can be converted to system with constants coefficients by using the assumption

$$x = e^t \qquad \ldots(2)$$

Which implies that $\ln x = t$

Also, derivative (2) with applying on $\eta(x)$, get :

$$\left.\begin{aligned} x^nD^n\eta(x) &= [D^n - \textstyle\sum_{k=1}^{n-1} D^{n-k}]\eta(t) \\ x^{n-1}D^{n-1}\eta(x) &= [D^{n-1} - \textstyle\sum_{k=1}^{n-2} D^{n-k}]\eta(t) \\ &\vdots \\ &\vdots \\ xD\eta(x) &= D\eta(t) \end{aligned}\right] \ldots(3)$$

3. The main result

The main result for converting system of order n in the following theorem

Theorem 1:

If $\eta(x) \in \mathbb{R}^m$ and has n-thderivative, then the system

$$x^n\overline{\eta^{(n)}}(x) = A\overline{\eta}(x) + \overline{B(x)}, \qquad \eta(x) \in \mathbb{R}^m \ldots(4)$$

Can be converted to

$$\overline{\eta^{(n)}(t)} = H_1\overline{\eta^{(n-1)}(t)} + H_2\overline{\eta^{(n-2)}(t)} + \cdots + A\overline{\eta}(t) + \overline{B(t)}, \quad \eta(t) \in \mathbb{R}^m \ldots.(5)$$

Where $A, H_1, H_2, \ldots$ is a matrix of $m \times m$ and B is a vector of $m \times 1$.

Proof:

We will prove only when n = 1 , 2 and using mathematical induction it is possible to complete the proof

Let $n = 1$, then system (4) directly from utilizing assumptions (2) and (3), yield:

$$\begin{pmatrix} \frac{d\eta_1(t)}{dt} \\ \vdots \\ \frac{d\eta_m(t)}{dt} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} \eta_1(t) \\ \vdots \\ \eta_m(t) \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_2 \end{pmatrix} \qquad \ldots(6)$$

Which represent system of dimension m with constants coefficients and the solution can be obtained by using Elzaki transform.

If $n = 2$, then

$$\begin{pmatrix} x^2\frac{d^2\eta_1(x)}{dx^2} \\ \vdots \\ x^2\frac{d^2\eta_m(x)}{dx^2} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} x\frac{d\eta_1(x)}{dx} \\ \vdots \\ x\frac{d\eta_m(x)}{dx} \end{pmatrix} +$$

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{pmatrix} \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \\ \vdots \\ \eta_m(x) \end{pmatrix} + \begin{pmatrix} c_1(x) \\ c_2(x) \\ \vdots \\ c_m(x) \end{pmatrix}, \ldots(7)$$

Where $c_j; j = 1, \ldots, m$ are functions of t or constants.

From (2) and (3)the above system converted as :

$$\begin{pmatrix} \ddot{\eta}_1(t) - \dot{\eta}_1(t) \\ \vdots \\ \ddot{\eta}_m(t) - \dot{\eta}_m(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} \dot{\eta}_1(t) \\ \vdots \\ \dot{\eta}_m(t) \end{pmatrix} +$$

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{pmatrix} \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \\ \vdots \\ \eta_m(t) \end{pmatrix} + \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_m(t) \end{pmatrix}, \ldots(8)$$

Therefore

$$\eta_1^{``}(t) = (1 + a_{11})\eta_1^{`}(t) + a_{12}\eta_2^{`}(t) + \cdots + a_{1m}\eta_m^{`}(t) + b_{11}\eta_1(t) + \cdots + b_{1m}\eta_m(t) + c_1(t)$$

$$\vdots \qquad\qquad \vdots$$

$$\eta_m^{``}(t) = a_{m1}\eta_1^{`}(t) + a_{m2}\eta_2^{`}(t) \ldots (1 + a_{mm})\eta_m^{`}(t) + b_{1m}\eta_1(t) + \cdots + b_{mm}\eta_m(t) + c_m(t)$$

Which represent system with constants coefficients and the proof is completed.

