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Lacunary interpolation by the Spline Function of Fractional Degree

Ridha G. Karem [a], Karwan H.F. Jwamer [b*], Faraidun K. Hamasalh [c]

[a*] *Department of Mathematics, College of Basic Education, University Sulaimani.Email:ridha.karem@univsul.edu.iq*

[b] *Department of Mathematics, College of Science, University of Sulaimani.Email: karwan.jwamer@univsul.edu.iq*

[c] *Department of Mathematics, College of Education, University Sulaimani.Email: faraidun.hamasalh@univsul.edu..iq*

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ABSTRACT

In almost all fields of numerical analysis, spline functions are the most effective tool for polynomials employed as the fundamental method of approximation theory. Additionally, existence, uniqueness, and error boundaries are required for the spline creation in the g-spline interpolation issue. Mathematics, physics, biology, engineering, signal processing, systems identification, control theory, finance, and fractional dynamics have all shown an interest in fractional differential equations, also work in social sciences including economics, finance, and dietary supplements. It is crucial to find both close and accurate solutions of fractional differential equations. To find solutions of fractional differential equations, several analytical and numerical techniques have been developed. In this paper, we extend the five-degree spline (0,4) lacunary interpolation on uniform meshes. The outcomes, uniqueness and error boundaries for generalize (0,4) Lacunary interpolation using five- degree splines. These generalizes outperform the usage of the (0,4) five splines for interpolation.

MSC..

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1.1 Introduction:

∗Corresponding author

Email addresses: karwan.jwamer@univsul.edu.iq

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Spline functions are the most effective polynomial tool for usage as the fundamental method of approximation theory in almost all branches of numerical analysis. Because they can be readily evaluated, differentiated, and integrated in a finite number of steps using the fundamental arithmetic operations of addition, subtraction, division, and multiplication, polynomials are used for approximation. Spline functions are a relatively recent analytical topic. The theories of splines and practical applications of their use in numerical analysis have both advanced significantly during the past 20 years. The theory and use of splines are covered in varying depths in the publications listed below (Ahlberg et al., 1967). In addition to the publications discussing the optimal interpolation or spline approximation already stated, Lacunary interpolation is used when parts of the sequences are broken. Finding the five -degree spline S(x) by interpolating data on the function value and fourth order in the interval [0,1] is the lacunary interpolation issue that we have looked at in this research. Additionally, the first derivative is required to meet an additional starting requirement.

The structure of this study is as follows: Consider first the degree five spline function that interpolates the Lacunary data (0,4). The existence, uniqueness, and error bounds of the degree five spline function are discussed theoretically, and convergence analysis is also explored. to show that the required Lacunary spline function converges.

1.2 Descriptions of the Method:

In this section, we offer a five- degree spline (0,4) interpolation for a one-dimensional, sufficiently smooth function $f(x)$ specified on $I = [0,1]$, define as following:

$$Q_n^{(j)}(x_i) = a_{i,j}\,, \qquad i = 1,2, \dots ,n\,; j = 0,1,2, \dots ,n \qquad (1)$$

If the order of the derivatives in (1) from an unbroken sequence is j for each i, then we have Hermite interpolation. We have lacunary interpolation if any of the sequences are broken:

$$\mathrm{I}_n: 0 = x_0 < x_1 < \cdots < x_n = 1$$

Use knots to represent the uniform division of I:

$$x_i = ih \quad \text{where } h = x_{i+1} - x_i\,,\ i = 1,2, \dots ,n-1.$$

The class of spline function $S_{n,5}^2$ is defined as follows, where $S_{n,5}^2$ indicates the class of all splines of degree six that are a part of $C^2[0,1]$, any element $S_I(x) \in S_{n,5}^2$ if both of the following two criteria are true:

(i) $S_I(x) \in C^2[0,1]$

(ii) $S_I(x)$ is a polynomial of degree least or equal to five in each $[x_i, x_{i+1}]\,, i = 0,1, \dots ,n-1$

(2)

1.2.1 The Lacunary five Splines Function is constructed as follows:

If $S(x)$ is a five-degree polynomial on [0, 1], then we get

$$S(x) = S(0)\,A_0(x) + S(\lambda)A_1(x) + S(1)A_2(x) + S^{(\frac{1}{2})}(0)A_3(x) + S^{(\frac{1}{2})}(1)A_4(x)$$

$$+S^{(4)}(\lambda)A_5(x),$$

(3)

Where, $\lambda \in (0,1)$

$$A_0(x) = \frac{-1}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left[\begin{matrix}6\left(4\,\lambda^{\frac{5}{2}} - 5\,\lambda^{\frac{3}{2}} + 1\right)x^{\frac{7}{2}} - \left(24\,\lambda^{\frac{7}{2}} - 35\,\lambda^{\frac{3}{2}} + 11\right)x^{\frac{5}{2}} + \left(30\lambda^{\frac{7}{2}} - 35\,\lambda^{\frac{5}{2}} + 5\right)x^{\frac{3}{2}} \\ -\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)\end{matrix}\right],$$

$$A_1(x) = \frac{-1}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left[-6\,x^{\frac{7}{2}} + 11\,x^{\frac{5}{2}} - 5\,x^{\frac{3}{2}}\right],$$

$$A_2(x) = \frac{1}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left[6\,\lambda^{\frac{3}{2}}\,(4\,\lambda-5)x^{\frac{7}{2}} - \lambda^{\frac{3}{2}}(24\,\lambda^2 - 35)x^{\frac{5}{2}} + 5\,\lambda^{\frac{5}{2}}(6\lambda-7)\,x^{\frac{3}{2}}\right],$$

$$A_3(x) = \frac{-1}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left[\begin{matrix}\frac{4}{\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)(4\lambda-3)x^{\frac{7}{2}} - \frac{2}{\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)(8\lambda^2+8\lambda-11)x^{\frac{5}{2}} + \frac{2}{\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)(14\lambda^2-5\lambda-5)x^{\frac{3}{2}} \\ -\frac{2}{\sqrt{\pi}}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)x^{\frac{1}{2}}\end{matrix}\right],$$

$$A_4(x) = \frac{-1}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left[\frac{32}{\sqrt{\pi}}\lambda^{\frac{3}{2}}(\lambda-1)\;x^{\frac{7}{2}} - \frac{32}{\sqrt{\pi}}\lambda^{\frac{3}{2}}(\lambda-1)(\lambda+1)x^{\frac{5}{2}} + \frac{32}{\sqrt{\pi}}\lambda^{\frac{5}{2}}(\lambda-1)x^{\frac{3}{2}}\right],$$

$$A_5(x) = \frac{-1}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left[\begin{matrix}\frac{-16}{945}\lambda(\lambda-1)(6\lambda-5)x^{\frac{9}{2}} + \frac{2}{315}\lambda(\lambda-1)(16\,\lambda^2+16\,\lambda-25)\;x^{\frac{7}{2}} \\ -\frac{2}{945}\lambda(\lambda-1)(88\,\lambda^2-35\,\lambda-35)x^{\frac{5}{2}} + \frac{10\lambda^2}{945}(\lambda-1)(8\,\lambda-7)x^{\frac{3}{2}}\end{matrix}\right],$$

$\forall\lambda \in (0,1)/\{\frac{5}{6}\}$.

(4)

We record that for future use:

