



# Eventual Fitting Shadowing Property for Hyperbolic Dynamical Systems

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## ABSTRACT

Let  $\mathcal{L}: M \rightarrow M$  be a diffeomorphism map on a closed smooth manifold  $M$  for dimension  $n \geq 2$ . We explain in this work any chain transitive set of  $C^1$  generic diffeomorphism  $\mathcal{L}$ , if a diffeomorphism  $\mathcal{L}$  has another type of shadowing property is called, the eventual shadowing property on locally maximal chain transitive set, then  $\mathcal{L}$  is hyperbolic. In general, the eventual fitting shadowing property is not fulfilled in hyperbolic dynamical systems (satisfy in case  $L$  is Anosov diffeomorphism map). In this paper, several concepts were presented. These concepts can be re-examined on other important spaces, and their impact on finding dynamical characteristics that may be employed in solving some mathematical problems.

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## 1. Introduction

The idea of shadowing in the dynamical systems (DS) theory boils down to the question is it possible to approximate Pseudo-orbit (PO) of a given DS by true orbits?

Naturally, the answer of this question depends on the type of the approximation. Due to the unavoidable presence of various errors and perturbations in the modeling of DS, the question arises about the relationship between the asymptotic properties of the simulated system and the simulation results; this problem was first posed by D.V. Anosov (1967) as a key step to analysis of structural stability[1]. The researchers developed the theory and performed the

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computations in dynamics systems in [2],[3],[4]and[5]. The answer about of this problem, researchers used several types of shadowing property for proof hyperbolic, see[6],[7],[8],[9]and [10].

We demonstrate in this work by another type, we used the eventual fitting shadowing property (EFSP) on locally maximal chain transitive (LMCT), then the map is hyperbolic.

let  $Diff(M)$  be the space of diffeomorphisms of  $M$ , where  $M$  is closed smooth manifold with  $\dim M \geq 2$  endowed with the  $C^1$  topology. The distance on  $M$  induced from a Riemannian metric  $\|\cdot\|$  on the tangent bundle  $TM$ . A closed  $\mathcal{L}$ -invariant set  $\mathfrak{B}$  admits a dominated splitting for  $\mathcal{L}$  if the tangent bundle  $T_{\mathfrak{B}}M$  has a continuous  $D\mathcal{L}$  invariant splitting  $E \oplus F$  and  $\exists C > 0, 0 < \mu < 1, \forall w \in \mathfrak{B}$  and  $\kappa \geq 0$ , then

$$\|D\mathcal{L}^\kappa|_{E(w)}\| \cdot \|D\mathcal{L}^{-\kappa}|_{F(\mathcal{L}^\kappa(w))}\| \leq C\mu^\kappa$$

$\mathfrak{B}$  is hyperbolic for  $\mathcal{L}$  if the tangent bundle  $T_{\mathfrak{B}}M$  has a  $D\mathcal{L}$  invariant splitting  $E^s \oplus F^u$  and  $\exists C > 0, 0 < \mu < 1, \forall w \in \mathfrak{B}$  and  $\kappa \geq 0$ , then

$$\|D_w \mathcal{L}^\kappa|_{E_w^s}\| \leq C\mu^\kappa \text{ and } \|D_w \mathcal{L}^{-\kappa}|_{E_w^u}\| \leq C\mu^\kappa$$

If  $\mathfrak{B} = M$ , then  $\mathcal{L}$  is called Anosov. In this paper, we assume that a chain transitive set  $\mathcal{C}(\mathcal{L})$  is nontrivial ( $\mathcal{C}(\mathcal{L})$  is not reduced to orbit). Denote a hyperbolic periodic point at  $w$  by  $Per_h(\mathcal{L})$  of  $\mathcal{L}$ .

## 2. The definitions and important results in the research:

### **Definition 2.1.[3]**

Let  $\mathcal{L} \in Diff(M)$ ,  $\alpha > 0$ , a sequence of points  $\{y_\kappa\}_{\kappa=a}^b, (-\infty \leq a < b \leq \infty)$  in  $M$  is said to be a  $\alpha$ - pseudo-orbit of  $\mathcal{L}$  if

$$d(\mathcal{L}(y_\kappa), (y_{\kappa+1})) < \alpha, \forall a \leq \kappa \leq b - 1.$$

We called  $\mathcal{L}$  has **shadowing property** on  $\mathfrak{B}$  if  $\forall \beta > 0, \exists \alpha > 0$  s.t.

$\forall \alpha$ -pseudo-orbit  $\{y_\kappa\}_{\kappa \in \mathbb{Z}} \subset \mathfrak{B}, \exists z \in M$  s.t.  $d(\mathcal{L}^\kappa(z_\kappa), (y_\kappa)) < \beta, \forall \kappa \in \mathbb{Z}$ .

### **Definition 2.2.**

Let  $(\mathfrak{S}, d)$  be a metric space and  $\mathcal{L}: \mathfrak{S} \rightarrow \mathfrak{S}$  is called have the **eventual fitting shadowing property** (EFSP) if  $\forall \beta > 0, \exists \alpha > 0$ , s.t.  $\forall \{y_\kappa\} \in \mathfrak{S}, \kappa \in \mathbb{Z}$  be sequence  $\exists \aleph = \aleph(\alpha) \in \mathbb{N}$  s.t.  $\forall n \geq \aleph, \exists z \in \mathfrak{S}$  to get:

$$\limsup_{n \rightarrow \infty} \sum_{\kappa=0}^{n-1} d(\mathcal{L}^\kappa(z), y_\kappa) < \beta \quad \forall \kappa \geq \aleph$$

and

$$\limsup_{n \rightarrow \infty} \sum_{\kappa=0}^{n-1} d(\mathcal{L}^\kappa(z), y_\kappa) < \beta \quad \forall \kappa \leq -\aleph$$

### **Definition 2.3.[11]**

A closed invariant set  $\mathfrak{B}$  is **transitive** if  $\exists \omega \in \mathfrak{B}$  s.t.  $\omega(\omega) = \mathfrak{B}$ , where  $\omega(\omega)$  is the omega limit set of  $\omega$ .