4. Solving System of Differential Equations by Using Elzaki Transform

Now, taking Elzaki transform to both sides of system (7) or (8) can get the set solution. In this section system (8) dependent to explain the method as following:

$$\frac{E(\eta_1(t))}{W^2} - W\eta_1^{`}(0) - \eta_1(0)$$
$$= (1 + a_{11})\left[\frac{E[\eta_1(t)]}{W} - W\eta_1(0)\right] + a_{12}\left[\frac{E[\eta_2(t)]}{W} - W\eta_2(0)\right] + \cdots + a_{1m}\left[\frac{E[\eta_m(t)]}{W} - W\eta_m(0)\right]$$
$$+ b_{11}E[\eta_1(t)] + \cdots + b_{1m}E[\eta_m(t)] + c_1(t)W^2$$

$$\vdots$$

$$\left[\frac{E[\eta_m(t)]}{W^2} - W\eta_m^{`}(0) - \eta_m(0)\right] = a_{m1}\left[\frac{E[\eta_1(t)]}{W} - W\eta_1(0)\right] + a_{m2}\left[\frac{E[\eta_2(t)]}{W} - W\eta_2(0)\right] + \cdots +$$

$$(1 + a_{mm})\left[\frac{E[\eta_m(t)]}{W} - W\eta_m(0)\right] + b_{m1}E[\eta_1(t)] + \cdots + b_{mm}E[\eta_m(t)] + c_m(t)W^2$$

Simple calculation

$$\left(\frac{1}{W^2} - \frac{1 + a_{11}}{W} - b_{11}\right)E[\eta_1(t)]$$
$$= a_{12}\left[\frac{E[\eta_2(t)]}{W} - W\eta_2(0)\right] + \cdots + a_{1m}\left[\frac{E[\eta_m(t)]}{W} - W\eta_m(0)\right]$$
$$+ \quad b_{12}E[\eta_2(t)] + \cdots + b_{1m}E[\eta_m(t)] + c_1W^2 + W\eta_1^{`}(0) + \eta_1(0)$$

$$\vdots \qquad\qquad \vdots$$

$$\left(\frac{1}{W^2} - \frac{1 + a_{mm}}{W} - b_{mm}\right)E[\eta_m(t)]$$
$$= a_{m1}\left[\frac{E[\eta_1(t)]}{W} - W\eta_1(0)\right] + a_{m2}\left[\frac{E[\eta_2(t)]}{W} - W\eta_2(0)\right] + \cdots \qquad + b_{m1}E[\eta_1(t)] + \cdots$$
$$+ b_{mm-1}E[\eta_{m-1}(t)] + C_mW^2 + W\eta_m^{`}(0) + \eta_m(0)$$

Using Cramer's rule to find $E[\eta_1(t)], \ldots, E[\eta_m(t)]$, then taking its inverse, we get the solutionof (8).

5. Application

In this section, some supported examples can be solved when $n = 1$ and m = 2, $b_1 = b_2 = 0$ as following:

$$\left.\begin{array}{ll} x\eta`(x) = 3\eta(x) - Z(x) & ,\eta(1) = \frac{1}{5} \\ xZ`(x) = 4\eta(x) - Z(x) & ,z(1) = \frac{1}{2} \end{array}\right] \quad (12)$$

Solving the aforementioned system using Theorem 1 results in:

$$\begin{pmatrix} \dot{\eta}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} \eta(t) \\ Z(t) \end{pmatrix} \quad (13)$$

And the initial conditions utilizing (2), get $\begin{pmatrix} \eta(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{2} \end{pmatrix}$