$$A_0^{\frac{1}{2}}(\lambda) = \frac{-\frac{15\sqrt{\pi}}{16}\left(4\,\lambda^{\frac{11}{2}} - 11\,\lambda^{\frac{9}{2}} + 7\,\lambda^{\frac{7}{2}} + 7\lambda^3 - 11\lambda^2 + 4\lambda\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}, \quad A_0^{\frac{3}{2}}(\lambda) = \frac{-\frac{15\sqrt{\pi}}{4}\left[9\,\lambda^{\frac{9}{2}} - \frac{81}{4}\,\lambda^{\frac{7}{2}} + \frac{21}{2}\,\lambda^{\frac{5}{2}} + \frac{21}{4}\lambda^2 - \frac{11}{2}\lambda + 1\right]}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_0''(0) = 0 \quad A_0''(1) = \frac{\frac{15}{4}\left(24\,\lambda^{\frac{7}{2}} - 56\,\lambda^{\frac{5}{2}} + 35\,\lambda^{\frac{3}{2}} - 3\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}, \quad A_0^{\frac{7}{2}}(0) = A_0^{\frac{7}{2}}(1) = A_0^{\frac{7}{2}}(\lambda) = \frac{-315\sqrt{\pi}\left(4\,\lambda^{\frac{5}{2}} - 5\,\lambda^{\frac{3}{2}} + 1\right)}{8\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_1^{\frac{1}{2}}(\lambda) = \frac{-\frac{15\sqrt{\pi}}{16}\lambda(\lambda-1)(-7\lambda+4)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}, \qquad A_1^{\frac{3}{2}}(\lambda) = \frac{\frac{15\sqrt{\pi}}{4}\left(\frac{21}{4}\,\lambda^2 - \frac{11}{2}\,\lambda + 1\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_1''(0) = 0, \quad A_1''(1) = \frac{\frac{45}{4}}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}, \quad A_1^{\frac{7}{2}}(0) = A_1^{\frac{7}{2}}(1) = A_1^{\frac{7}{2}}(\lambda) = \frac{315\sqrt{\pi}}{8\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_2^{\frac{1}{2}}(\lambda) = \frac{\frac{15\sqrt{\pi}}{16}\lambda^{\frac{7}{2}}(\lambda-1)(4\lambda-7)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}, \qquad A_2^{\frac{3}{2}}(\lambda) = \frac{\frac{15\sqrt{\pi}}{4}\lambda^{\frac{5}{2}}\left(9\,\lambda^2 - \frac{81}{4}\lambda + \frac{21}{2}\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_2''(0) = 0, \quad A_2''(1) = \frac{-\frac{15}{4}\,\lambda^{\frac{3}{2}}\left(24\,\lambda^2 - 56\,\lambda + 35\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}, \quad A_2^{\frac{7}{2}}(0) = A_2^{\frac{7}{2}}(1) = A_2^{\frac{7}{2}}(\lambda) = \frac{\frac{315\sqrt{\pi}}{8}\,\lambda^{\frac{3}{2}}\,(4\,\lambda-5)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_3^{\frac{1}{2}}(\lambda) = \frac{-\lambda^{\frac{3}{2}}(\lambda-1)\left(\frac{5}{2}\lambda^3 - \frac{57}{8}\lambda^2 + \frac{57}{8}\lambda - \frac{5}{2}\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}, \qquad A_3^{\frac{3}{2}}(\lambda) = \frac{-\lambda^{\frac{1}{2}}(\lambda-1)\left(\frac{45}{2}\lambda^3 - \frac{387}{8}\lambda^2 + \frac{135}{4}\lambda - \frac{15}{2}\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_3''(0) = 0, \quad A_3''(1) = \frac{\frac{5}{2\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left(24\lambda^2 - 32\lambda + 9\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}, \quad A_3^{\frac{7}{2}}(0) = A_3^{\frac{7}{2}}(1) = A_3^{\frac{7}{2}}(\lambda) = \frac{-\frac{105}{4}\lambda^{\frac{1}{2}}(\lambda-1)(4\lambda-3)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_4^{\frac{1}{2}}(\lambda) = \frac{-\lambda^{\frac{7}{2}}(\lambda-1)(5\,\lambda-6)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}, \qquad A_4^{\frac{3}{2}}(\lambda) = \frac{-\lambda^{\frac{3}{2}}(\lambda-1)\left(45\lambda^2 - 36\lambda\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_4''(0) = 0, \quad A_4''(1) = \frac{\frac{40}{\sqrt{\pi}}\lambda^{\frac{3}{2}}(\lambda-1)(3\lambda-4)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}, \quad A_4^{\frac{7}{2}}(0) = A_4^{\frac{7}{2}}(1) = A_4^{\frac{7}{2}}(\lambda) = \frac{-210\lambda^{\frac{3}{2}}(\lambda-1)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_5^{\frac{1}{2}}(\lambda)=\frac{-\sqrt{\pi}\lambda^3(\lambda-1)\left(-\frac{53}{9}\lambda^3+\frac{41}{1008}\lambda^2-\frac{41}{1008}\lambda+\frac{1}{72}\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},\quad A_5^{\frac{3}{2}}(\lambda)=\frac{-\frac{-\sqrt{\pi}}{12}\lambda(\lambda-1)\left[2\lambda^4-\frac{101}{21}\lambda^3+\frac{449}{84}\lambda^2-\frac{7}{3}\lambda\right]}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_5''(0)=0,\qquad A_5''(1)=\frac{\frac{1}{6}\lambda(\lambda-1)\left(-\frac{24}{3}\lambda^2+\frac{39}{15}\lambda-\frac{4}{3}\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},\quad A_5^{\frac{7}{2}}(0)=\frac{-\frac{\sqrt{\pi}}{24}\lambda(\lambda-1)\left(16\lambda^2+16\lambda-25\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)},$$

$$A_5^{\frac{7}{2}}(1)=\frac{-\frac{\sqrt{\pi}}{24}\lambda(\lambda-1)\left(16\lambda^2-56\lambda+35\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\ ,\qquad A_5^{\frac{7}{2}}(\lambda)=\frac{\frac{\sqrt{\pi}}{24}\lambda(\lambda-1)\left(56\lambda^2-76\lambda+25\right)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}. \tag{5}$$

The following expansions on $[x_i,x_{i+1}]$ exist for $f\in C^5[0,1]$.

$$f(x_{i-1})=f(x_i)-hf'(x_i)+\frac{h^2}{2!}f''(x_i)-\frac{h^3}{3!}f^{(3)}(x_i)+\frac{h^4}{4!}f^{(4)}(x_i)-\frac{h^5}{5!}f^{(5)}(\theta_{1,i}),$$

for $x_{i-1}<\theta_{1,i}<x_i$,

$$f(x_{i+1})=f(x_i)+hf'(x_i)+\frac{h^2}{2!}f''(x_i)+\frac{h^3}{3!}f'''(x_i)+\frac{h^4}{4!}f^{(4)}(x_i)+\frac{h^5}{5!}f^{(5)}(\theta_{2,i}),$$

for $x_i<\theta_{2,i}<x_{i+1}$,

$$f(x_{i-1+\lambda})=f(x_i)+(\lambda-1)hf'(x_i)+\frac{(\lambda-1)^2h^2}{2!}f''(x_i)+\frac{(\lambda-1)^3h^3}{3!}f'''(x_i)$$

$$+\frac{(\lambda-1)^4h^4}{4!}f^{(4)}(x_i)+\frac{(\lambda-1)^5h^5}{5!}f^{(5)}(\theta_{3,i}),\text{ for }\ x_i<\theta_{3,i}<x_{i-1+\lambda},$$

$$f(x_{i+\lambda})=f(x_i)+\lambda hf'(x_i)+\frac{\lambda^2h^2}{2!}f''(x_i)+\frac{\lambda^3h^3}{3!}f'''(x_i)+\frac{\lambda^4h^4}{4!}f^{(4)}(x_i)+\frac{\lambda^5h^5}{5!}f^{(5)}(\theta_{4,i}),$$

for $x_i<\theta_{4,i}<x_{i+\lambda}$,

$$f^{(4)}(x_{i-1+\lambda})=f^{(4)}(x_i)+(\lambda-1)hf^{(5)}(\theta_{5,i}),\text{ for }\ x_i<\theta_{5,i}<x_{i-1+\lambda},$$

$$f^{(4)}(x_{i+\lambda})=f^{(4)}(x_i)+\lambda hf^{(5)}(\theta_{6,i}),\quad\text{for }\ x_i<\theta_{6,i}<x_{i+\lambda},$$

$$f^{\left(\frac{1}{2}\right)}(x_{i-1})=f^{\left(\frac{1}{2}\right)}(x_i)-\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}f'(x_i)-hf^{\left(\frac{3}{2}\right)}(x_i)+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}f''(x_i)+\frac{h^2}{2!}f^{\left(\frac{5}{2}\right)}(x_i)$$

$$-\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}f^{(3)}(x_i)-\frac{h^3}{3!}f^{\left(\frac{7}{2}\right)}(x_i)+\frac{16}{105\sqrt{\pi}}h^{\frac{7}{2}}f^{(4)}(x_i)$$

$$+\frac{h^4}{4!}f^{\left(\frac{9}{2}\right)}(x_i)-\frac{32}{945\sqrt{\pi}}h^{\frac{9}{2}}f^{(5)}(\theta_{7,i}), \text{ for}, x_{i-1}<\theta_{7,i}<x_i$$

$$f^{\left(\frac{1}{2}\right)}(x_{i+1})=f^{\left(\frac{1}{2}\right)}(x_i)+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}f'(x_i)+hf^{\left(\frac{3}{2}\right)}(x_i)+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}f''(x_i)+\frac{h^2}{2!}f^{\left(\frac{5}{2}\right)}(x_i)$$

$$+\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}f^{(3)}(x_i)+\frac{h^3}{3!}f^{\left(\frac{7}{2}\right)}(x_i)+\frac{16}{105\sqrt{\pi}}h^{\frac{7}{2}}f^{(4)}(x_i)$$