**Definition 2.4.[11]**

$\forall \kappa_1, \kappa_2 \in M$ , we write  $\kappa_1 \rightsquigarrow \kappa_2$  if  $\forall \alpha > 0, \exists \{y_j\}_{j=0}^n$  ( $n \geq 1$ ) of  $\mathcal{L}$  s.t.  $y_0 = \kappa_1$  and  $y_n = \kappa_2$ .  $\forall \kappa_1, \kappa_2 \in \mathfrak{B}$ , can write  $\kappa_1 \rightsquigarrow_{\mathfrak{B}} \kappa_2$  if  $\kappa_1 \rightsquigarrow \kappa_2$  and  $\{y_j\}_{j=0}^n \subset \mathfrak{B} (n \geq 1)$ . Symbolize the set  $C(\mathcal{L})$  is **chain transitive (CT)** if  $\forall \kappa_1, \kappa_2 \in C(\mathcal{L}), \kappa_1 \rightsquigarrow_{C(\mathcal{L})} \kappa_2$ .

**Definition 2.5.**

Assume  $\mathfrak{B}$  is called **locally maximal (LM)** where  $\mathfrak{B}$  a closed invariant set if  $\exists U$  is neighborhood of  $\mathfrak{B}$  s.t.

$$\mathfrak{B} = \bigcap_{n \in \mathbb{Z}} \mathcal{L}^n(U)$$

**Definition 2.6.[12]**

A subset  $\mathcal{R} \subset Diff(M)$  is said to be **residual** if it contains the intersection of a countable family of dense and open subset of  $Diff(M)$ ; then,  $\mathcal{R}$  is dense in  $Diff(M)$ .

**Definition 2.7.[12]**

A property  $\rho$  is called  **$C^1$ generic** if  $\rho$  hold for any diffeomorphisms that leads to some residual subset of  $Diff(M)$ .

**Theorem 2.8.**

There is  $\mathcal{R} \subset Diff(M)$ , which is if  $\mathcal{L} \in \mathcal{R}$  has the EFSP on LMCT set  $C(\mathcal{L})$ , then  $\mathcal{L}$  is hyperbolic on  $C(\mathcal{L})$ .

**Lemma 2.9.[13]**

There is  $\mathcal{R} \subset Diff(M)$  such that  $\forall \mathcal{L} \in \mathcal{R}$ , any CT set  $C(\mathcal{L})$ , there is a sequence  $Orb(\kappa_n)$  of periodic orbits of  $\mathcal{L}$  such that

$$\lim_{n \rightarrow \infty} Orb(\kappa_n) = C(\mathcal{L})$$

**Proposition 2.10.**

For every  $C(\mathcal{L})$  of  $\mathcal{L} \in \mathcal{R}_1$ , if  $C(\mathcal{L})$  is LM, then

$$C(\mathcal{L}) \cap Per(\mathcal{L}) \neq \emptyset.$$

**Proof:** Suppose  $\mathcal{L} \in \mathcal{R}_1$ , and let  $C(\mathcal{L})$  of  $\mathcal{L}$  be LM in  $\mathfrak{U}$ .

To proof this Proposition by contradiction, if  $C(\mathcal{L}) \cap Per(\mathcal{L}) = \emptyset$

Because  $C(\mathcal{L})$  is compact,  $\exists \alpha > 0$  such that  $C(\mathcal{L}) \subset B_\alpha(C(\mathcal{L})) \subset \mathfrak{U}$ .

(By Lemma 2.9),  $\exists$  sequence  $Orb(\kappa_n)$  of periodic orbits of  $\mathcal{L}$  such that  $n$  is large  $d(Orb(\kappa_n), C(\mathcal{L})) < \frac{\alpha}{2}$

Then it is  $Orb(\kappa_n) \subset B_\alpha(C(\mathcal{L})) \subset \mathfrak{U}$ .

Because  $C(\mathcal{L})$  is LM in  $\mathfrak{U}, \forall n \in \mathbb{Z}$

$$\mathcal{L}^n(Orb(\kappa_n)) \subset \mathcal{L}^n(\mathfrak{U})$$

If  $C(\mathcal{L})$  is LM, then  $C(\mathcal{L}) \cap \text{Per}(\mathcal{L}) \neq \emptyset$ ,

But this is contradiction. ■

**Proposition 2.11.**

Assume  $\mathfrak{B}$  is compact  $\mathcal{L}$ -invariant set of  $\mathcal{L}$ . If  $\mathcal{L}$  has EFSP on LM ( $\mathfrak{B}$ ), then the EFS points are get it from  $\mathfrak{B}$ .

**Proof:** Take  $\mathfrak{B}$  is LM, a neighborhood  $\mathcal{U}$  of  $\mathfrak{B}$ , since  $\mathfrak{B}$  is compact,

$\exists \alpha > 0$  such that  $\mathfrak{B} \subset B_\alpha(\mathfrak{B}) \subset \mathcal{U}$ .