Taking Elzaki transform for (13)

$$\frac{E[\eta(t)]}{W} - W\eta(0) = 3E[\eta(t)] - E[z(t)]$$

$$\frac{E[Z(t)]}{W} - W\eta(0) = 4E[\eta(t)] - E[z(t)]$$

Then

$$E[\eta(t)] - \frac{1}{5}W^2 = 3WE[\eta(t)] - WE[z(t)]$$

$$E[z(t)] - \frac{1}{2}W^2 = 4WE[\eta(t)] - WE[z(t)]$$

Simplification

$$(1 - 3W)E[\eta(t)] + WE[z(t)] = \frac{1}{5}W^2$$

$$-4WE[\eta(t)] + (1 + W)E[z(t)] = \frac{1}{2}W^2$$

Using Gramer's rule

$$\Delta = \begin{vmatrix} 1 - 3W & W \\ -4W & 1 + W \end{vmatrix} = (1 - 3W)(1 + W) + 4W^2 = 1 - 2W + W^2$$

Moreover

$$E[\eta(t)] = \frac{1}{\Delta}\begin{vmatrix} \frac{1}{5}W^2 & W \\ \frac{1}{2}W^2 & 1 + W \end{vmatrix}, \quad \text{And} E[z(t)] = \frac{1}{\Delta}\begin{vmatrix} 1 - 3W & \frac{1}{5}W^2 \\ -4W & \frac{1}{2}W^2 \end{vmatrix}$$

Then, taking inverse of Elzaki transform:

$$\eta(t) = \frac{1}{5}e^{t} - \frac{1}{10}te^{t}$$

$$z(t) = \frac{1}{2}e^{t} - \frac{1}{5}te^{t}$$

Represent the set solution of system (13). Also, using (2) again can be found the set solution of system (12)

$$\eta(x) = \frac{1}{5}x - \frac{1}{10}x\ln x$$

$$z(x) = \frac{1}{2}x - \frac{1}{5}x\ln x$$

If the vector b_i is not equal to zero, the system is non homogeneous:

$$\left.\begin{array}{ll} x\eta`(x) = z(x) + x, & \eta(1) = 2 \\ xz`(x) = -z\eta(x) + 3z(x) + z \quad , & z(1) = 2 \end{array}\right] \qquad \ldots(14)$$

To solve the above system when $n = 1$and by Theorem 1, yield:

$$\begin{pmatrix}\eta(t)\\ z(t)\end{pmatrix}' = \begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}\begin{pmatrix}\eta(t)\\ z(t)\end{pmatrix} + \begin{pmatrix}e^{t}\\ 2\end{pmatrix} \qquad \ldots (15)$$

The initial condition by using (2) become $\begin{pmatrix}\eta(0)\\ z(0)\end{pmatrix} = \begin{pmatrix}2\\ 2\end{pmatrix}$

Taking Elzaki to (15):

$$\frac{E[\eta(t)]}{W} - W\eta(0) = E[z(t)] + E[e^{t}]$$

$$\frac{E[z(t)]}{W} - Wz(0) = -2E[\eta(t)] + 3E[z(t)] + E[2]$$

Then

$$E[\eta(t)] - WE[z(t)] = \frac{W^3}{1-W} + 2W^2$$

$$2W[\eta(t)] + (1-3W)E[z(t)] = 2W^3 + 2W^2$$

Using Gramer's rule

$$\Delta = \begin{vmatrix}1 & -W\\ 2W & 1-3W\end{vmatrix} = 1 - 3W + 2W^2 = (1-2W)(1-W)$$

Moreover

$$E[\eta(t)] = \frac{1}{\Delta}\begin{vmatrix}\frac{-W^3+2W^2}{(1-W)} & -W\\ 2(W^3+W^2) & 1-3W\end{vmatrix}, \quad \text{And} E[z(t)] = \frac{1}{\Delta}\begin{vmatrix}1 & \frac{-W^3+2W^2}{(1-W)}\\ 2W & 2(W^3+W^2)\end{vmatrix}$$

Taking inverse of Elzaki transform, get

$$\eta(t) = 2te^{t} + e^{t} + 1$$

$$z(t) = 2te^t + e^t$$

After using (2), we get the set solution of (14):

$$\eta(x) = 2x \ln x + x + 1$$

$$z(x) = 2x \ln x + 2x$$

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