$$+\frac{h^4}{4!}f^{\left(\frac{9}{2}\right)}(x_i)+\frac{32}{945\sqrt{\pi}}h^{\frac{9}{2}}f^{(5)}(\theta_{8,i}), \text{ for } x_i<\theta_{8,i}<x_{i+1},$$

$$f^{\left(\frac{1}{2}\right)}(x_{i+\lambda})=f^{\left(\frac{1}{2}\right)}(x_i)+\lambda\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}f'(x_i)+\lambda hf^{\left(\frac{3}{2}\right)}(x_i)+\frac{4\lambda^2}{3\sqrt{\pi}}h^{\frac{3}{2}}f''(x_i)+\frac{\lambda^2h^2}{2!}f^{\left(\frac{5}{2}\right)}(x_i)$$

$$+\frac{8\lambda^3}{15\sqrt{\pi}}h^{\frac{5}{2}}f'''(x_i)+\frac{\lambda^3h^3}{3!}f^{\left(\frac{7}{2}\right)}(x_i)+\frac{16\lambda^4h^{\frac{7}{2}}}{105\sqrt{\pi}}f^{(4)}(x_i)+\frac{\lambda^4h^4}{4!}f^{\left(\frac{9}{2}\right)}(x_i)$$

$$+\frac{\lambda^5h^{\frac{9}{2}}}{5!}\frac{256}{63\sqrt{\pi}}f^{(5)}(\theta_{9,i}), \quad \text{for } x_i<\theta_{9,i}<x_{i+\lambda},$$

$$f^{\left(\frac{3}{2}\right)}(x_{i+\lambda})=f^{\left(\frac{3}{2}\right)}(x_i)+\lambda hf^{\left(\frac{5}{2}\right)}(x_i)+\frac{2}{\sqrt{\pi}}\lambda^2h^{\frac{1}{2}}f''(x_i)+\frac{\lambda^2h^2}{2!}f^{\left(\frac{7}{2}\right)}(x_i)+\frac{4\lambda^3}{3\sqrt{\pi}}h^{\frac{3}{2}}f'''(x_i)$$

$$+\frac{\lambda^3h^3}{3!}f^{\left(\frac{9}{2}\right)}(x_i)+\frac{8}{15\sqrt{\pi}}\lambda^4h^{\frac{5}{2}}f^{(4)}(x_i)+\frac{128}{7\sqrt{\pi}}\frac{\lambda^5h^{\frac{7}{2}}}{5!}f^{(5)}(\theta_{10,i}) \text{ for } x_i<\theta_{10,i}<x_{i+\lambda},$$

(6)

1.3 Theorems of Existence and Uniqueness:

This section presents and analyzes the existence and uniqueness theorem for degree five spline functions that interpolate the lacunary data (0,4)

Theorem 1: The Spline Function is New and Unique

for specified random numbers $f(x_j), j=0,1,\dots,n$, $f^{(4)}(x_{i+\lambda}), i=0,1,\dots,n-1$ and $f^{\left(\frac{1}{2}\right)}(x_0)$, $f^{\left(\frac{1}{2}\right)}(x_n)$, there is a special spline $S_n(x)\in S_{n,5}^2$ like that:

$$S_n(x_i)=f(x_i), i=0,1,\dots,n,$$

$$S_n^{(4)}(x_{j+\lambda})=f^{(4)}(x_{j+\lambda}), j=0,1,\dots,n-1,$$

$$S_n^{\left(\frac{1}{2}\right)}(x_0)=f^{\left(\frac{1}{2}\right)}(x_0), S_n^{\left(\frac{1}{2}\right)}(x_n)=f^{\left(\frac{1}{2}\right)}(x_n).$$

(7)

Theorem 2:

Let $f(x) \in C^5[0,1]$ and $S_n(x) \in S_{n,5}^2$ be a singular spline fulfilling Theorem 1 prerequisites, then

$$\left\| S_n^{(r)}(x) - f^{(r)}(x) \right\| \le V \frac{l^{r-5}}{5!}\ w(f^{(5)}; \frac{1}{m})\ ;\ r = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}\ ,$$

$$V = \left[\begin{array}{c} \frac{\frac{15\sqrt{\pi}}{16}\lambda[(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)\left(\frac{6847}{72}\lambda^7 - \frac{48509}{504}\lambda^6 + \frac{227}{126}\lambda^5 - \frac{55}{63}\lambda^4 + \frac{2}{9}\lambda^3 + \frac{1}{2}\lambda^{\frac{9}{2}} - \frac{15}{2}\lambda^{\frac{7}{2}} + 7\lambda^{\frac{5}{2}}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)} \\ - \frac{\left(-12\lambda^{\frac{5}{2}} + 21\lambda^{\frac{3}{2}} - \frac{35}{4}\lambda^{\frac{1}{2}} + \frac{65}{252}\lambda^4 - \frac{155}{84}\lambda^3 + \frac{110}{84}\lambda^2 - \frac{329}{126}\lambda + \frac{665}{252}\right)\left(10\lambda^4 + \frac{103}{2}\lambda^3 - \frac{141}{2}\lambda^2 + 180\lambda - \frac{140}{3}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)} \\ - \frac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)[(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)\left(\frac{3}{2}\lambda^7 - \frac{5}{4}\lambda^6 - \frac{5}{8}\lambda^4 + 2\lambda^3 - \lambda^2 + \frac{1}{81}\lambda + \frac{1}{324}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)} \\ + \frac{\left(15\lambda^2 + 15\lambda - \frac{45}{8}\right)\left(\frac{15}{2}\lambda^4 + \frac{15}{8}\lambda^3 - \frac{93}{8}\lambda^2 + 13\lambda - \frac{11}{3}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)} \end{array} \right]$$

And $\|S_n(x) - f(x)\| \le V \frac{l^{r-5}}{5!} \frac{63\sqrt{\pi}}{256} w(f^{(5)}; \frac{1}{m})$,where $l = \frac{1}{h}$

Theorem 1's first proof:

The evidence is dependent on the following examples of $S_n(x)$, for $x_i \le x \le x_{i+1}$

$i = 0,1, \ldots, m-1$, we have

$$S_n(x) = f(x_i)\, A_0(t) + f(x_{i+\lambda})\, A_1(t) + f(x_{i+1})\, A_2(t) + h^{\frac{1}{2}} S_n^{\left(\frac{1}{2}\right)}(x_i)\, A_3(t)$$

$$+ h^{\frac{1}{2}} S_n^{\left(\frac{1}{2}\right)}(x_{i+1})\, A_4(t) + h^4 f^{(4)}(x_{i+\lambda})\, A_5(t) \text{ Where } t = \frac{x - x_i}{h} \tag{8}$$

Using the circumstances and equation (8)

$$S_n^{\left(\frac{3}{2}\right)}(0) = f^{\left(\frac{3}{2}\right)}(0), S_n^{\left(\frac{3}{2}\right)}(1) = f^{\left(\frac{3}{2}\right)}(1) \tag{9}$$

As we can see, as provided by (8), it fulfills (2) and is finite in equation (1), i=0, 1,..., m-1. Additionally, we must demonstrate if it is feasible to determine

$S_n^{\left(\frac{3}{2}\right)}(x_i)$, i = 0, 1, …, m − 1, anything uniquely. We employ the knowledge that

$S_n(x) \in C^5[0,1]$ and the following criteria to this end:

$$S_n^{(\frac{3}{2})}(x_{i+}) = S_n^{(\frac{3}{2})}(x_{i-}), \mathrm{i} = 0, 1, \ldots, \mathrm{m}-1, \tag{10}$$

Where $S_n^{(\frac{3}{2})}(x_{i+}) = \lim_{n\to x_i^+} S_n^{(\frac{3}{2})}(x)$ and $S_n^{(\frac{3}{2})}(x_{i-}) = \lim_{n\to x_i^-} S_n^{(\frac{3}{2})}(x)$

then decreased to with the aid of (8) and (9) to get:

$$\frac{3}{8}\lambda^{\frac{1}{2}}(\lambda-1)[-24\lambda^2+40\lambda-15\,]h^{\frac{1}{2}}S_{i-1}^{(\frac{1}{2})}(x_{i-1}) + \frac{3}{2}\lambda^{\frac{1}{2}}(\lambda-1)(16\lambda^3-70\lambda^2+35\lambda+5)h^{\frac{1}{2}}S_i^{(\frac{1}{2})}(x_i)$$

$$-24\lambda^{\frac{5}{2}}(\,\lambda-1)h^{\frac{1}{2}}\,S_i^{(\frac{1}{2})}(x_{i+1}) = -\frac{15\sqrt{\pi}}{4}\left(-6\,\lambda^{\frac{7}{2}}+14\,\lambda^{\frac{5}{2}}-\frac{35}{4}\,\lambda^{\frac{3}{2}}+\frac{3}{4}\right)f(x_{i-1})$$

$$+\frac{15\sqrt{\pi}}{16}\left(28\,\lambda^{\frac{5}{2}}-35\lambda^{\frac{3}{2}}+4\right)f(x_i) - \frac{15\sqrt{\pi}}{4}\,\lambda^{\frac{5}{2}}(6\lambda-7)\,f(x_{i+1}) + \frac{45\sqrt{\pi}}{16}f(x_{i-1+\lambda})$$

$$-\frac{15\sqrt{\pi}}{4}f(x_{i+\lambda}) - \frac{\sqrt{\pi}}{252}\lambda(\lambda-1)\left[12\,\lambda^2-21\lambda+\frac{35}{4}\right]h^4 f^{(4)}(x_{i-1+\lambda})$$

$$+(\frac{\sqrt{\pi}}{126}\lambda^2(\lambda-1)(\,8\,\lambda-7))h^4 f^{(4)}(x_{i+\lambda}) \quad \text{Where } \ i = 1,2,\ldots,m-1 \tag{11}$$

Equation (11) has a single solution and is a strictly tridiagonal dominating system.