Let  $0 < \varepsilon \leq \alpha$  be the number of the EFSP, and

Suppose  $\{y_j\}_{j \in \mathbb{Z}} \subset \mathfrak{B}$  a  $\varepsilon$ -pseudo-orbit of  $\mathcal{L}$

By the EFSP on  $\mathfrak{B}$ ,  $\exists \kappa \in M, \forall \kappa \geq \aleph$  and  $\aleph \in \mathbb{Z}$  such that

$$\limsup_{\kappa \rightarrow \infty} \sum_{j=0}^{\kappa-1} d(\mathcal{L}^j(\kappa), y_j) < \alpha \quad \forall j \geq \aleph \quad \text{and}$$

$$\limsup_{\kappa \rightarrow \infty} \sum_{j=0}^{\kappa-1} d(\mathcal{L}^{-j}(\kappa), y_{-j}) < \alpha \quad \forall -j \leq -\aleph$$

Then, take

$$\forall j \geq \aleph, \mathcal{L}^j(\kappa) \in B_\alpha(\mathfrak{B}) \text{ and}$$

$$\forall -j \leq -\aleph, \mathcal{L}^{-j}(\kappa) \in B_\alpha(\mathfrak{B}) \text{ and so,}$$

$$\mathcal{L}^j(\mathcal{L}^\aleph(\kappa)) \in B_\alpha(\mathfrak{B}) \text{ and } \mathcal{L}^{-j}(\mathcal{L}^{-\aleph}(\kappa)) \in B_\alpha(\mathfrak{B})$$

Since  $\mathfrak{B}$  is LM, we know that

$$\bigcap_{\kappa \in \mathbb{Z}} \mathcal{L}^\kappa(\mathcal{L}^{\aleph+j}(\kappa)) \in \bigcap_{\kappa \in \mathbb{Z}} \mathcal{L}^\kappa(B_\alpha(\mathfrak{B})) \subset \bigcap_{\kappa \in \mathbb{Z}} \mathcal{L}^\kappa(\mathcal{U}) = \mathfrak{B}$$

Then  $\mathcal{L}^{\aleph+j}(\kappa) \in \mathfrak{B}$

$\kappa \in \mathcal{L}^{-\aleph-j}(\mathfrak{B}) = \mathfrak{B}$ , since  $\mathfrak{B}$  is an  $\mathcal{L}$ -invariant set

Thus the EFS points are get it from  $\mathfrak{B}$ . ■

Let  $v$  be a  $\text{Per}_\eta$  of  $\mathcal{L}$ .  $\exists \varepsilon = \varepsilon(v) > 0$ , s.t.

$$W_\varepsilon^s(v) = \{w \in M : d(\mathcal{L}^\kappa(v), d(\mathcal{L}^\kappa(w)) \leq \varepsilon, \kappa \geq 0\}$$

$$W_\varepsilon^u(v) = \{w \in M : d(\mathcal{L}^\kappa(v), d(\mathcal{L}^\kappa(w)) \leq \varepsilon, \kappa \leq 0\}$$

Then  $W_\varepsilon^s(v), W_\varepsilon^u(v)$  are said the local stable (unstable) manifold of  $v$  respectively.

If  $\mathfrak{B}$  is hyperbolic closed  $\mathcal{L}$ -invariant set, then  $\exists \varepsilon > 0$  s.t. ,  $\forall 0 < \delta \leq \varepsilon$ , the above sets are  $C^1$ - embedded disks .

**Proposition 2.12.**

Let  $C(\mathcal{L})$  be locally maximal,  $\mathcal{L}$  has the EFSP. Then  $\forall w_1, w_2 \in C(\mathcal{L}) \cap \text{Per}(\mathcal{L})$ , we get  $W^s(w_1) \cap W^u(w_2) \neq \emptyset$  and  $W^u(w_1) \cap W^s(w_2) \neq \emptyset$

**Proof:** By [14],  $\mathcal{L}$  does not contain sources or sinks since  $C(\mathcal{L})$  is a CT set of  $\mathcal{L}$ .

Let  $w_1, w_2$  be  $\text{Per}_h$  of  $\mathcal{L}$ ,

Put  $\beta = \min \{\beta(w_1), \beta(w_2)\}$  and let  $0 < \alpha \leq \beta$  be the number of the EFSP for  $\mathcal{L}$ . To simple expression, we may assume that  $\mathcal{L}(w_1) = w_1$  and  $\mathcal{L}(w_2) = w_2$

Since  $\mathcal{L}$  is chain transitive,  $\exists \{y_j\}_{j=0}^{\kappa}$  ( $\kappa \geq 1$ )  $\subset C(\mathcal{L})$  is a finite  $\alpha$ -pseudo-orbit such that

$y_0 = w_1$  and  $y_{\kappa} = w_2$  and

$$d(\mathcal{L}(y_j), y_{j+1}) < \alpha \quad \forall 0 \leq j \leq \kappa - j$$

Take  $y_j = \mathcal{L}^j(w_1)$ ,  $\forall j \leq 0$  and  $y_{j+1} = \mathcal{L}^j(w_2)$ ,  $\forall j \geq 0$ ,

Then the sequence

$\{\dots, w_1 (= y_{-1}), w_1 (= y_0), y_1, y_2, \dots, w_2 (= y_{\kappa}), w_2 (= y_{\kappa+1}), \dots\} = \{y_j\}_{j \in \mathbb{Z}} \subset C(\mathcal{L})$  is  $\alpha$ -pseudo-orbit of  $\mathcal{L}$

Since, the EFSP on  $C(\mathcal{L})$ ,  $\exists \kappa \in \mathbb{M}, \forall \kappa \geq \aleph$  and  $\aleph \in \mathbb{Z}$  such that

$$d(\mathcal{L}^j(\kappa), y_j) < \beta \quad \forall j \geq \aleph \text{ and } d(\mathcal{L}^j(\kappa), y_j) < \beta \quad \forall j \leq -\aleph$$

i.e.  $\limsup_{\kappa \rightarrow \infty} \sum_{j=0}^{\kappa-1} d(\mathcal{L}^j(\kappa), y_j) < \beta \quad \forall j \geq \aleph$  and

$$\limsup_{\kappa \rightarrow \infty} \sum_{j=0}^{\kappa-1} d(\mathcal{L}^j(\kappa), y_j) < \beta \quad \forall j \leq -\aleph$$