The system (11) that proved Theorem 1 may thus obtain i=1,2,..., m-1 $S_n^{(\frac{1}{3})}(x_i)$, in a single way.

1.4. Convergence and Bounds on Error:

The findings of the following tests are supported in this section by the upper limits for mistakes explored first:

Lemma 1:

Let us write $C_i = \left|S^{(\frac{1}{2})}(x_i) - f^{(\frac{1}{2})}(x_i)\right|$, then for $f(x) \in C^5[0,1]$, we have

$$Max\ C_i \le \left[\begin{array}{c}\dfrac{[\frac{15\sqrt{\pi}}{4}\lambda\left(-12\,\lambda^{\frac{5}{2}}+21\,\lambda^{\frac{3}{2}}-\frac{35}{4}\,\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)}{\frac{1}{8}\lambda^{\frac{1}{2}}(\lambda-1)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ \dfrac{+\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left(15\lambda^2+15\lambda-\frac{45}{8}\right)]}{\frac{1}{8}\lambda^{\frac{1}{2}}(\lambda-1)(192\lambda^3-1104\lambda^2+540\lambda+15)}\end{array}\right]\frac{h^{\frac{9}{2}}}{5!}W(f^{(5)};h)$$

,i=1,2,..., m-1 (12)

Where $\lambda \in (0,1)/\{0.569374\}$

Proof:

From (6) and (11), we get

$$\frac{3}{8}\lambda^{\frac{1}{2}}(\lambda-1)[-24\lambda^2+40\lambda-15\,]h^{\frac{1}{2}}\left(S_{i-1}^{\left(\frac{1}{2}\right)}(x_{i-1})-f^{\left(\frac{1}{2}\right)}(x_{i-1})\right)$$

$$+\frac{3}{2}\lambda^{\frac{1}{2}}(\lambda-1)(16\lambda^3-70\lambda^2+35\lambda+5)h^{\frac{1}{2}}\left(S_{i}^{\left(\frac{1}{2}\right)}(x_{i})-f^{\left(\frac{1}{2}\right)}(x_{i})\right)$$

$$-24\lambda^{\frac{5}{2}}(\,\lambda-1)\,h^{\frac{1}{2}}\left(S_{i}^{\left(\frac{1}{2}\right)}(x_{i+1})-f^{\left(\frac{1}{2}\right)}(x_{i+1})\right)$$

$$=-\,\frac{15\sqrt{\pi}}{4}\left(-6\,\lambda^{\frac{7}{2}}+14\,\lambda^{\frac{5}{2}}-\frac{35}{4}\,\lambda^{\frac{3}{2}}+\frac{3}{4}\right)f(x_{i-1})+\frac{15\sqrt{\pi}}{16}\left(28\,\lambda^{\frac{5}{2}}-35\lambda^{\frac{3}{2}}+4\right)f(x_i)$$

$$-\frac{15\sqrt{\pi}}{4}\,\lambda^{\frac{5}{2}}(6\lambda-7)\,f(x_{i+1})+\frac{45\sqrt{\pi}}{16}f(x_{i-1+\lambda})-\frac{15\sqrt{\pi}}{4}f(x_{i+\lambda})$$

$$-\frac{\sqrt{\pi}}{252}\lambda(\lambda-1)\left[12\,\lambda^2-21\lambda+\frac{35}{4}\right]h^4f^{(4)}(x_{i-1+\lambda})+\left(\frac{\sqrt{\pi}}{126}\lambda^2(\lambda-1)(\,8\,\lambda-7)\right)h^4f^{(4)}(x_{i+\lambda})$$

$$-\frac{3}{8}\lambda^{\frac{1}{2}}(\lambda-1)[-24\lambda^2+40\lambda-15\,]h^{\frac{1}{2}}f^{\left(\frac{1}{2}\right)}(x_{i-1})$$

$$-\frac{3}{2}\lambda^{\frac{1}{2}}(\lambda-1)(16\lambda^3-70\lambda^2+35\lambda+5)h^{\frac{1}{2}}f^{\left(\frac{1}{2}\right)}(x_i)+24\lambda^{\frac{5}{2}}(\,\lambda-1)\,h^{\frac{1}{2}}f^{\left(\frac{1}{2}\right)}(x_{i+1})$$

$$=\frac{15\sqrt{\pi}}{4}\left[-6\,\lambda^{\frac{7}{2}}+14\,\lambda^{\frac{5}{2}}-\frac{35}{4}\,\lambda^{\frac{3}{2}}+\frac{3}{4}\right]\frac{h^5}{5!}f^{(5)}(\theta_{1,i})-\frac{15\sqrt{\pi}}{4}\,\lambda^{\frac{5}{2}}(6\lambda-7)\,\frac{h^5}{5!}f^{(5)}(\theta_{2,i})$$

$$+\frac{45\sqrt{\pi}}{16}\frac{(\lambda-1)^5h^5}{5!}f^{(5)}(\theta_{3,i})-\frac{15\sqrt{\pi}}{4}\frac{\lambda^5h^5}{5!}f^{(5)}(\theta_{4,i})$$

$$-\frac{\sqrt{\pi}}{252}\lambda(\lambda-1)(12\,\lambda^2-21\lambda+\frac{35}{4})h^5(\lambda-1)f^{(5)}(\theta_{5,i})$$

$$+\left(\frac{\sqrt{\pi}}{126}\lambda^2(\lambda-1)(\,8\,\lambda-7)\right)h^5\lambda f^{(5)}(\theta_{6,i})$$

$$+\frac{3}{8}\lambda^{\frac{1}{2}}(\lambda-1)[-24\lambda^2+40\lambda-15\,]\frac{32}{945\sqrt{\pi}}h^5f^{(5)}(\theta_{7,i})+24\lambda^{\frac{5}{2}}(\,\lambda-1)\,\frac{32}{945\sqrt{\pi}}h^5f^{(5)}(\theta_{8,i})$$

$$\leq \left[\frac{15\sqrt{\pi}}{4}\lambda\left(-12\,\lambda^{\frac{5}{2}}+21\,\lambda^{\frac{3}{2}}-\frac{35}{4}\,\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^{4}-\frac{155}{84}\lambda^{3}+\frac{110}{84}\lambda^{2}-\frac{329}{126}\lambda+\frac{665}{252}\right)\right.$$
$$\left.+\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)(15\lambda^{2}+15\lambda-\frac{45}{8})\right]\frac{h^{5}}{5!}\varphi_{1}W\left(f^{(5)};\frac{1}{m}\right)$$

$$Max\ C_i \leq \left[\begin{array}{c}\dfrac{[\frac{15\sqrt{\pi}}{4}\lambda\left(-12\,\lambda^{\frac{5}{2}}+21\,\lambda^{\frac{3}{2}}-\frac{35}{4}\,\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^{4}-\frac{155}{84}\lambda^{3}+\frac{110}{84}\lambda^{2}-\frac{329}{126}\lambda+\frac{665}{252}\right)}{\frac{1}{8}\lambda^{\frac{1}{2}}(\lambda-1)(192\lambda^{3}-1104\lambda^{2}+540\lambda+15)} \\ \dfrac{+\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left(15\lambda^{2}+15\lambda-\frac{45}{8}\right)]}{\frac{1}{8}\lambda^{\frac{1}{2}}(\lambda-1)(192\lambda^{3}-1104\lambda^{2}+540\lambda+15)}\end{array}\right]\frac{h^{\frac{9}{2}}}{5!}\varphi_{1}W(f^{(5)};h)$$

this is the complete of the proof.