Since  $y_{-j} = w_1 = \mathcal{L}^{-j}(w_1)$ ,  $\forall j \geq 0$  and  $y_{\kappa+j} = w_2 = \mathcal{L}^{\kappa+j}(w_2)$ ,  $\forall j \geq 0$

If  $\aleph \geq \kappa$ , then we know

$$\mathcal{L}^{-\aleph}(\kappa) \in B_{\beta}(y_{-\aleph}) = B_{\beta}(w_1)$$

And

$$\mathcal{L}^{\aleph}(\kappa) \in B_{\beta}(y_{\aleph}) = B_{\beta}(w_2)$$

Thus  $\forall j \geq \aleph$

(1)  $\mathcal{L}^{j+\aleph}(\kappa) = \mathcal{L}^j(\mathcal{L}^{\aleph}(\kappa)) \in B_{\beta}(y_{\aleph+j}) = B_{\beta}(w_2)$ , and

Thus  $\forall -j \leq -\aleph$

(2)  $\mathcal{L}^{-\aleph-j}(\kappa) \in B_{\beta}(y_{-\aleph-j}) = B_{\beta}(w_1)$

By (1), we obtain on  $d(\mathcal{L}^j(\mathcal{L}^{\aleph}(\kappa)), w_2) < \beta$ ,  $\forall j \geq 0$ , and

By (2),  $d(\mathcal{L}^{-j}(\mathcal{L}^{-\aleph}(\kappa)), w_1) < \beta$ ,  $\forall j \geq 0$

Then  $\mathcal{L}^{\aleph}(\kappa) \in W_{\beta}^s(w_2)$ ,  $\mathcal{L}^{-\aleph}(\kappa) \in W_{\beta}^u(w_1)$

So  $\kappa \in \mathcal{L}^{\aleph}(W_{\beta}^u(w_1))$  and  $\kappa \in \mathcal{L}^{-\aleph}(W_{\beta}^s(w_2))$

Since  $\mathcal{L}^{\aleph}(W_{\beta}^u(w_1)) \subset W^u(w_1)$  and  $\mathcal{L}^{-\aleph}(W_{\beta}^s(w_2)) \subset W^s(w_2)$ ,

Gradually,  $\kappa \in W^u(w_1) \cap W^s(w_2)$ .

Thus,  $W^u(w_1) \cap W^s(w_2) \neq \emptyset$ .

Now, to prove other case when  $W^s(w_1) \cap W^u(w_2) \neq \emptyset$

In fact, the proof of this case has the same to the above case. ■

**Definition 2.13.**

A  $Per_h(\mathcal{L})$  at  $w$ , suppose that  $w_1, w_2$  are two homoclinically related, and denote by  $w_1 \sim w_2$  if verification:

- $W^s(w_1) \cap W^u(w_2) \neq \emptyset$ .
- $W^u(w_1) \cap W^s(w_2) \neq \emptyset$ .

By above definition it is obviously that if  $w_1 \sim w_2$ , then

$\text{index}(w_1) = \text{index}(w_2)$ , i.e.

$\dim W^s(w_1) = \dim W^s(w_2)$ .

**Definition 2.14.**

A diffeomorphism  $\mathcal{L} \in Diff(M)$  is said to be **Kupka- Smale** ( $\mathcal{KS}$ ) if  $\exists Per_h(\mathcal{L})$  at  $w$ , also if  $w_1, w_2 \in Per(\mathcal{L})$ , then  $W^s(w_1)$  is transversal to  $W^u(w_2)$ .

The set of all Kupka- Smale diffeomorphisms is  $C^1$ -residual in  $Diff(M)$ .

**Proposition 2.15.**

Any chain transitive  $C(\mathcal{L})$  of  $\mathcal{L}$ ,  $\exists \mathcal{R}_2 \subset Diff(M)$  is a residual set s.t.  $\mathcal{L} \in \mathcal{R}_2$ , if  $\mathcal{L}$  has the EFSP on locally maximal  $C(\mathcal{L})$ , then  $\forall w_2 \in C(\mathcal{L}) \cap Per(\mathcal{L})$ , it have  $\text{index}(w_1) = \text{index}(w_2)$ .

**Proof:** Let  $\mathcal{L} \in \mathcal{R}_2 = \mathcal{R}_1 \cap \mathcal{KS}$  and let  $C(\mathcal{L})$  be a LMCT set of  $\mathcal{L}$ .

Assume that  $\mathcal{L}$  has the EFSP on  $C(\mathcal{L})$

Because  $C(\mathcal{L})$  is LM of  $\mathcal{L}$ , then by Proposition 2.10

It known  $C(\mathcal{L}) \cap Per(\mathcal{L}) \neq \emptyset$ , to proof by contradiction .

Assume that  $\exists w_1, w_2 \in C(\mathcal{L})$  are two  $Per_h$  such that.

$\text{index}(w_1) \neq \text{index}(w_2)$ .

Since  $\text{index}(w_1) \neq \text{index}(w_2)$ , it know

$\dim W^s(w_1) + \dim W^u(w_2) < \dim M$  or

$\dim W^u(w_1) + \dim W^s(w_2) < \dim M$

Then, take the case in which  $\dim W^s(w_1) + \dim W^u(w_2) < \dim M$ , the other case has the same proof.

Since  $\mathcal{L} \in \mathcal{KS}$  and  $\dim W^s(w_1) + \dim W^u(w_2) < \dim M$ , it know that

$W^s(w_1) \cap W^u(w_2) = \emptyset$ , this is contradiction.

Since  $\mathcal{L}$  has the EFSP on  $C(\mathcal{L})$ ,

By Proposition 2.12  $W^s(w_1) \cap W^u(w_2) \neq \emptyset$ ,

Thus, if  $\mathcal{L} \in \mathcal{R}_2$  has the EFSP on a LMCT set  $C(\mathcal{L})$ , then

$\forall w_2 \in C(\mathcal{L}) \cap Per(\mathcal{L})$

Then  $\text{index}(w_1) = \text{index}(w_2)$ .

**Definition 2.16.**

We write  $x \leftrightarrow y$  and  $y \leftrightarrow x$ , the set of points  $\{x \in M : x \leftrightarrow x\}$  is called the **chain recurrent set (CR)** of  $\mathcal{L}$

the **chain recurrent class** of  $\mathcal{L}$  is the set of equivalent classes  $\sim$  on  $CR(\mathcal{L})$ , denoted by  $C(w, \mathcal{L}) = \{x \in M: x \rightsquigarrow w \text{ and } w \rightsquigarrow x\}$ , which is a closed invariant set.

Suppose  $Per_h(\mathcal{L})$  at  $w_1$ , we say that  $w_1$  and  $w_2$  are homoclinically related, denote by  $w_1 \sim w_2$  if

$$W^s(w_1) \cap W^u(w_2) \neq \emptyset$$

$$W^u(w_1) \cap W^s(w_2) \neq \emptyset$$

It is obviously that in the case of verification if  $w_1 \sim w_2$ , then

$\text{index}(w_1) = \text{index}(w_2)$ , leads to  $\dim W^s(w_1) = \dim W^s(w_2)$  is denote by  $H(w, \mathcal{L}) = \overline{\{w_1 \sim w_2\}}$  such that  $H(w, \mathcal{L}) \subset C(w, \mathcal{L})$  [15].

**Lemma 2.17.[11]**

Every  $\mathcal{L} \in \mathcal{R}_3$ , where  $\mathcal{R}_3$  is a residual set such that is satisfies:

- $\mathfrak{B}$  Locally maximal transitive set,  $\exists w \in \mathfrak{B}$  periodic point such that  $\mathfrak{B}$  is locally maximal  $H((w, \mathcal{L}))$ .
- $\exists Per_h(w) \in \mathfrak{B}$  such that  $H(w, \mathcal{L}) = C(w, \mathcal{L})$ .
- Every chain transitive  $C(\mathcal{L})$  of  $\mathcal{L}$  is a transitive  $\mathfrak{B}$  of  $\mathcal{L}$ .
- If  $C_L(w)$  is locally maximal, then  $C_L(w)$  is robustly isolated,  $\exists \mathcal{U}(\mathcal{L})$  a  $C^1$  neighborhood of  $\mathcal{L}$  and a neighborhood  $\mathcal{U}$  of  $C_L(w)$  such that  $\forall g \in \mathcal{U}(\mathcal{L}), C_g(w_g) = CR(g) \cap \mathcal{U} = \mathfrak{B}_g(\mathcal{U}) = \bigcap_{n \in \mathbb{Z}} g^n(\mathcal{U})$ .

**Lemma 2.18. [Franks' Lemma (16)]**

Assume  $\mathcal{U}(\mathcal{L})$  any  $C^1$  neighborhood of  $\mathcal{L}$ . Then  $\exists \theta > 0$  and  $C^1$  neighborhood  $\mathcal{U}(\mathcal{L}) \subset \mathcal{U}(\mathcal{L})$  of  $\mathcal{L}$  such that  $\forall g \in \mathcal{U}(\mathcal{L})$ , a finite set  $\{y_i\}_{i=1}^k$ , a neighborhood  $\mathcal{W}$  of  $\{y_i\}_{i=1}^k$  and linear maps  $L_i: T_{y_i}M \rightarrow T_{g(y_i)}M$  satisfying  $\|L_i - D_{y_i}g\| \leq \theta, \forall 1 \leq i \leq k, \exists g^* \in \mathcal{U}(\mathcal{L})$  s.t.  $g^*(y) = g(y)$  if  $y \in \{y_i\}_{i=1}^k \cup (M \setminus \mathcal{W})$  and  $D_{y_i}g^* = L_i, \forall 1 \leq i \leq k$ .

**Definition 2.19.**

Let  $w \in \mathcal{L}$  be  $Per_h(\mathcal{L}), \forall \theta > 0$ , with period  $\chi(w)$  is a  **$\theta$  weak  $Per_h$**  if  $\exists \mu$  an eigenvalue of  $D\mathcal{L}^{\chi(w)}(w)$  such that

$$(1 - \theta)^{\chi(w)} < |\mu| < (1 + \theta)^{\chi(w)}$$

**Proposition 2.20.**

For any  $\mathcal{L} \in \mathcal{R}_4$ , where  $\mathcal{R}_4 \subset Diff(M)$  a residual set,  $\forall \theta > 0$ , if a CT set  $C(\mathcal{L})$  of  $\mathcal{L}$  is LM and  $C(\mathcal{L})$  contains a  $\theta$  weak  $Per_h(\mathcal{L})$  at  $w_1, \exists g$  is  $C^1$  closed to  $\mathcal{L}$  s.t. has  $w_1, w_2 \in C(g)$  two  $Per_h$  such that  $\text{index}(w_1) \neq \text{index}(w_2)$ , where  $C(g)$  is the CT set of  $g$ .