Where $|\varphi_1| \leq 1, i = 1,2,\ldots,m-1$

Using the diagonal dominating property, the outcome (12) is what comes next.

Lemma 2:

Let $f(x) \in C^{5}[0,1]$ then

$$\left|S_n^{\left(\frac{7}{2}\right)}(x_{i+}) - f^{\left(\frac{7}{2}\right)}(x)\right| \leq \left[\begin{array}{c}\dfrac{\frac{5\sqrt{\pi}}{8}\lambda\,[\left(-\frac{65}{8}\lambda^{4}+41\lambda^{2}-25\lambda+\frac{63}{2}\,\lambda^{\frac{3}{2}}-\frac{315}{8}\lambda^{\frac{1}{2}}\right)(192\lambda^{3}-1104\lambda^{2}+540\lambda+15)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^{3}-1104\lambda^{2}+540\lambda+15)} \\ \dfrac{-1890\left(\lambda-\frac{1}{4}\right)\left(-12\,\lambda^{\frac{5}{2}}+21\,\lambda^{\frac{3}{2}}-\frac{35}{4}\,\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^{4}-\frac{155}{84}\lambda^{3}+\frac{110}{84}\lambda^{2}-\frac{329}{126}\lambda+\frac{665}{252}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^{3}-1104\lambda^{2}+540\lambda+15)} \\ -\dfrac{\frac{320}{3\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left[(192\lambda^{4}-1104\lambda^{3}+540\lambda^{2}+15\lambda)+180\left(\lambda-\frac{1}{4}\right)\left(\lambda^{2}+\lambda-\frac{3}{8}\right)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^{3}-1104\lambda^{2}+540\lambda+15)}\end{array}\right]\frac{h^{\frac{3}{2}}}{5!}W\left(f^{(5)};h\right) \quad (13)$$

$$\left|S_n^{\left(\frac{7}{2}\right)}(x_{i+\lambda}) - f^{\left(\frac{7}{2}\right)}(x)\right| \leq \left[\begin{array}{c}\dfrac{\frac{15\sqrt{\pi}}{4}[\frac{21}{2}\left(\frac{73}{9}\,\lambda^{5}-\frac{352}{21}\,\lambda^{4}+\frac{808}{63}\lambda^{3}-\frac{200}{63}\lambda^{2}+4\,\lambda^{\frac{5}{2}}-5\lambda^{\frac{3}{2}}\right)(192\lambda^{3}-1104\lambda^{2}+540\lambda+15)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^{3}-1104\lambda^{2}+540\lambda+15)} \\ \dfrac{-\lambda\left(315\lambda+\frac{315}{4}\right)\left(-12\,\lambda^{\frac{5}{2}}+21\,\lambda^{\frac{3}{2}}-\frac{35}{4}\,\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^{4}-\frac{155}{84}\lambda^{3}+\frac{110}{84}\lambda^{2}-\frac{329}{126}\lambda+\frac{665}{252}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^{3}-1104\lambda^{2}+540\lambda+15)} \\ -\dfrac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left[\left(15\lambda^{2}+15\lambda-\frac{45}{8}\right)\left(315\,\lambda-\frac{315}{4}\right)+\frac{105}{4}\lambda\left(192\lambda^{3}-1104\lambda^{2}+540\lambda+15\right)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^{3}-1104\lambda^{2}+540\lambda+15)}\end{array}\right]\frac{h^{\frac{3}{2}}}{5!}W\left(f^{(5)};h\right) \quad (14)$$

$$\left|S_n^{\left(\frac{7}{2}\right)}(x_{i-})-f^{\left(\frac{7}{2}\right)}(x)\right|\le\left[\begin{array}{c}\dfrac{\frac{15\sqrt{\pi}}{4}\lambda[(192\lambda^3-1104\lambda^2+540\lambda+15)\left(\frac{21}{4}\lambda^{\frac{3}{2}}-\frac{105}{16}\lambda^{\frac{1}{2}}+\frac{191}{48}\lambda^4-\frac{1019}{48}\lambda^3+\frac{1934}{48}\lambda^2-\frac{1638}{48}\lambda+\frac{959}{48}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\\ \dfrac{-\left(315\lambda-\frac{315}{4}\right)\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\\ +\dfrac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left[\frac{105}{32}(4\lambda-3)(192\lambda^3-1104\lambda^2+540\lambda+15)-\left(315\lambda-\frac{315}{4}\right)\left(15\lambda^2+15\lambda-\frac{45}{8}\right)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\end{array}\right]\frac{h^{\frac{3}{2}}}{5!}W\left(f^{(5)};h\right)\qquad(15)$$

$$\left|S_n^{\left(\frac{3}{2}\right)}(x_{i+\lambda})-f^{\left(\frac{3}{2}\right)}(x_{i+\lambda})\right|\le\left[\begin{array}{c}\dfrac{\frac{15\sqrt{\pi}}{4}\lambda[\left(192\lambda^3-1104\lambda^2+540\lambda+15\right)\left(\frac{21}{32}\lambda^6-\frac{1}{48}\lambda^5-\frac{1081}{504}\lambda^4+\frac{853}{252}\lambda^3-\frac{215}{84}\lambda^2+\frac{7}{9}\lambda+\frac{9}{8}\lambda^{\frac{7}{2}}-\frac{81}{32}\lambda^{\frac{5}{2}}+\frac{21}{16}\lambda^{\frac{3}{2}}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\\ -\dfrac{\left(\frac{135}{2}\lambda^3-\frac{675}{8}\lambda^2+\frac{135}{4}\lambda-\frac{15}{2}\right)\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\\ -\dfrac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)[\left(15\lambda^2+15\lambda-\frac{45}{8}\right)\left(\frac{135}{2}\lambda^3-\frac{675}{8}\lambda^2+\frac{135}{4}\lambda-\frac{15}{2}\right)+(\frac{27}{8}\lambda^7-\frac{45}{16}\lambda^6+\frac{45}{8}\lambda^3-\frac{9}{2}\lambda^2)\left(192\lambda^3-1104\lambda^2+540\lambda+15\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\end{array}\right]\frac{h^{\frac{7}{2}}}{5!}W\left(f^{(5)};h\right)\qquad(16)$$

and

$$\left|S_n^{\left(\frac{1}{2}\right)}(x_{i+\lambda})-f^{\left(\frac{1}{2}\right)}(x_{i+\lambda})\right|\le\left[\begin{array}{c}\dfrac{\frac{15\sqrt{\pi}}{16}\lambda^2[\frac{15\sqrt{\pi}}{16}\lambda^2[\lambda(\lambda-1)\left(192\lambda^3-1104\lambda^2+540\lambda+15\right)\left(\frac{6847}{72}\lambda^4-\frac{145}{126}\lambda^3+\frac{41}{63}\lambda^2-\frac{2}{9}\lambda+\frac{1}{2}\lambda^{\frac{3}{2}}-7\lambda^{\frac{1}{2}}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\\ -\dfrac{\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)\left(10\lambda^3-\frac{17}{2}\lambda^2+\frac{9}{2}\lambda-10\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\\ -\dfrac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{3}{2}}(\lambda-1)[\lambda^2\left(192\lambda^3-1104\lambda^2+540\lambda+15\right)(\frac{3}{4}\lambda^4-\frac{5}{8}\lambda^3-\frac{5}{8}\lambda+\frac{3}{4})+\left(15\lambda^2+15\lambda-\frac{45}{8}\right)\left(\frac{15}{2}\lambda^3-\frac{105}{8}\lambda^2+\frac{57}{8}\lambda-\frac{5}{2}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\end{array}\right]\frac{h^{\frac{9}{2}}}{5!}W\left(f^{(5)};h\right)\qquad(17)$$

Proof:

What we get from (5), (6), and (8) is:

$$h^{\frac{7}{2}}S_n^{\left(\frac{7}{2}\right)}(x_{i+})=\frac{-315\sqrt{\pi}\left(4\lambda^{\frac{5}{2}}-5\lambda^{\frac{3}{2}}+1\right)}{8\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}f(x_i)+\frac{315\sqrt{\pi}}{8\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}f(x_{i+\lambda})+\frac{\frac{315\sqrt{\pi}}{8}\lambda^{\frac{3}{2}}(4\lambda-5)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}f(x_{i+1})$$
$$-\frac{\frac{105}{4}\lambda^{\frac{1}{2}}(\lambda-1)(4\lambda-3)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_i)-\frac{210\lambda^{\frac{3}{2}}(\lambda-1)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_{i+1})-\frac{\frac{\sqrt{\pi}}{24}\lambda(\lambda-1)(16\lambda^2+16\lambda-25)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^4f^{(4)}(x_{i+\lambda})$$