**Proof:** Let  $\mathcal{U}$  be locally maximal neighborhood of  $C(\mathcal{L})$ , and

$\mathcal{L} \in \mathcal{R}_4 = \mathcal{R}_2 \cap \mathcal{R}_3, \exists w \in C(\mathcal{L}) \cap Per(\mathcal{L})$  such that  $\forall \theta > 0, w$  is a  $\theta$  weak  $Per_h$  since  $\mathcal{L} \in \mathcal{R}_3$  and  $C(\mathcal{L})$  is LM,  $C(\mathcal{L})$  is transitive set  $\mathfrak{B}$  of  $\mathcal{L}$  and so,

$C(\mathcal{L}) = H_L(w) = C_L(w)$  and  $C(\mathcal{L})$  is robustly isolated.

For simplify, assume that  $C^{\chi(w)}(w) = C(\mathcal{L}) = w$ .

$w \in C(\mathcal{L}) \cap Per(\mathcal{L})$  is a  $\theta$  weak  $Per_h$ ,  $\forall \theta > 0, \exists \mu \in D_w \mathcal{L}$  is an eigenvalue such that

$$(1 - \theta) < |\mu| < (1 + \theta)$$

By Franks' Lemma [16]  $\exists g$  is  $C^1$  close to  $\mathcal{L}$  s.t.  $g(w) = \mathcal{L}(w) = w$  and  $D_w g$  has an eigenvalue  $\mu$  such that  $|\mu| = 1$ .

By Franks' Lemma [16]  $\exists g_1$  is  $C^1$  close to  $\mathcal{L}$  s.t.  $D_w g_1$  has only one eigenvalue  $\mu$  such that  $|\mu| = 1$ .

Denote by  $E_w^c$  the eigenspace corresponding to  $\mu$

The proof consist of two cases :

Case i\ Assume  $\mu$  is an real number

By Lemma 2.18,  $\exists \theta > 0, B_\theta(w) \subset \mathcal{U}$  and  $\hbar$   $C^1$  close to  $g(\hbar \in \mathcal{U}(\mathcal{L}))$  s.t.

- $\hbar(w) = g(w) = w$
- $\hbar(y) = \exp_w \circ D_w g \circ \exp_w^{-1}(y)$  for  $y \in B_\theta(w)$ , and
- $\hbar(y) = g(y)$  for  $y \in B_{2\theta}(w)$ .

Let  $\gamma = \theta/2$ . Take a nonzero vector  $v \in \exp_w(E_w^c(\theta))$  that corresponds to  $\mu$  s.t.  $\|v\| = \gamma$ .

Here,  $E_w^c(\theta)$  is the  $\theta$ -ball in  $E_w^c$  with its center at  $\overrightarrow{0_w}$ .

Then we have

$$\hbar(\exp_w(v)) = \exp_w \circ D_w g \circ \exp_w^{-1}(\exp_w(v)) = \exp_w(v).$$

Put  $\mathfrak{I}_w = \exp_w(\{\tau v : -\gamma/2 \leq \tau \leq \gamma/2\})$ .

Then,  $\mathfrak{I}_w$  is centered at  $w$  and  $\hbar(\mathfrak{I}_w) = \mathfrak{I}_w$

Know that  $\mathfrak{I}_w \subset \mathfrak{B}_\hbar(\mathcal{U}) = \bigcap_{\ell \in \mathbb{Z}} \hbar^\ell(\mathcal{U})$  since  $B_\theta(w) \subset \mathcal{U}$ .

Take  $w_1, w_2$  are two endpoints of  $\mathfrak{I}_w$  since  $\hbar(\mathfrak{I}_w) = \mathfrak{I}_w$ .

Then, know that

$$D_{w_1} \hbar|_{E_w^c} = D_{w_2} \hbar|_{E_w^c} = 1$$

By Lemma 2.18,  $\exists \Psi$  is  $C^1$  close to  $\hbar(\Psi \in \mathcal{U}(\mathcal{L}))$  s.t.

$\text{index}(w_{1\Psi}) \neq \text{index}(w_{2\Psi})$ , where  $w_{1\Psi}$  and  $w_{2\Psi}$  are hyperbolic points in  $\mathcal{U}$  with respect to  $\Psi$ .

$w_{1\Psi}, w_{2\Psi} \in C(\Psi) = \mathfrak{B}_\Psi(\mathcal{U}) = \bigcap_{\ell \in \mathbb{Z}} \Psi^\ell(\mathcal{U})$ , where  $C(\Psi)$  is the CT set of  $\Psi$ .

Case ii\ If  $\mu$  is an complex number, assume  $\mathcal{L}(w) = w$

By Lemma (2.7),  $\exists \theta > 0, B_\theta(w) \subset \mathcal{U}$  and  $g(g \in \mathcal{U}(\mathcal{L}))$  such that

$$g(w) = \mathcal{L}(w) = w \text{ and } g(y) = \exp_w \circ D_w g \circ \exp_w^{-1}(y) \quad \forall y \in B_\theta(w).$$

$\exists \ell > 0, \forall v \in \exp_w^{-1}(E_w^c(\theta))$  s.t.  $D_w g^\ell(v) = v$  since  $\mu = 1$ .

Assume  $v \in \exp_w(E_w^c(\theta))$  such that  $\|v\| = \theta/2$ .

Then there exists a small arc

$$\exp_w(\{\tau v : 0 \leq \tau \leq 1 + \gamma/2\}) = \zeta_w \subset \mathfrak{B}_g(\mathcal{U}) = \bigcap_{\ell \in \mathbb{Z}} g^\ell(\mathcal{U}) \text{ s.t.}$$

- $g^j(\zeta_w) \cap g^i(\zeta_w) = \emptyset \quad \forall 0 \leq j \neq i \leq \ell - 1$



- $g^\ell(\zeta_w) = \zeta_w$ , and
- $g^\ell_{|\zeta_w}: \zeta_w \rightarrow \zeta_w$  is identity map.