Hence

$$h^{\frac{7}{2}}\left(S_n^{\left(\frac{7}{2}\right)}(x_{i+})-f^{\left(\frac{7}{2}\right)}(x)\right)=\frac{315\sqrt{\pi}\lambda^5}{8\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\frac{h^5}{5!}f^{(5)}(\theta_{4,i})+\frac{\frac{315\sqrt{\pi}}{8}\lambda^{\frac{3}{2}}(4\lambda-5)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\frac{h^5}{5!}f^{(5)}(\theta_{2,i})$$
$$+\frac{-\frac{105}{4}\lambda^{\frac{1}{2}}(\lambda-1)(4\lambda-3)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^{\frac{1}{2}}\left(S_n^{\left(\frac{1}{2}\right)}(x_i)-f^{\left(\frac{1}{2}\right)}(x_i)\right)+\frac{-210\lambda^{\frac{3}{2}}(\lambda-1)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^{\frac{1}{2}}\left(S_n^{\left(\frac{1}{2}\right)}(x_{i+1})-f^{\left(\frac{1}{2}\right)}(x_{i+1})\right)$$

$$+\frac{-210}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left[\lambda^{\frac{3}{2}}(\lambda-1)\right]\frac{32}{945\sqrt{\pi}}h^5 f^{(5)}(\theta_{8,i})+\frac{-\frac{\sqrt{\pi}}{24}\lambda^2(\lambda-1)(16\lambda^2+16\lambda-25)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^5 f^{(5)}(\theta_{6,i})$$

$$\leq\left[\begin{array}{c}\frac{\frac{5\sqrt{\pi}}{8}\lambda\left[\left(-\frac{65}{8}\lambda^4+41\lambda^2-25\lambda+\frac{63}{2}\lambda^{\frac{3}{2}}-\frac{315}{8}\lambda^{\frac{1}{2}}\right)(192\lambda^3-1104\lambda^2+540\lambda+15)\right.}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\\ \frac{\left.-1890\left(\lambda-\frac{1}{4}\right)\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\\ -\frac{\frac{320}{3\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left[(192\lambda^4-1104\lambda^3+540\lambda^2+15\lambda)+180\left(\lambda-\frac{1}{4}\right)\left(\lambda^2+\lambda-\frac{3}{8}\right)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\end{array}\right]\frac{h^{\frac{3}{2}}}{5!}\varphi_2 W\left(f^{(5)};h\right)$$

Where $|\varphi_2|\leq 1$

By using (12) , we get (13). The proofs of (14)-(17) are similar, and we only mention that

$$h^{\frac{7}{2}}S_n^{\left(\frac{7}{2}\right)}(x_{i-})=\frac{-315\sqrt{\pi}\left(4\lambda^{\frac{5}{2}}-5\lambda^{\frac{3}{2}}+1\right)}{8\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}f(x_{i-1})+\frac{315\sqrt{\pi}}{8\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}f(x_{i-1+\lambda})+\frac{\frac{315\sqrt{\pi}}{8}\lambda^{\frac{3}{2}}(4\lambda-5)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}f(x_i)$$

$$-\frac{\frac{105}{4}\lambda^{\frac{1}{2}}(\lambda-1)(4\lambda-3)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_{i-1})-\frac{210\lambda^{\frac{3}{2}}(\lambda-1)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_i)-\frac{\frac{\sqrt{\pi}}{24}\lambda(\lambda-1)(16\lambda^2+16\lambda-35)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^4 f^{(4)}(x_{i-1+\lambda})$$

$$h^{\frac{7}{2}}S_n^{\left(\frac{7}{2}\right)}(x_{i+\lambda})=\frac{-315\sqrt{\pi}}{8\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left(4\lambda^{\frac{5}{2}}-5\lambda^{\frac{3}{2}}+1\right)f(x_i)+\frac{315\sqrt{\pi}}{8\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}f(x_{i+\lambda})+\frac{\frac{315\sqrt{\pi}}{8}\lambda^{\frac{3}{2}}(4\lambda-5)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}f(x_{i+1})$$

$$-\frac{\frac{105}{4}\lambda^{\frac{1}{2}}(\lambda-1)(4\lambda-3)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_i)-\frac{210\lambda^{\frac{3}{2}}(\lambda-1)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_{i+1})+\frac{\frac{\sqrt{\pi}}{24}\lambda(\lambda-1)(56\lambda^2-76\lambda+25)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^4 f^{(4)}(x_{i+\lambda})$$

$$h^{\frac{3}{2}}S_n^{\left(\frac{3}{2}\right)}(x_{i+\lambda})=\frac{-\frac{15\sqrt{\pi}}{4}}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left[9\lambda^{\frac{9}{2}}-\frac{81}{4}\lambda^{\frac{7}{2}}+\frac{21}{2}\lambda^{\frac{5}{2}}+\frac{21}{4}\lambda^2-\frac{11}{2}\lambda+1\right]f(x_i)+\frac{\frac{15\sqrt{\pi}}{4}}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left(\frac{21}{4}\lambda^2-\frac{11}{2}\lambda+1\right)f(x_{i+\lambda})$$

$$+\frac{\frac{15\sqrt{\pi}}{4}\lambda^{\frac{5}{2}}}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left(9\lambda^2-\frac{81}{4}\lambda+\frac{21}{2}\right)f(x_{i+1})-\frac{\lambda^{\frac{1}{2}}(\lambda-1)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left(\frac{45}{2}\lambda^3-\frac{387}{8}\lambda^2+\frac{135}{4}\lambda-\frac{15}{2}\right)h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_i)$$

$$-\frac{\lambda^{\frac{3}{2}}(\lambda-1)(45\lambda^2-36\lambda)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_{i+1})+\frac{\frac{\sqrt{\pi}}{12}\lambda(\lambda-1)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left[2\lambda^4-\frac{101}{21}\lambda^3+\frac{449}{84}\lambda^2-\frac{7}{3}\lambda\right]h^4 f^{(4)}(x_{i+\lambda})$$

and

$$h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_{i+\lambda})=\frac{-\frac{15\sqrt{\pi}}{16}}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left(4\lambda^{\frac{11}{2}}-11\lambda^{\frac{9}{2}}+7\lambda^{\frac{7}{2}}+7\lambda^3-11\lambda^2+4\lambda\right)f(x_i)-\frac{\frac{15\sqrt{\pi}}{16}\lambda(\lambda-1)(-7\lambda+4)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}f(x_{i+\lambda})$$

$$+\frac{\frac{15\sqrt{\pi}}{16}\lambda^{\frac{7}{2}}(\lambda-1)(4\lambda-7)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}f(x_{i+1})-\frac{\lambda^{\frac{3}{2}}(\lambda-1)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left(\frac{5}{2}\lambda^3-\frac{57}{8}\lambda^2+\frac{57}{8}\lambda-\frac{5}{2}\right)h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_i)$$

$$-\frac{\lambda^{\frac{7}{2}}(\lambda-1)(5\lambda-6)}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}h^{\frac{1}{2}}S_n^{\left(\frac{1}{2}\right)}(x_{i+1})-\frac{\lambda^3(\lambda-1)\sqrt{\pi}}{\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)}\left(-\frac{53}{9}\lambda^3+\frac{41}{1008}\lambda^2-\frac{41}{1008}\lambda+\frac{1}{72}\right)h^4 f^{(4)}(x_{i+\lambda})$$

Proof of the Theorem 2:

For $0 \le y \le 1$, we obtain

$$A_0(y) + A_1(y) + A_2(y) = 1. \tag{18}$$

Let $x_i \le x \le x_{i+1}$, on using (18) and (8) we obtain

$$S_n^{\left(\frac{7}{2}\right)}(x) - f^{\left(\frac{7}{2}\right)}(x) = \left(S_n^{\left(\frac{7}{2}\right)}(x_{i+}) - f^{\left(\frac{7}{2}\right)}(x)\right) A_0(t) + \left(S_n^{\left(\frac{7}{2}\right)}(x_{i+\lambda}) - f^{\left(\frac{7}{2}\right)}(x)\right) A_1(t)$$
$$+ \left(S_n^{\left(\frac{7}{2}\right)}(x_{i-}) - f^{\left(\frac{7}{2}\right)}(x)\right) A_2(t) = U_1 + U_2 + U_3 \tag{19}$$