Take  $w_1, w_2$  are two endpoints of  $\zeta_w$ ,  $\exists g^*$  is  $C^1$  close to  $g$  such that  $\text{index}(w_{1g^*}) \neq \text{index}(w_{2g^*})$ , where  $w_{1g^*}$  and  $w_{2g^*}$  are hyperbolic points with respect to  $g^*$ .  
 $w_{1g^*}, w_{2g^*} \in C_{g^*}(w_{g^*}) = \mathfrak{B}_{g^*}(\mathcal{U}) = \bigcap_{\ell \in \mathbb{Z}} g^{*\ell}(\mathcal{U}) = C(g^*)$ , where  $C(g^*)$  is the CT set of  $g^*$ . ■

**Lemma 2.21.[17]**

For any  $\mathcal{L} \in \mathcal{R}_5$ , where  $\mathcal{R}_5 \subset \text{Diff}(M)$  a residual set, s.t. if for any  $C^1$  neighborhood  $\mathcal{U}(\mathcal{L})$  of  $\mathcal{L}$ ,  $\exists g \in \mathcal{U}(\mathcal{L})$  s.t.  $g$  has  $w_1, w_2$  are two periodic point with  $\text{index}(w_1) \neq \text{index}(w_2)$ , then  $\mathcal{L}$  has  $w_{1\mathcal{L}}$  and  $w_{2\mathcal{L}}$  with  $\text{index}(w_{1\mathcal{L}}) \neq \text{index}(w_{2\mathcal{L}})$ .

**Proposition 2.22.**

For any  $\mathcal{L} \in \mathcal{R}_6$ , where  $\mathcal{R}_6 \subset \text{Diff}(M)$  a residual set, such that if  $\mathcal{L}$  has the EFSP on locally maximal  $C(\mathcal{L})$ ,  $\exists \theta > 0$  such that  $\forall w \in C(\mathcal{L}) \cap \text{Per}(\mathcal{L})$ ,  $w$  is not a  $\theta$  weak  $\text{Per}_h$  of  $\mathcal{L}$ .

**Proof:** Let  $\mathcal{L} \in \mathcal{R}_6 = \mathcal{R}_4 \cap \mathcal{R}_5$  and let  $C(\mathcal{L})$  be a LMCT set of  $\mathcal{L}$ .

To proof this Proposition by contradiction,  $\forall \theta > 0, \exists w \in C(\mathcal{L}) \cap \text{Per}(\mathcal{L})$ ,  $w$  is a  $\theta$  weak  $\text{Per}_h$  of  $\mathcal{L}$ .

$C(\mathcal{L})$  is robustly isolated, since  $\mathcal{L} \in \mathcal{R}_3$  and  $C(\mathcal{L})$  is LM.

Since  $\mathcal{L} \in \mathcal{R}_4$  and  $w \in C(\mathcal{L}) \cap \text{P}(\mathcal{L})$  is a  $\theta$  weak  $\text{Per}_h$  of  $\mathcal{L}$ .

By Proposition 2.20,  $\exists g$  is  $C^1$  close to  $\mathcal{L}$  such that  $g$  has  $w_1, w_2 \in C(g)$  two  $\text{Per}_h$

So,  $\text{index}(w_1) \neq \text{index}(w_2)$ .

Since  $\mathcal{L} \in \mathcal{R}_5$ ,  $\mathcal{L}$  has two  $\text{Per}_h$   $w_{1\mathcal{L}}, w_{2\mathcal{L}} \in C(\mathcal{L})$  with

$\text{index}(w_{1\mathcal{L}}) \neq \text{index}(w_{2\mathcal{L}})$ .

This contradiction, since  $\mathcal{L}$  has the EFSP on  $C(\mathcal{L})$  by Proposition 2.15,

$\forall w_1, w_2 \in C(\mathcal{L}) \cap \text{Per}(\mathcal{L})$  with  $\text{index}(w_1) = \text{index}(w_2)$  ■

**Definition 2.23.**

$\mathcal{L}$  satisfies a **star condition** on  $C(\mathcal{L})$  if  $\exists C^1$  neighborhood  $\mathcal{U}$  of  $C$  such that  $\forall g \in \mathcal{U}(\mathcal{L})$ , every  $w \in \mathfrak{B}_g \cap \text{Per}(g)$  is hyperbolic.

Denote  $\mathcal{T}(C(\mathcal{L}))$  for the set of all diffeomorphisms that satisfy the local star condition on  $C(\mathcal{L})$ .

**Lemma 2.24.[14]**

For any  $\mathcal{L} \in \mathcal{R}_7$ , where  $\mathcal{R}_7 \subset \text{Diff}(M)$  a residual set, such that  $\forall \theta > 0$  and any  $C^1$  neighborhood  $\mathcal{U}$  of  $C$ , if  $\exists g \in \mathcal{U}(\mathcal{L})$  and  $w \in \text{Per}(g)$  is hyperbolic point such that  $w$  is a  $\theta$  weak  $\text{Per}_h$  then  $\exists w_{\mathcal{L}} \in \text{Per}(g)$  is hyperbolic point with  $2\theta$  weak  $\text{Per}_h$ .

**Proposition 2.25.**

For any  $\mathcal{L} \in \mathcal{R}_8$ , where  $\mathcal{R}_8 \subset Diff(M)$  a residual set, any CT set  $C(\mathcal{L})$  of  $\mathcal{L}$ , if  $\mathcal{L}$  has the EFSP on locally maximal  $C(\mathcal{L})$ , then  $\mathcal{L} \in \mathcal{T}(C(\mathcal{L}))$ .