From (4), it is evident that:

$$U_1 = \left(S_n^{\left(\frac{7}{2}\right)}(x_{i+}) - f^{\left(\frac{7}{2}\right)}(x)\right) A_0(t) \to |U_1| = \left|\left(S_n^{\left(\frac{7}{2}\right)}(x_{i+}) - f^{\left(\frac{7}{2}\right)}(x)\right) A_0(t)\right|$$
$$= \left|S_n^{\left(\frac{7}{2}\right)}(x_{i+}) - f^{\left(\frac{7}{2}\right)}(x)\right| |\, A_0(t)| \le \left|S_n^{\left(\frac{7}{2}\right)}(x_{i+}) - f^{\left(\frac{7}{2}\right)}(x)\right|$$

$|\, A_0(t)| \le 1$, $|\, A_1(t)| \le 1$ and $|\, A_2(t)| \le 1$ on $0 \le x \le 1$

Since $f^{\left(\frac{7}{2}\right)}(x) = f^{\left(\frac{7}{2}\right)}(x_i) + 2\frac{(x-x_i)^{1/2}}{\sqrt{\pi}} f^{(4)}(x_i) + (x - x_i) f^{\left(\frac{9}{2}\right)}(x_i) + \frac{4}{3}\frac{(x-x_i)^{3/2}}{\sqrt{\pi}} f^{(5)}\big(\theta_{11,i}\big)$

for $x_i < \theta_{10,i} < x_{i+1}$,

Therefore, on using (13) and $|x - x_i| \le h$. We obtain

$$|U_1| \le \left[\frac{\frac{5\sqrt{\pi}}{8}\lambda\left[\left(-\frac{65}{8}\lambda^4+41\lambda^2-25\lambda+\frac{63}{2}\lambda^{\frac{3}{2}}-\frac{315}{8}\lambda^{\frac{1}{2}}\right)(192\lambda^3-1104\lambda^2+540\lambda+15)\right.}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \right.$$
$$\frac{\left.-1890\left(\lambda-\frac{1}{4}\right)\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}$$
$$\left. - \frac{\frac{320}{3\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left[(192\lambda^4-1104\lambda^3+540\lambda^2+15\lambda)+180\left(\lambda-\frac{1}{4}\right)\left(\lambda^2+\lambda-\frac{3}{8}\right)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)}\right]\frac{h^{\frac{3}{2}}}{5!}W\left(f^{(5)};h\right) \tag{20}$$

Similarly,

$$|U_3| \le \left[\begin{array}{c} \frac{\frac{15\sqrt{\pi}}{4}\lambda[(192\lambda^3-1104\lambda^2+540\lambda+15)\left(\frac{21}{4}\lambda^{\frac{3}{2}}-\frac{105}{16}\lambda^{\frac{1}{2}}+\frac{191}{48}\lambda^4-\frac{1019}{48}\lambda^3+\frac{1934}{48}\lambda^2-\frac{1638}{48}\lambda+\frac{959}{48}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ \frac{-\left(315\lambda-\frac{315}{4}\right)\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ +\frac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left[\frac{105}{32}(4\lambda-3)(192\lambda^3-1104\lambda^2+540\lambda+15)-\left(315\lambda-\frac{315}{4}\right)\left(15\lambda^2+15\lambda-\frac{45}{8}\right)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \end{array}\right] \frac{h^{\frac{3}{2}}}{5!} W\left(f^{(5)};h\right) \qquad (21)$$

and

$$U_2 = \left(S_n^{\left(\frac{7}{2}\right)}(x_{i+\lambda}) - f^{\left(\frac{7}{2}\right)}(x)\right) A_1(t) = \left(S_n^{\left(\frac{7}{2}\right)}(x_{i+\lambda}) - f^{\left(\frac{7}{2}\right)}(x)\right) A_1\left(\frac{x-x_i}{h}\right) \qquad (22)$$

$$|U_2| \le \left[\begin{array}{c} \frac{\frac{15\sqrt{\pi}}{4}[\frac{21}{2}\left(\frac{73}{9}\lambda^5-\frac{352}{21}\lambda^4+\frac{808}{63}\lambda^3-\frac{200}{63}\lambda^2+4\lambda^{\frac{5}{2}}-5\lambda^{\frac{3}{2}}\right)(192\lambda^3-1104\lambda^2+540\lambda+15)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ \frac{-\lambda\left(315\lambda+\frac{315}{4}\right)\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\frac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left[\left(15\lambda^2+15\lambda-\frac{45}{8}\right)\left(315\lambda-\frac{315}{4}\right)+\frac{105}{4}\lambda(192\lambda^3-1104\lambda^2+540\lambda+15)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \end{array}\right] \frac{h^{\frac{3}{2}}}{5!} W\left(f^{(5)};h\right) \qquad (23)$$

Therefore, by using (20)-(23) and putting in (19) we obtain

$$\left|S_n^{\left(\frac{7}{2}\right)}(x) - f^{\left(\frac{7}{2}\right)}(x)\right| \le \left[\begin{array}{c} \frac{\frac{15\sqrt{\pi}}{4}\lambda[(192\lambda^3-1104\lambda^2+540\lambda+15)(\frac{511}{6}\lambda^5-\frac{1387}{8}\lambda^4+\frac{1815}{16}\lambda^3+\frac{331}{24}\lambda^2-\frac{919}{24}\lambda+42\lambda^{\frac{5}{2}}-42\lambda^{\frac{3}{2}}-\frac{105}{8}\lambda^{\frac{1}{2}}+\frac{959}{48})}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\frac{1575\left(\lambda-\frac{1}{4}\right)\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\frac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left[4725\left(\lambda-\frac{1}{4}\right)\left(\lambda^2+\lambda-\frac{3}{8}\right)+\frac{315}{8}\left(\lambda+\frac{1}{4}\right)(192\lambda^3-1104\lambda^2+540\lambda+15)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \end{array}\right] \frac{h^{\frac{3}{2}}}{5!} W\left(f^{(5)};h\right) \qquad (24)$$

This proves Theorem 2 for $r = \frac{7}{2}$. To Prove the Theorem 2 for $r = \frac{5}{2}$:

$$S_n^{\left(\frac{5}{2}\right)}(x) - f^{\left(\frac{5}{2}\right)}(x) = \int_{x_{i+\lambda}}^{x}\left(S_n^{\left(\frac{7}{2}\right)}(x) - f^{\left(\frac{7}{2}\right)}(x)\right)dt + S_n^{\left(\frac{5}{2}\right)}(x_{i+\lambda}) - f^{\left(\frac{5}{2}\right)}(x_{i+\lambda}) = \int_{x_{i+\lambda}}^{x}\left(S_n^{\left(\frac{7}{2}\right)}(x) - f^{\left(\frac{7}{2}\right)}(x)\right)dt$$

since $\quad S_n^{\left(\frac{5}{2}\right)}(x_{i+\lambda}) = f^{\left(\frac{5}{2}\right)}(x_{i+\lambda})$

On using (24) we get

$$\left|S_n^{\left(\frac{5}{2}\right)}(x)-f^{\left(\frac{5}{2}\right)}(x)\right| \le \begin{bmatrix} \dfrac{\frac{3\sqrt{\pi}}{2}\lambda[(192\lambda^3-1104\lambda^2+540\lambda+15)(\frac{511}{6}\lambda^5-\frac{1387}{8}\lambda^4+\frac{1815}{16}\lambda^3+\frac{331}{24}\lambda^2-\frac{919}{24}\lambda+42\lambda^{\frac{5}{2}}-42\lambda^{\frac{3}{2}}-\frac{105}{8}\lambda^{\frac{1}{2}}+\frac{959}{48})}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\dfrac{1575\left(\lambda-\frac{1}{4}\right)\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\dfrac{\frac{512}{315\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)\left[4725\left(\lambda-\frac{1}{4}\right)\left(\lambda^2+\lambda-\frac{3}{8}\right)+\frac{315}{8}\left(\lambda+\frac{1}{4}\right)(192\lambda^3-1104\lambda^2+540\lambda+15)\right]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \end{bmatrix} \frac{h^{\frac{5}{2}}}{5!}W\left(f^{(5)};h\right) \qquad (25)$$

which proves Theorem 2 for $r=\frac{5}{2}$. To Prove the Theorem 2 for $r=\frac{3}{2}$:

Since

$$S_n^{\left(\frac{3}{2}\right)}(x)-f^{\left(\frac{3}{2}\right)}(x)=\int_{x_{i+\lambda}}^{x}\left(S_n^{\left(\frac{5}{2}\right)}(x)-f^{\left(\frac{5}{2}\right)}(x)\right)dt+S_n^{\left(\frac{3}{2}\right)}(x_{i+\lambda})-f^{\left(\frac{3}{2}\right)}(x_{i+\lambda})$$

and using (16) and (25) we obtain

$$\left|S_n^{\left(\frac{3}{2}\right)}(x)-f^{\left(\frac{3}{2}\right)}(x)\right| \le \begin{bmatrix} \dfrac{\frac{15\sqrt{\pi}}{4}\lambda[(192\lambda^3-1104\lambda^2+540\lambda+15)\left(\frac{21}{32}\lambda^6+\frac{777}{80}\lambda^5-\frac{55337}{2520}\lambda^4+\frac{1030}{63}\lambda^3-\frac{413}{420}\lambda^2-\frac{2267}{630}\lambda+\frac{9}{8}\lambda^{\frac{7}{2}}+\frac{363}{160}\lambda^{\frac{5}{2}}-\frac{279}{80}\lambda^{\frac{3}{2}}-\frac{3}{2}\lambda^{\frac{1}{2}}+\frac{137}{60}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\dfrac{\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)\left(\frac{135}{2}\lambda^3-\frac{675}{8}\lambda^2+\frac{855}{4}\lambda-\frac{105}{2}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\dfrac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)[\left(15\lambda^2+15\lambda-\frac{45}{8}\right)\left(\frac{135}{2}\lambda^3-\frac{675}{8}\lambda^2+\frac{279}{4}\lambda-\frac{33}{2}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ \dfrac{+(192\lambda^3-1104\lambda^2+540\lambda+15)\left(\frac{27}{8}\lambda^7-\frac{45}{16}\lambda^6+\frac{45}{8}\lambda^3-\frac{9}{2}\lambda^2+\frac{1}{18}\lambda+\frac{1}{72}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \end{bmatrix} \frac{h^{\frac{7}{2}}}{5!}W\left(f^{(5)};h\right)$$

(26)

which proves Theorem 2 for $r=\frac{3}{2}$. To Prove the Theorem 2 for $r=\frac{1}{2}$:

Since

$$S_n^{\left(\frac{1}{2}\right)}(x)-f^{\left(\frac{1}{2}\right)}(x)=\int_{x_{i+\lambda}}^{x}\left(S_n^{\left(\frac{3}{2}\right)}(x)-f^{\left(\frac{3}{2}\right)}(x)\right)dt+S_n^{\left(\frac{1}{2}\right)}(x_{i+\lambda})-f^{\left(\frac{1}{2}\right)}(x_{i+\lambda})$$

and using (17) and (26) we get

$$S_n^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x) = \left[\begin{array}{c} \dfrac{\frac{15\sqrt{\pi}}{16}\lambda[(192\lambda^3-1104\lambda^2+540\lambda+15)\left(\frac{6847}{72}\lambda^7-\frac{48509}{504}\lambda^6+\frac{227}{126}\lambda^5-\frac{55}{63}\lambda^4+\frac{2}{9}\lambda^3+\frac{1}{2}\lambda^{\frac{9}{2}}-\frac{15}{2}\lambda^{\frac{7}{2}}+7\lambda^{\frac{5}{2}}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\dfrac{\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)\left(10\lambda^4+\frac{103}{2}\lambda^3-\frac{141}{2}\lambda^2+180\lambda-\frac{140}{3}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\dfrac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)[(192\lambda^3-1104\lambda^2+540\lambda+15)\left(\frac{3}{2}\lambda^7-\frac{5}{4}\lambda^6-\frac{5}{8}\lambda^4+2\lambda^3-\lambda^2+\frac{1}{81}\lambda+\frac{1}{324}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ \dfrac{+\left(15\lambda^2+15\lambda-\frac{45}{8}\right)\left(\frac{15}{2}\lambda^4+\frac{15}{8}\lambda^3-\frac{93}{8}\lambda^2+13\lambda-\frac{11}{3}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \end{array}\right]\frac{h^{\frac{9}{2}}}{5!}W\left(f^{(5)};h\right) \tag{27}$$

which proves Theorem 2 for $r=\frac{1}{2}$. To Prove the Theorem 2 for $r=0$:

Since $S_n(x_{i+\lambda}) = f(x_{i+\lambda})$

$$S_n(x) - f(x) = \int_{x_{i+\lambda}}^{x}\left(S_n^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x)\right)dt + S_n(x_{i+\lambda}) - f(x_{i+\lambda}) = \int_{x_{i+\lambda}}^{x}\left(S_n^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x)\right)dt$$

and using (27) we get

$$|S(x)-f(x)| \le \left[\begin{array}{c} \dfrac{\frac{15\sqrt{\pi}}{16}\lambda[(192\lambda^3-1104\lambda^2+540\lambda+15)\left(\frac{6847}{72}\lambda^7-\frac{48509}{504}\lambda^6+\frac{227}{126}\lambda^5-\frac{55}{63}\lambda^4+\frac{2}{9}\lambda^3+\frac{1}{2}\lambda^{\frac{9}{2}}-\frac{15}{2}\lambda^{\frac{7}{2}}+7\lambda^{\frac{5}{2}}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\dfrac{\left(-12\lambda^{\frac{5}{2}}+21\lambda^{\frac{3}{2}}-\frac{35}{4}\lambda^{\frac{1}{2}}+\frac{65}{252}\lambda^4-\frac{155}{84}\lambda^3+\frac{110}{84}\lambda^2-\frac{329}{126}\lambda+\frac{665}{252}\right)\left(10\lambda^4+\frac{103}{2}\lambda^3-\frac{141}{2}\lambda^2+180\lambda-\frac{140}{3}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ -\dfrac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)[(192\lambda^3-1104\lambda^2+540\lambda+15)\left(\frac{3}{2}\lambda^7-\frac{5}{4}\lambda^6-\frac{5}{8}\lambda^4+2\lambda^3-\lambda^2+\frac{1}{81}\lambda+\frac{1}{324}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \\ +\dfrac{\left(15\lambda^2+15\lambda-\frac{45}{8}\right)\left(\frac{15}{2}\lambda^4+\frac{15}{8}\lambda^3-\frac{93}{8}\lambda^2+13\lambda-\frac{11}{3}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3-1104\lambda^2+540\lambda+15)} \end{array}\right]\frac{63\sqrt{\pi}}{256}\frac{h^5}{5!}W\left(f^{(5)};h\right)$$

$$\left\|S_n^{(r)}(x) - f^{(r)}(x)\right\| \le V\,\frac{l^{r-5}}{5!}\; w\left(f^{(5)};\frac{1}{m}\right)\;;\; r=\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2}$$

$$V = \left[\begin{array}{c} \frac{\frac{15\sqrt{\pi}}{16}\lambda[(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)\left(\frac{6847}{72}\lambda^7 - \frac{48509}{504}\lambda^6 + \frac{227}{126}\lambda^5 - \frac{55}{63}\lambda^4 + \frac{2}{9}\lambda^3 + \frac{1}{2}\lambda^{\frac{9}{2}} - \frac{15}{2}\lambda^{\frac{7}{2}} + 7\lambda^{\frac{5}{2}}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)} \\ - \frac{\left(-12\lambda^{\frac{5}{2}} + 21\lambda^{\frac{3}{2}} - \frac{35}{4}\lambda^{\frac{1}{2}} + \frac{65}{252}\lambda^4 - \frac{155}{84}\lambda^3 + \frac{110}{84}\lambda^2 - \frac{329}{126}\lambda + \frac{665}{252}\right)\left(10\lambda^4 + \frac{103}{2}\lambda^3 - \frac{141}{2}\lambda^2 + 180\lambda - \frac{140}{3}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)} \\ - \frac{\frac{256}{63\sqrt{\pi}}\lambda^{\frac{1}{2}}(\lambda-1)[(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)\left(\frac{3}{2}\lambda^7 - \frac{5}{4}\lambda^6 - \frac{5}{8}\lambda^4 + 2\lambda^3 - \lambda^2 + \frac{1}{81}\lambda + \frac{1}{324}\right)}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)} \\ + \frac{\left(15\lambda^2 + 15\lambda - \frac{45}{8}\right)\left(\frac{15}{2}\lambda^4 + \frac{15}{8}\lambda^3 - \frac{93}{8}\lambda^2 + 13\lambda - \frac{11}{3}\right)]}{\frac{1}{8}\lambda^{\frac{3}{2}}(\lambda-1)(6\lambda-5)(192\lambda^3 - 1104\lambda^2 + 540\lambda + 15)} \end{array} \right]$$

and $\|S_n(x) - f(x)\| \leq V \frac{l^{r-5}}{5!} \frac{63\sqrt{\pi}}{256} w(f^{(5)}; \frac{1}{m})$

This completes the proof of theorem 2

Conclusions

We were able to demonstrate gap interpolation in the general case (0,4) with a quintic spline and also showed that this is the case. It exists, unique, and the margin of error for this case has been found. Spline functions have been showed that the developed scheme produced better results in terms of errors and performance norms. As a result, the fractional spline interpolation function was proposed as a way to identify the fractional initial value problem.

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