**Proof:** Let  $\mathcal{L} \in \mathcal{R}_8 = \mathcal{R}_6 \cap \mathcal{R}_7$  and let  $C(\mathcal{L})$  be a LMCT set of  $\mathcal{L}$ .

To proof this Proposition by contradiction,  $\mathcal{L} \notin \mathcal{T}(C(\mathcal{L}))$  then  $\exists C^1$  close to  $\mathcal{L}$  such that

$\forall \theta > 0, g$  possess a  $\theta/2$  weak  $Per_h(\mathcal{L}) \in C(g)$ .

Because  $\mathcal{L} \in \mathcal{R}_7, \exists w_{\mathcal{L}} \in C(\mathcal{L}) \cap Per(\mathcal{L})$  s.t.  $w_{\mathcal{L}}$  is a  $\theta$  weak  $Per_h$ . This is a contradiction;

Since  $\mathcal{L}$  has the EFSP on  $C(\mathcal{L})$ , by Proposition 2.22 every periodic point in  $C(\mathcal{L})$  is not a  $\theta$  weak  $Per_h$ .

Thus, if  $\mathcal{L}$  has the EFSP on  $C(\mathcal{L})$ ,

Lead to  $\mathcal{L} \in \mathcal{T}(C(\mathcal{L}))$ . ■

**Proposition 2.26.[18]**

Let  $C(\mathcal{L})$  be LM and  $\mathcal{L} \in \mathcal{T}(C(\mathcal{L}))$ , any CT set  $C(\mathcal{L})$  of  $\mathcal{L} \in \mathcal{R}_8$ , then  $\exists i > 0$  and  $0 < \theta < 1$  such that  $\forall w \in \mathfrak{B} \cap Per(\mathcal{L})$ , we have the following :

$$(1) \prod_{j=0}^{\chi(w)-1} \|D\mathcal{L}^j|_{E^s(\mathcal{L}^j(w))}\| < \theta^{\chi(w)}, \prod_{j=0}^{\chi(w)-1} \|D\mathcal{L}^{-j}|_{E^s(\mathcal{L}^{-j}(w))}\| < \theta^{\chi(w)}$$

$$(2) \|D\mathcal{L}^i|_{E^s(w)}\| \cdot \|D\mathcal{L}^{-1}|_{E^u(\mathcal{L}^i(w))}\| < \theta^2$$

Where  $\chi(w)$  denote the period of  $w$ .

**Theorem 2.27.[19]**

For any  $\mathcal{L} \in \mathcal{R}_9, \exists \mathcal{R}_9 \subset Diff(M)$  a residual set, s.t.  $\forall \mu_n$  ergodic measure of  $\mathcal{L}, \exists$  a sequence of periodic orbit  $Orb(w_n)$  such that  $\mu_n \rightarrow \mu$  in weak topology and  $Orb(w_n) \rightarrow Supp(\mu)$  in the Hausdorff metric.

**Lemma 2.28.[11]**

Let  $E \subset T_{\mathfrak{B}}M$  be a continuous invariant subbundle,  $\mathfrak{B} \subset M$  be closed  $\mathcal{L}$ -invariant set. If  $\exists \theta > 0$  that is.

$$\sum \log \|D\mathcal{L}^\theta|_E\| < 0$$

**Proof of Theorem 2.8:** Let  $C(\mathcal{L})$  be LMCT set of  $\mathcal{L}$ , and  $\mathcal{L} \in \mathcal{R} = \mathcal{R}_8 \cap \mathcal{R}_9$

Assume  $\mathcal{L}$  has the EFSP on  $C(\mathcal{L})$ , where  $\mathcal{L}$  is measurable map.

Since  $\mathcal{L} \in \mathcal{R}$  and  $C(\mathcal{L})$  is LM,  $\exists w$  is  $Per_h$ , we know that  $C(\mathcal{L}) = H_{\mathcal{L}}(w)$

Then by Proposition 2.25,  $\mathcal{L} \in \mathcal{T}(C(\mathcal{L})) = \mathcal{T}(H_{\mathcal{L}}(w))$

Thus by Proposition 2.26,  $T_{\mathfrak{B}}M = E \oplus F$ , satisfy  $E(w) = E^s(w)$  and  $F(w) = F^u(w)$ , it is enough to show that  $D\mathcal{L}$  is contracting on  $E$  and expanding on  $F$

By contradiction that  $E$  is not contracting, such that  $\mu$  is an ergodic measure supported on  $H_{\mathcal{L}}(w)$ . Take  $w_n \in \text{Orb}(w_n)$  with period  $\chi(w_n)$

Assume that  $\mathcal{L}^{\chi(w_n)}(w_n) = \mathcal{L}(w_n) = w_n$

Then by Theorem 2.27, we have

$$\sum \|D\mathcal{L}|_E\| = \lim_{n \rightarrow \infty} \sum \|D\mathcal{L}|_{E^S(w_n)}\| < 0$$

By Lemma 2.28,  $E$  is contracting, this is contradiction

Then  $C(\mathcal{L})$  is hyperbolic. ■

## Conclusion:

In general, the eventual fitting shadowing property is not fulfilled in hyperbolic dynamical systems (satisfy in case  $\mathcal{L}$  is Anosov diffeomorphisms). In this paper, several concepts were presented. These concepts can be re-examined on other important spaces. Future studies can also be conducted on hyperbolic dynamical systems using the concept of strongly fitting shadowing property, recording the difference between the two studies and their impact on finding dynamical characteristics that may be employed in solving some mathematical problems.

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